PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 1, 1974

REPRESENTATIONS OF B^*-ALGEBRAS ON BANACH SPACES

BRUCE A. BARNES

This paper deals with continuous irreducible representations of a B^*-algebra on a Banach space. The main result is that if π is a continuous irreducible representation of a B^*-algebra A on a reflexive Banach space X, and if there is a subset S of A such that the intersection of the null spaces of the operators $\pi(a)$ for all $a \in S$ is a nonzero, finite dimensional subspace of X, then X is a Hilbert space in an equivalent norm and π is similar to a $*$-representation of A on this Hilbert space.

In [7], R. V. Kadison raised the question of whether every continuous representation of a B^*-algebra on a Hilbert space is similar to a $*$-representation. Recently, J. Bunce, in [4], answered Kadison's question affirmatively for a class of B^*-algebras which includes the GCR algebras of Kaplansky. Also, the present author proved an affirmative result concerning this question in [2] (a new proof of this result is given in § 2; see Corollary 2.3). However, the general question remains open. In this paper we consider this question of Kadison in the special case where the representation is assumed to be irreducible. Actually, the problem we consider (in the irreducible case) is more general than Kadison's problem, since we allow the representation space to be a Banach space X. Then we prove under certain conditions on the representation that X is a Hilbert space in an equivalent norm, and that the given representation of the B^*-algebra is similar to a $*$-representation of the algebra on this Hilbert space. A precise statement of our main result is in the abstract above. It is an open question whether the existence of a continuous irreducible representation of a B^*-algebra on a Banach space X necessitates that X is a Hilbert space in an equivalent norm.

At this point we introduce some notation and terminology. *Throughout this paper X is a Banach space and A is a B^*-algebra.* All norms, except for particular norms introduced in context, are denoted by $\|\cdot\|$. The normed dual of X is denoted by X^*. If $x \in X$ and $\alpha \in X^*$, we often use the notation $\langle x, \alpha\rangle$ for $\alpha(x)$. $\mathscr{B}(X)$ is the algebra of all bounded linear operators on X. If $T \in \mathscr{B}(X)$, then $\mathscr{R}(T)$ and $\mathscr{N}(T)$ denote the range and null space of T, respectively.

A nonzero subalgebra B of $\mathscr{B}(X)$ is irreducible (or acts irreducibly) on X if the only closed B-invariant subspaces of X are $\{0\}$ and X. B is strictly irreducible if the only B-invariant subspaces of X are $\{0\}$ and X. If π is a nonzero representation of A into $\mathscr{B}(X)$, then π is [strictly] irreducible if the image algebra $\pi(A)$ is [strictly] irreducible on X. *Throughout this paper* π *is a continuous representation of* A *into* $\mathscr{B}(X)$.

2. Representation of a B^*-algebra on a Banach space. As we stated previously, π will always denote a continuous representation of A into $\mathscr{B}(X)$. In this section we find general conditions which imply that X is a Hilbert space in an equivalent norm, and that π is similar to a $*$-representation of A on this Hilbert space (Proposition 2.2 and Proposition 2.5). Proposition 2.5 is used in the next section to prove the main result of the paper.

LEMMA 2.1. *Let H be a Hilbert space, and let B be a $*$-subalgebra of $\mathscr{B}(H)$ acting strictly irreducibly on H. Let $\|\cdot\|_2$ denote the Hilbert space norm on H, and assume that $\|\cdot\|$ is a norm on H with the properties*:

(1) *there exists $K > 0$ such that $K\|x\|_2 \geqq \|x\|$ for all $x \in H$, and*

(2) *every $T \in B$ is continuous on the normed linear space $(H, \|\cdot\|)$.*

Then $\|\cdot\|$ is equivalent to $\|\cdot\|_2$ on H.

