

LOCAL LIMITS AND TRIPLEABILITY

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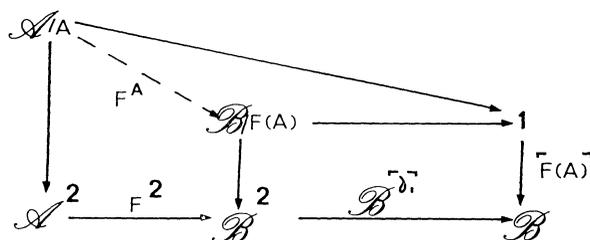
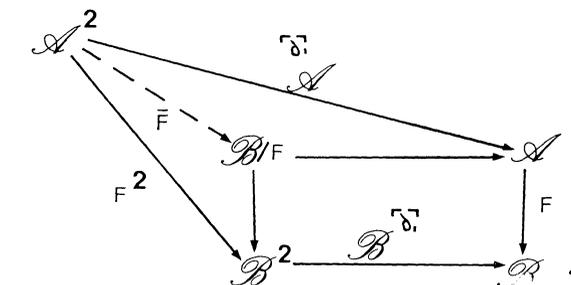
If A is an object in a category \mathcal{A} , the properties of \mathcal{A}/A (the category of objects over A) may be considered as local properties of \mathcal{A} . Using 'local' in this sense, the notion of local universality is defined and some of its basic properties developed. These ideas are then applied in a brief discussion of local adjunction and local limits. Finally two local tripleability theorems are given.

The Lawvere comma category of the diagram

$$1_{\mathcal{B}}: \mathcal{B} \longrightarrow \mathcal{B} \longleftarrow \mathcal{A}: F$$

is denoted by \mathcal{B}/F , in particular \mathcal{B}/B denotes the category of objects over B , when B is an object of \mathcal{B} .

Given a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ we define [3] $\bar{F}: \mathcal{A}^2 \rightarrow \mathcal{B}/F$ and, for each object A of \mathcal{A} , $F^A: \mathcal{A}/A \rightarrow \mathcal{B}/F(A)$ by the following pull-back diagrams in $\mathcal{C}\mathcal{A}\mathcal{T}$:-



LEMMA 1. For any category \mathcal{C} there are isomorphisms making

$$\begin{array}{ccc} (\mathcal{B}^2)^{\mathcal{C}} & \xrightarrow{\bar{F}^{\mathcal{C}}} & (\mathcal{B}/F)^{\mathcal{C}} \\ \cong \downarrow & & \cong \downarrow \\ (\mathcal{B}^{\mathcal{C}})^2 & \xrightarrow{\bar{F}^{\mathcal{C}}} & \mathcal{B}^{\mathcal{C}}/F^{\mathcal{C}} \end{array} \quad \text{commute.}$$

Proof. $\mathcal{C}\mathcal{A}\mathcal{I}$ is cartesian-closed.

Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and $f: B \rightarrow F(A)$ a map in \mathcal{B} . The pair $\langle f_0, f_1 \rangle$ is a *locally-universal pair* (dual colocally-couniversal) from f to F when $f_0: f \rightarrow F^A(f_1) \in \mathcal{B}/F(A)$ is universal from f to F^A in the sense of MacLane [6]. Similarly a *locally-couniversal pair* (dual colocally-universal) from F to f is a pair $\langle f_0, f_1 \rangle$ for which $f_0: F^A(f_1) \rightarrow f$ is couniversal from F^A to f .

LEMMA 2. *The pair $\langle f_0, f_1 \rangle$ is locally-couniversal from F to f if and only if $(f_0, 1_A): \bar{F}(f_1) \rightarrow (B, f: B \rightarrow F(A), A) \in \mathcal{B}/F$ is couniversal from \bar{F} to f .*

THEOREM 1. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and $f: B \rightarrow F(A)$ a map in \mathcal{B} . There is a locally-couniversal pair from F to f if and only if there is a couniversal map from \bar{F} to $(B, f: B \rightarrow F(A), A) \in \mathcal{B}/F$.*

Proof. If $(f_0, f_1): \bar{F}(f_2) \rightarrow (B, f: B \rightarrow F(A), A)$ is couniversal from \bar{F} to f , then so is $(f_0, 1_A): \bar{F}(f_1, f_2) \rightarrow (B, f: B \rightarrow F(A), A)$. The result now follows by Lemma 2.

The corresponding result for locally-universal pairs is false in general, although Kaput [3] showed that a functor F is locally-adjunctable if and only if \bar{F} has an adjoint.

The functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is *locally-adjunctable* if a locally-universal pair exists for every $f: B \rightarrow F(A) \in \mathcal{B}$ and *locally-coadjunctable* if a locally-couniversal pair exists for every $g: F(A) \rightarrow B \in \mathcal{B}$.

Leroux [5] stated the following proposition, which is an immediate consequence of Theorem 1.

