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LOCAL LIMITS AND TRIPLEABILITY

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If A is an object in a category $\mathscr{A}$, the properties of $\mathscr{A}/A$ (the category of objects over A) may be considered as local properties of $\mathscr{A}$. Using 'local' in this sense, the notion of local universality is defined and some of its basic properties developed. These ideas are then applied in a brief discussion of local adjunction and local limits. Finally two local tripleability theorems are given.

The Lawvere comma category of the diagram

$$1_{\mathscr{B}}: \mathscr{B} \longrightarrow \mathscr{B} \longleftarrow \mathscr{A}: F$$

is denoted by $\mathscr{B}/F$, in particular $\mathscr{B}/B$ denotes the category of objects over B, when B is an object of $\mathscr{B}$.

Given a functor $F: \mathscr{A} \to \mathscr{B}$ we define [3] $\bar{F}: \mathscr{A}^2 \to \mathscr{B}/F$ and, for each object A of $\mathscr{A}$, $F^A: \mathscr{A}/A \to \mathscr{B}/F(A)$ by the following pull-back diagrams in $\mathscr{C}\mathscr{A}\mathscr{T}$:—

$$\begin{tikzcd}
\mathscr{A}^2 \arrow[dr, dashed, "\bar{F}"] \arrow[drr, "\ulcorner\delta\urcorner_{\mathscr{A}}"] \arrow[ddr, "F^2"'] & & \\
& \mathscr{B}/F \arrow[r] \arrow[d] & \mathscr{A} \arrow[d, "F"] \\
& \mathscr{B}^2 \arrow[r, "\ulcorner\delta\urcorner_{\mathscr{B}}"] & \mathscr{B}
\end{tikzcd} .$$

$$\begin{tikzcd}
\mathscr{A}/A \arrow[dr, dashed, "F^A"] \arrow[drr] \arrow[dd] & & \\
& \mathscr{B}/F(A) \arrow[r] \arrow[d] & 1 \arrow[d, "\ulcorner F(A)\urcorner"] \\
\mathscr{A}^2 \arrow[r, "F^2"] & \mathscr{B}^2 \arrow[r, "\ulcorner\delta\urcorner_{\mathscr{B}}"] & \mathscr{B}
\end{tikzcd}$$

LEMMA 1. *For any category $\mathscr{C}$ there are isomorphisms making*

$$\begin{tikzcd}
(\mathscr{B}^2)^{\mathscr{C}} \arrow[r, "\bar{F}^{\mathscr{C}}"] \arrow[d, "\wr\wr"] & (\mathscr{B}/F)^{\mathscr{C}} \arrow[d, "\wr\wr"] \\
(\mathscr{B}^{\mathscr{C}})^2 \arrow[r, "\overline{F^{\mathscr{C}}}"] & \mathscr{B}^{\mathscr{C}}/F^{\mathscr{C}}
\end{tikzcd}$$

commute.

Proof. $\mathscr{CAT}$ is cartesian-closed.

Let $F: \mathscr{A} \to \mathscr{B}$ be a functor and $f: B \to F(A)$ a map in $\mathscr{B}$. The pair $\langle f_0, f_1 \rangle$ is a *locally-universal pair* (dual colocally-couniversal) from f to F when $f_0: f \to F^A(f_1) \in \mathscr{B}/F(A)$ is universal from f to F^A in the sense of MacLane [6]. Similarly a *locally-couniversal pair* (dual colocally-universal) from F to f is a pair $\langle f_0, f_1 \rangle$ for which $f_0: F^A(f_1) \to f$ is couniversal from F^A to f.

LEMMA 2. *The pair $\langle f_0, f_1 \rangle$ is locally-couniversal from F to f if and only if $(f_0, 1_A): \bar{F}(f_1) \to (B, f: B \to F(A), A) \in \mathscr{B}/F$ is couniversal from $\bar{F}$ to f.*

THEOREM 1. *Let $F: \mathscr{A} \to \mathscr{B}$ be a functor and $f: B \to F(A)$ a map in $\mathscr{B}$. There is a locally-couniversal pair from F to f if and only if there is a couniversal map from $\bar{F}$ to $(B, f: B \to F(A), A) \in \mathscr{B}/F$.*

Proof. If $(f_0, f_1): \bar{F}(f_2) \to (B, f: B \to F(A), A)$ is couniversal from $\bar{F}$ to f, then so is $(f_0, 1_A): \bar{F}(f_1 f_2) \to (B, f: B \to F(A), A)$. The result now follows by Lemma 2.

The corresponding result for locally-universal pairs is false in general, although Kaput [3] showed that a functor F is locally-adjunctable if and only if $\bar{F}$ has an adjoint.