Proof. Assume that $\alpha \in (H, \|\cdot\|)^*$, the dual space of $(H, \|\cdot\|)$, and $\alpha \neq 0$. By (1), α is continuous on H with respect to $\|\cdot\|_2$. Therefore, there exists $z \in H$, $z \neq 0$, such that $\alpha(x) = (x, z)$ for all $x \in H$ where $(\cdot, \cdot)$ is the inner product on H. Given $w \in H$, there exists $T \in B$ such that $T^*z = w$. Then the functional $\beta(x) = (x, w)$ is continuous with respect to $\|\cdot\|$, since $\beta(x) = (x, w) = (Tx, z) = \alpha(Tx)$, and T is continuous on $(H, \|\cdot\|)$ by (2). Thus the continuous linear functionals on $(H, \|\cdot\|)$ are exactly the functionals α of the form $\alpha(x) = (x, w)$ for some $w \in H$.

Now if $\alpha \in (H, \|\cdot\|)^*$, let $\Phi(\alpha)$ be the unique $w \in H$ such that $\alpha(x) = (x, w)$ for all $x \in H$. By the previous argument, Φ is a conjugate linear isomorphism of $(H, \|\cdot\|)^*$ onto H. Furthermore, since $|(x, w)| \leqq \|\alpha\|\,\|x\| \leqq K\|\alpha\|\,\|x\|_2$ for all $x \in H$, then $\|\Phi(\alpha)\|_2 = \|w\|_2 \leqq K\|\alpha\|$. Then by the Closed Graph Theorem, there exists $J > 0$ such that $J\|\Phi(\alpha)\|_2 \geqq \|\alpha\|$ for all $\alpha \in (H, \|\cdot\|)^*$. It follows that for every $w \in H$,

$$J\|w\|_2 \geqq \sup_{x \in H, x \neq 0} \frac{|(x, w)|}{\|x\|}.$$

In particular, $J\|w\|_2 \geqq \|w\|^{-1}(w, w)$, which implies that $J\|w\| \geqq \|w\|_2$. Therefore, $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on H.

Let K be a modular maximal left ideal of A. Then there exists a positive functional α on A such that

$$K = \{a \in A: \alpha(a^*a) = 0\}$$

(see [5, Théorème (2.9.5)]). The quotient space $A - K = \{a + K: a \in A\}$ is an inner product space with inner product $(a + K, b + K) = \alpha(b^*a)$. Furthermore, by [10, Theorem 2], the norms

$$|a + K| = \alpha(a^*a)^{1/2}$$

and

$$\|a + K\|_2 = \inf\{\|a + k\|: k \in K\}$$

coincide on $A - K$. It follows that $\|a + K\|_2$ is a Hilbert space norm on $A - K$, and that the left regular representation of A on $A - K$ (i.e., $a \in A$ acts on $b + K$ by $a(b + K) = ab + K$) is a $*$-representation on this Hilbert space.

We use these remarks in the proof of the next proposition.

PROPOSITION 2.2. *Assume that π is irreducible, and assume that there exists $x \in X$ such that the left ideal*

$$K = \{a \in A: \pi(a)x = 0\}$$

is modular maximal in A. Then X is a Hilbert space in an equivalent norm, amd π is similar to a $*$-representation of A on this Hilbert space.

Proof. Let K be as in the statement of the proposition, and let

$$\|a + K\|_2 = \inf\{\|a + k\|: k \in K\}\ .$$

As noted above, $A - K$ is a Hilbert space in the norm $\|\cdot\|_2$, and the left regular representation of A on $A - K$ is a $*$-representation on this Hilbert space. Since K is a modular maximal left ideal of A, then A acts strictly irreducibly on $A - K$. Define a norm, $\|\cdot\|$, on $A - K$ by $\|a + K\| = \|\pi(a)x\|$. If $a \in A$ and $k \in K$, then

$$\|a + K\| = \|a + k + K\| = \|\pi(a + k)x\| \leqq \|\pi\|\,\|x\|\,\|a + k\|\ .$$

Therefore, $\|a + K\| \leqq (\|\pi\|\,\|x\|)\,\|a + K\|_2$ for any $a \in A$. Thus (1) of Proposition 2.1 holds. If $a \in A$,

$$\|a(b + K)\| = \|\pi(ab)x\| \leqq \|\pi(a)\|\,\|\pi(b)x\| = \|\pi(a)\|\,\|b + K\|\ .$$