PROPOSITION 1. *A functor F is locally-coadjunctable if and only if \bar{F} has a coadjoint.*

From which, by Lemma 1, we obtain

PROPOSITION 2. *If a functor $F: \mathcal{A} \rightarrow \mathcal{B}$, is locally-coadjunctable then so is $F^{\mathcal{C}}: \mathcal{A}^{\mathcal{C}} \rightarrow \mathcal{B}^{\mathcal{C}}$ for any category \mathcal{C} .*

THEOREM 2. *Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and $f: B \rightarrow F(A)$, $c: F(A') \rightarrow B$, $c': F(A'') \rightarrow F(A)$ maps in \mathcal{B} with c' couniversal from F to $F(A)$. By couniversality there are (unique) maps s, t for which $c'F(s) = 1_{F(A)}$ and $c'F(t) = fc$.*

(A) *Suppose c is couniversal from F to B . The diagram $s: A \rightarrow A'' \leftarrow A': t$ has a pullback in \mathcal{A} if and only if there is a locally-couniversal pair from F to f .*

(B) Suppose c' is an identity map. The map c is couniversal from F to B if and only if $\langle c, t \rangle$ is locally-couniversal from F to f .

Proof. (A) If

$$\begin{array}{ccc}
 \tilde{A} & \xrightarrow{s'} & A' \\
 f_1 \downarrow & (*) & \downarrow t \\
 A & \xrightarrow{s} & A''
 \end{array}$$

is a pullback in \mathcal{A} , then $\langle cF(s'), f_1 \rangle$ is locally-couniversal from F to f . Conversely, if $\langle f_0, f_1 \rangle$ is locally-couniversal from F to f then (*) is a pullback where s' is the unique map for which $cF(s') = f_0$.

(B) Suppose $\langle c, t \rangle$ is locally-couniversal from F to f . For any $g: F(\tilde{A}) \rightarrow B$, by the couniversality of $1_{F(A)}$, there is a unique h for which $F(h) = fg$. Thus there is a unique p for which (i) $tp = h$ and (ii) $cF(p) = g$. Again by the couniversality of $1_{F(A)}$, (ii) implies (i) and so p is unique in satisfying (ii).

The converse follows directly from (A).

COROLLARY 1. (Leroux) *If a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ has a coadjoint and \mathcal{A} has pullbacks then F is locally-coadjunctable.*

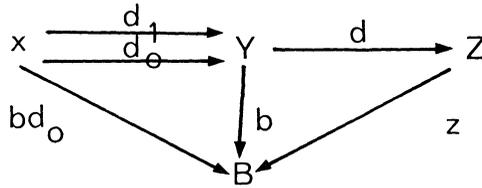
Let $\Delta: \mathcal{C} \rightarrow \mathcal{C}^\infty$ be the canonical embedding. If $D: \mathcal{Z} \rightarrow \mathcal{C}$ is a functor and $\rho: D \rightarrow \Delta(X) \in \mathcal{C}^\infty$, then a locally-universal pair from ρ to Δ is called a *local colimit* for D at ρ and a locally-couniversal pair from Δ to ρ is called a *local limit* for D at ρ .

COROLLARY 2. (Folklore) *When \mathcal{Z} is a connected category, D has a limit if and only if it has a local limit at ρ .*

The definitions of local limit and colimit are the obvious ones to make (given the definition of local universality) but they are essentially renameings of limits and colimits in \mathcal{C}/X . Thus the standard theorems on limits, colimits and adjointness have local counterparts; moreover, in most cases the local result is a trivial corollary of the global. Propositions 3, 4, 5, and 6 are immediate consequences of this observation.

PROPOSITION 3. *If a functor is locally-adjunctable then it preserves \mathcal{Z} -limits for any connected category \mathcal{Z} . In particular it preserves pullbacks and equalizers.*

is a contractible coequalizer diagram in $\mathcal{A}/U(B)$. The category \mathcal{B}/B has and U^B preserves coequalizers of U^B -contractible pairs so, without loss of generality, $U(b)$ s may be taken to be $U(z)$ for some $z: Z \rightarrow B \in \mathcal{B}$ for which



is a coequalizer diagram in \mathcal{B}/B .

Thus the existence of a cobound implies that $dd_0 = dd_1$, and the remaining properties of a coequalizer follow as $b: (Y \rightarrow B) \rightarrow (B \rightarrow B) \in \mathcal{B}/B$.

(ii) If the U -contractible pair $d_0, d_1: X \rightarrow Y$ has a cobound $b: Y \rightarrow B$ then $d_0, d_1: (X \xrightarrow{bd_0} B) \rightarrow (Y \xrightarrow{b} B)$ is a U^B -contractible pair and thus has a coequalizer, $c: (Y \rightarrow B) \rightarrow (C \rightarrow B) \in \mathcal{B}/B$. The map $U^B(c)$ is a coequalizer of $U^B(d_0)$ and $U^B(d_1)$ and thus a contractible coequalizer. Therefore $U(c)$ is a contractible coequalizer of $U(d_0)$ and $U(d_1)$. By (i) c is a coequalizer of d_0 and d_1 .

(iii) The preceding construction also serves to prove (iii).

By the same reasoning the following local version of Duskin's tripleability theorem [2] may be obtained.

THEOREM 4. *Let $U: \mathcal{B} \rightarrow \mathcal{A}$ be a locally-adjunctable functor and suppose that \mathcal{A} has kernel pairs. The functor U is locally-tripleable if and only if*

- (i) U reflects coequalizers of cobounded U -contractible equivalence pairs
- (ii) U preserves coequalizers of U -contractible equivalence pairs
- (iii) \mathcal{B} has coequalizers of cobounded U -contractible equivalence pairs
- (iv) \mathcal{B} has kernel pairs.

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