The functor $F: \mathscr{A} \to \mathscr{B}$ is *locally-adjunctable* if a locally-universal pair exists for every $f: B \to F(A) \in \mathscr{B}$ and *locally-coadjunctable* if a locally-couniversal pair exists for every $g: F(A) \to B \in \mathscr{B}$.

Leroux [5] stated the following proposition, which is an immediate consequence of Theorem 1.

PROPOSITION 1. *A functor F is locally-coadjunctable if and only if $\bar{F}$ has a coadjoint.*

From which, by Lemma 1, we obtain

PROPOSITION 2. *If a functor $F: \mathscr{A} \to \mathscr{B}$, is locally-coadjunctable then so is $F^{\mathscr{C}}: \mathscr{A}^{\mathscr{C}} \to \mathscr{B}^{\mathscr{C}}$ for any category $\mathscr{C}$.*

THEOREM 2. *Let $F: \mathscr{A} \to \mathscr{B}$ be a functor and $f: B \to F(A)$, $c: F(A') \to B$, $c': F(A'') \to F(A)$ maps in $\mathscr{B}$ with c' couniversal from F to $F(A)$. By couniversality there are (unique) maps s, t for which $c'F(s) = 1_{F(A)}$ and $c'F(t) = fc$.*

(A) *Suppose c is couniversal from F to B. The diagram $s: A \to A'' \leftarrow A': t$ has a pullback in $\mathscr{A}$ if and only if there is a locally-counivearsal pair from F to f.*

(B) *Suppose c' is an identity map. The map c is couniversal from F to B if and only if $\langle c, t\rangle$ is locally-couniversal from F to f.*

Proof. (A) If

$$\begin{tikzcd} \tilde{A} \arrow[r, "s'"] \arrow[d, "f_1"'] \arrow[dr, phantom, "(*)"] & A' \arrow[d, "t"] \\ A \arrow[r, "s"] & A'' \end{tikzcd}$$

is a pullback in $\mathscr{A}$, then $\langle cF(s'), f_1\rangle$ is locally-couniversal from F to f. Conversely, if $\langle f_0, f_1\rangle$ is locally-couniversal from F to f then (*) is a pullback where s' is the unique map for which $cF(s') = f_0$.

(B) Suppose $\langle c, t\rangle$ is locally-couniversal from F to f. For any $g: F(\tilde{A}) \to B$, by the couniversality of $1_{F(A)}$, there is a unique h for which $F(h) = fg$. Thus there is a unique p for which (i) $tp = h$ and (ii) $cF(p) = g$. Again by the couniversality of $1_{F(A)}$, (ii) implies (i) and so p is unique in satisfying (ii).

The converse follows directly from (A).

COROLLARY 1. (Leroux) *If a functor $F: \mathscr{A} \to \mathscr{B}$ has a coadjoint and $\mathscr{A}$ has pullbacks then F is locally-coadjunctable.*

Let $\Delta: \mathscr{C} \to \mathscr{C}^{\mathscr{X}}$ be the canonical embedding. If $D: \mathscr{X} \to \mathscr{C}$ is a functor and $\rho: D \to \Delta(X) \in \mathscr{C}^{\mathscr{X}}$, then a locally-universal pair from ρ to Δ is called a *local colimit* for D at ρ and a locally-couniversal pair from Δ to ρ is called a *local limit* for D at ρ.

COROLLARY 2. (Folklore) *When $\mathscr{X}$ is a connected category, D has a limit if and only if it has a local limit at ρ.*

The definitions of local limit and colimit are the obvious ones to make (given the definition of local universality) but they are essentially renameings of limits and colimits in $\mathscr{C}/X$. Thus the standard theorems on limits, colimits and adjointness have local counterparts; moreover, in most cases the local result is a trivial corollary of the global. Propositions 3, 4, 5, and 6 are immediate consequences of this observation.

PROPOSITION 3. *If a functor is locally-adjunctable then it preserves $\mathscr{X}$-limits for any connected category $\mathscr{X}$. In particular it preserves pullbacks and equalizers.*

PROPOSITION 4. *If a functor is locally-coadjunctable it commutes with colimits.*

A functor $U: \mathscr{B} \to \mathscr{A}$ is *locally-tripleable* if $U^B: \mathscr{B}/B \to \mathscr{A}/U(B)$ is tripleable for every object B of $\mathscr{B}$.