Thus a acts continuously on $(A - K, \|\cdot\|)$, and this verifies that (2) of Proposition 2.1 holds. Then by Proposition 2.1, $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent on $A - K$. It follows that $\{\pi(a)x: a \in A\}$ is a closed $\pi(A)$-invariant subspace of X, and since $\pi(A)$ acts irreducibly on X, then $\{\pi(a)x: a \in A\} = X$. Define a norm $\|\cdot\|'$ on X by $\|\pi(a)x\|' = \|a + K\|_2$.

Then $\|\cdot\|'$ is a Hilbert space norm on X, and $\|\cdot\|'$ is equivalent to the given norm $\|\cdot\|$ on X. Finally, it follows from [3, Theorem 4.1] that π is similar to a *-representation of A on this Hilbert space.

As a corollary we have a new proof of [2, Theorem 8].

COROLLARY 2.3. *Let A be a B^*-algebra, and let π be a continuous irreducible representation of A into $\mathscr{B}(X)$. Assume that $A/\ker(\pi)$ contains a minimal left ideal. Then the conclusion of Proposition 2.2 holds.*

Proof. Denote by γ the natural quotient map of A onto $A/\ker(\pi)$. Let N be a minimal left ideal of $A/\ker(\pi)$. Choose $b \in \gamma^{-1}(N)$ and $y \in X$ such that $\pi(b)y \neq 0$. The set $Y = \{\pi(a)y\colon a \in \gamma^{-1}(N)\}$ is a nonzero $\pi(A)$-invariant subspace of X. We prove that $\pi(A)$ acts strictly irreducibly on Y. Assume that $y_1, y_2 \in Y$ and $y_1 \neq 0$. There exist $a_1, a_2 \in \gamma^{-1}(N)$ such that $y_k = \pi(a_k)y$, $k = 1, 2$. Since N is a minimal left ideal of $A/\ker(\pi)$, there exists $a \in A$ such that $a_2 - aa_1 \in \ker(\pi)$. Then $\pi(a_2) = \pi(a)\pi(a_1)$, so that $\pi(a)y_1 = y_2$. Now an easy algebraic argument using the fact that $\pi(A)$ acts strictly irreducibly on Y shows that for any nonzero vector $x \in Y$, $K = \{a \in A\colon \pi(a)x = 0\}$ is a modular maximal left ideal of A. Then Proposition 2.2 applies.

REMARK. Proposition 2.2 and Corollary 2.3 hold under the weaker hypothesis that A is a Banach *-algebra with the property that every modular maximal left ideal of A is the left kernel of a strictly pure state of A. For examples of algebras with this property see [3].

Next we apply Corollary 2.3 to the case where A is a GCR algebra as defined by I. Kaplansky.

COROLLARY 2.4. *If A is a GCR algebra and π is a continuous irreducible representation of A into $\mathscr{B}(X)$, then X is a Hilbert space in an equivalent norm and π is similar to a *-representation of A on this Hilbert space.*

Proof. The quotient algebra $A/\ker(\pi)$ has no ideal divisors of zero by [8, Lemma 2.5]. Therefore, by [8, Lemma 7.4], $A/\ker(\pi)$ contains a minimal left ideal. Then the result follows from Corollary 2.3.

As mentioned in the Introduction, J. Bunce has proved that every continuous representation of a GCR algebra on a Hilbert space is similar to a *-representation [4, Theorem 1].

Now for each positive integer n, let $A_n = \{a \in A\colon \|a\| \leqq n\}$. If $x \in X$, let $[\pi(A_n)x]$ denote the weak closure of the set $\pi(A_n)x$ in X.

PROPOSITION 2.5. *Assume that π is irreducible and there exists $x \in X$ such that*

$$X = \bigcup_{n=1}^{+\infty} [\pi(A_n)x] .$$

*Then X is a Hilbert space in an equivalent norm, and π is similar to a *-representation of A on this Hilbert space.*

Proof. By the Baire Category Theorem, $[\pi(A_n)x]$ must have nonempty interior for some n. Note that since $\pi(A_n)x$ is a convex subset of X, then by [6, Corollary 14, p. 418] the norm closure of $\pi(A_n)x$ is $[\pi(A_n)x]$. Then an easy computation shows that there exists an integer m such that $[\pi(A_m)x]$ contains the closed unit ball of X. Define a map $T: A \to X$ by $T(a) = \pi(a)x$, $a \in A$. We have shown that if y is in the closed unit ball of X, then there exists $a \in A$ such that $\| a \| \leqq m$ and

$$\| T(a) - y \| = \| \pi(a)x - y \| < 1/2 .$$

Then a direct application of a theorem of W. Bade and P. Curtis [1, Theorem 1.2] proves that T is onto. Therefore $\pi(A)x = X$.

Let $K = \{a \in A: \pi(a)x = 0\}$. Let $A - K$ be the quotient space $\{a + K: a \in A\}$ equipped with the usual quotient norm. Define a map $\Phi: A - K \to X$ by $\Phi(a + K) = \pi(a)x$. Then Φ is one-to-one, continuous, and onto (since $\pi(A)x = X$). Therefore, by the Open Mapping Theorem Φ maps closed subsets of $A - K$ onto closed sets in X. By [5, Théorème 2.9.5] there exists a modular maximal left ideal J of A such that $K \subset J$. Then $\Phi(J)$ is a closed $\pi(A)$-invariant subspace of X. Therefore $K = J$. By the remarks preceding Proposition 2.2, $A - K$ is a Hilbert space in the usual quotient norm. Therefore, X is a Hilbert space in an equivalent norm. Then π is similar to a *-representation of A on this Hilbert space by [3, Theorem 4.1].

3. **Some preliminary lemmas.** In this section we prove several lemmas which we apply subsequently in the proof of the main result.

LEMMA 3.1. *Assume that X is reflexive and that A has an identity. Assume that $a = a^* \in A$. Then*

$$X = \overline{\mathscr{R}(\pi(a))} \oplus \mathscr{N}(\pi(a)) .$$

Let E be the projection in $\mathscr{B}(X)$ with $\mathscr{R}(E) = \mathscr{N}(\pi(a))$ and $\mathscr{N}(E) = \overline{\mathscr{R}(\pi(a))}$. If $\mathscr{N}(\pi(a)) \neq \{0\}$, then there exists a sequence $\{a_n\} \subset A$ such that $\| a_n \| = 1$, $n \geqq 1$, and

$$\text{st. op.} \lim \pi(a_n) = E .$$

In this case $\|E\| \leqq \|\pi\|$.

Proof. Let $\operatorname{sp}(a)$ denote the spectrum of a in A. If $\lambda \notin \operatorname{sp}(a)$, let $R(\lambda) = (\lambda - a)^{-1}$. Since $a = a^*$, then $\operatorname{sp}(a)$ is real and $\|R(\lambda)\| = (d(\lambda))^{-1}$ where $d(\lambda) = \inf\{|\lambda - \alpha| : \alpha \in \operatorname{sp}(a)\}$. Set $\lambda_n = i(1/n)$, $n \geqq 1$. Then $\lambda_n \notin \operatorname{sp}(a)$, $n \geqq 1$, and $\{\lambda_n R(\lambda_n)\}$ is a bounded sequence in A. It follows from [6, Corollary 5, p. 597] that

$$X = \overline{\mathscr{R}(\pi(a))} \oplus \mathscr{N}(\pi(a)) ,$$

and that $\{\lambda_n \pi(R(\lambda_n))\}$ converges in the strong operator topology to the projection E defined above. For $n \geqq 1$, let $a_n = \lambda_n R(\lambda_n)$. If $\mathscr{N}(\pi(a)) \neq \{0\}$, then $0 \in \operatorname{sp}(a)$, so that $d(\lambda_n) = 1/n$, $n \geqq 1$. Therefore, $\|a_n\| = |\lambda_n| d(\lambda_n)^{-1} = 1$ and $\|\pi(a_n)\| \leqq \|\pi\|$ for $n \geqq 1$. Then $\|E\| \leqq \|\pi\|$.