PROPOSITION 5. *If a functor is locally-tripleable then it preserves and reflects $\mathscr{X}$-limits for any connected category $\mathscr{X}$.*

PROPOSITION 6. *Suppose that $U: \mathscr{B} \to \mathscr{A}$ is locally-tripleable and that $\mathscr{A}$ and $\mathscr{B}$ have finite products. If U preserves $\mathscr{X}$-colimits then it reflects $\mathscr{X}$-colimits. (U reflects the colimits that it preserves.)*

For each map $f: X \to Y \in \mathscr{C}$ the functor $f^!: \mathscr{C}/X \to \mathscr{C}/Y$ defined by $f^!(X' \to X) = (X' \to X \to Y)$, and the forgetful functor $\mathscr{C}/X \to \mathscr{C}$ are trivially locally-tripleable for any category $\mathscr{C}$. It is also clear that every full locally-reflective subcategory has a locally-tripleable inclusion functor. The following theorem serves to show that every tripleable functor is locally-tripleable and also that the category of fields is locally-tripleable over the category of sets.

THEOREM 3. *Let $U: \mathscr{B} \to \mathscr{A}$ be a locally-adjunctable functor. U is locally-tripleable if and only if*

(i) *U reflects coequalizers of cobounded U-contractible pairs*

(ii) *$\mathscr{B}$ has coequalizers of cobounded U-contractible pairs*

(iii) *U preserves coequalizers of U-contractible pairs.*

(Here a cobound for a pair $d_0, d_1: X \to Y \in \mathscr{B}$ is a map $b: Y \to B \in \mathscr{B}$ for which $bd_0 = bd_1$.)

The proof relies heavily on Beck's tripleability theorem [1].

Proof. If $d_0, d_1: (X \to B) \to (Y \to B) \in \mathscr{B}/B$ is a U^B-contractible pair, then $d_0, d_1: X \to Y \in \mathscr{B}$ is clearly a cobounded U-contractible pair. Thus (i), (ii), and (iii) are together sufficient for local tripleability.

Conversely, suppose U is locally-tripleable and $d_0, d_1: X \to Y \in \mathscr{B}$ is a U-contractible pair.

(i) Suppose that $U(d)$ is a coequalizer of $U(d_0)$ and $U(d_1)$, and thus a contractible coequalizer. For any cobound $b: Y \to B$,

$$\begin{tikzcd} U(X) \arrow[r, shift left, "U(d_1)"] \arrow[r, shift right, "U(d_0)"'] \arrow[dr, "U(bd_0)"'] & U(Y) \arrow[r, "U(d)"] \arrow[d, "U(b)"] \arrow[l, bend left, "s_0"] & U(Z) \arrow[l, bend left, "s"] \arrow[dl, "U(b)s"] \\ & U(B) & \end{tikzcd}$$

is a contractible coequalizer diagram in $\mathscr{A}/U(B)$. The category $\mathscr{B}/B$ has and U^B preserves coequalizers of U^B-contractible pairs so, without loss of generality, $U(b)s$ may be taken to be $U(z)$ for some $z\colon Z \to B \in \mathscr{B}$ for which

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[dr, "bd_0"'] & Y \arrow[r, "d"] \arrow[d, "b"] & Z \arrow[dl, "z"] \\
& B &
\end{tikzcd}$$

is a coequalizer diagram in $\mathscr{B}/B$.

Thus the existence of a cobound implies that $dd_0 = dd_1$, and the remaining properties of a coequalizer follow as $b\colon (Y \to B) \to (B \to B) \in \mathscr{B}/B$.

(ii) If the U-contractible pair $d_0, d_1\colon X \to Y$ has a cobound $b\colon Y \to B$ then $d_0, d_1\colon (X \xrightarrow{bd_0} B) \to (Y \xrightarrow{b} B)$ is a U^B-contractible pair and thus has a coequalizer, $c\colon (Y \to B) \to (C \to B) \in \mathscr{B}/B$. The map $U^B(c)$ is a coequalizer of $U^B(d_0)$ and $U^B(d_1)$ and thus a contractible coequalizer. Therefore $U(c)$ is a contractible coequalizer of $U(d_0)$ and $U(d_1)$. By (i) c is a coequalizer of d_0 and d_1.

(iii) The preceding construction also serves to prove (iii).

By the same reasoning the following local version of Duskin's tripleability theorem [2] may be obtained.

THEOREM 4. *Let $U\colon \mathscr{B} \to \mathscr{A}$ be a locally-adjunctable functor and suppose that $\mathscr{A}$ has kernel pairs. The functor U is locally-tripleable if and only if*

(i) *U reflects coequalizers of cobounded U-contractible equivalence pairs*

(ii) *U preserves coequalizers of U-contractible equivalence pairs*

(iii) *$\mathscr{B}$ has coequalizers of cobounded U-contractible equivalence pairs*

(iv) *$\mathscr{B}$ has kernel pairs.*

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