Let A^+ denote the set of positive elements of A. If α is a positive functional on A, the left kernel of α is the set

$$K_\alpha = \{a \in A : \alpha(a^*a) = 0\} .$$

LEMMA 3.2.
(1) *Let M be a closed left (or right) ideal of A. If $a, b \in A^+$ and $a + b \in M$, then $a \in M$.*
(2) *If $a, b \in A^+$, then $\overline{\mathscr{R}(\pi(a))} \subset \overline{\mathscr{R}(\pi(a + b))}$ and $\mathscr{N}(\pi(a + b)) \subset \mathscr{N}(\pi(a))$.*
(3) *If $a \in A$, $\mathscr{N}(\pi(a)) = \mathscr{N}(\pi(a^*a))$.*

Proof. Let a, b, and M be as in the statement of (1). Let α be an arbitrary positive functional on A such that $M \subset K_\alpha$, where K_α is the left kernel of α. Then $\alpha(a + b) = 0$, so that $\alpha(a) = 0$. Then $a \in K_\alpha$. By [5, Théorème (2.9.5)], M is the intersection of the collection of all K_α such that $M \subset K_\alpha$. Therefore $a \in M$.

Now assume that $a, b \in A^+$. Let

$$M = \{c \in A : \mathscr{R}(\pi(c)) \subset \overline{\mathscr{R}(\pi(a + b))}\} .$$

M is a closed right ideal of A and $a + b \in M$. By (1), $a \in M$. Therefore, $\overline{\mathscr{R}(\pi(a))} \subset \overline{\mathscr{R}(\pi(a + b))}$. Similarly, $a + b$ is in the closed left ideal

$$N = \{c \in A \mid \mathscr{N}(\pi(a + b)) \subset \mathscr{N}(\pi(c))\} .$$

Thus $\mathscr{N}(\pi(a + b)) \subset \mathscr{N}(\pi(a))$.

If $a \in A$, then $\mathscr{N}(\pi(a)) \subset \mathscr{N}(\pi(a^*a))$. Consider the closed left ideal

$$N = \{b \in A : \pi(b)\mathscr{N}(\pi(a^*a)) = \{0\}\} .$$

Since $a^*a \in N$, then by [9, Corollary (4.9.3)], $a \in N$. This proves (3).

If $x \in X$ and $f \in X^*$, let $(f \mid x)$ denote the operator defined on X by $(f \mid x)(y) = f(y)x$. If $x \neq 0$ and $f \neq 0$, then the operator $(f \mid x)$ has one dimensional range. Conversely, every bounded operator on X with one dimensional range is of the form $(f \mid x)$ for some $x \in X$ and $f \in X^*$. Note that $\|(f \mid x)\| = \|f\| \|x\|$.

LEMMA 3.3. *Let X be reflexive. Assume that $\{E_n\}$ is a sequence of projections in $\mathscr{B}(X)$ such that*

(i) *there exists $M > 0$ such that $\|E_n\| \leqq M$, $n \geqq 1$, and*

(ii) *there exists a positive integer m such that* $\dim(\mathscr{R}(E_n)) = m$, $n \geqq 1$.

(1) *Then there exists a subsequence $\{E_{n_k}\}$ of $\{E_n\}$ and an operator $E \in \mathscr{B}(X)$ with finite dimensional range such that*

$$\text{wk. op.} \lim_k (E_{n_k}) = E .$$

(2) *Furthermore, if $\{y_n\} \subset X$, $y_n \to y$, and $E_n(y_n) = y_n$ for $n \geqq 1$, then $E(y) = y$.*

Proof. First assume that $m = 1$. Then there exist sequences $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$ such that $E_n = (f_n \mid x_n)$, $n \geqq 1$. By hypothesis, $M \geqq \|(f_n \mid x_n)\| = \|f_n\| \|x_n\|$. Therefore, we may assume that $\|x_n\| = 1$ and $\|f_n\| \leqq M$ for $n \geqq 1$. Then, since X is reflexive, we can choose a subsequence $\{n_k\}$ of the positive integers such that wk. $\lim_k x_{n_k} = x$ and wk.* $\lim_k f_{n_k} = f$ for some $x \in X$ and $f \in X^*$ [6, Theorem 28, p. 68]. Then it is immediate that $(f_{n_k} \mid x_{n_k})$ converges in the weak operator topology to $(f \mid x)$. Thus the conclusion of the lemma holds when $m = 1$.

Now we proceed to prove (1) by induction. Let $m > 1$ be a fixed integer, and assume that (1) holds for $m - 1$. Let $\{E_n\}$ be a sequence of projections satisfying (i) and (ii). For each $n \geqq 1$, choose $x_n \in \mathscr{R}(E_n)$ such that $\|x_n\| = 1$. By (i), for an arbitrary $z \in \mathscr{N}(E_n)$ we have

$$M \geqq \left\| E_n\left(\frac{x_n + z}{\|x_n + z\|}\right)\right\| = \frac{1}{\|x_n + z\|} .$$

This implies that $d_n \geqq 1/M$ where d_n is the distance from x_n to $\mathscr{N}(E_n)$. Choose $g_n \in X^*$ such that $g_n(x_n) = d_n$, $\|g_n\| = 1$, and $g_n(\mathscr{N}(E_n)) = \{0\}$. Let $f_n = d_n^{-1} g_n$. Then $\|(f_n \mid x_n)\| = \|f_n\| = d_n^{-1} \leqq M$ for $n \geqq 1$. Set $G_n = E_n - (f_n \mid x_n)$. By the choice of f_n and x_n, G_n is a projection. Furthermore, $\dim(\mathscr{R}(G_n)) = m - 1$ and $\|G_n\| \leqq 2M$ for all $n \geqq 1$. If follows from the induction hypothesis and the argument in the first paragraph of the proof that there exists a subsequence $\{E_{n_k}\}$ of $\{E_n\}$ and an operator E with finite dimensional range, such that $\{E_{n_k}\}$

converges to E in the weak operator topology.

Now we prove (2). Assume that $E_n(y_n) = y_n$ for $n \geqq 1$ and $y_n \to y$ in X. Let α be an arbitrary functional in X^*. Then

$$\begin{aligned} |\langle Ey - y, \alpha\rangle| &\leqq |\langle E(y) - E_{n_k}(y_{n_k}), \alpha\rangle| + |\langle y_{n_k} - y, \alpha\rangle| \\ &\leqq |\langle (E - E_{n_k})y, \alpha\rangle| + |\langle E_{n_k}(y - y_{n_k}), \alpha\rangle| \\ &\quad + |\langle y_{n_k} - y, \alpha\rangle| \\ &\leqq |\langle (E - E_{n_k})y, \alpha\rangle| + (M + 1)\|y - y_{n_k}\|\,\|\alpha\| . \end{aligned}$$

Since the right hand side of this inequality approaches zero as $k \to +\infty$, then $E(y) = y$.

4. The main result.

In this section we prove the main result, the statement of which follows.

THEOREM 4.1. *Let X be a reflexive Banach space. Assume that A is a B^*-algebra and that π is a continuous irreducible representation of A into $\mathscr{B}(X)$. Assume that there is a nonempty subset S of A such that the intersection of $\mathscr{N}(\pi(a))$ for $a \in S$ is a nonzero finite dimensional subspace of X. Then X is a Hilbert space in an equivalent norm and π is similar to a *-representation of A on this Hilbert space.*

We assume throughout the proof of the theorem that A has an identity. It is not difficult to verify that there is no loss of generality in making this assumption.

Now let S be the nonempty subset of A hypothesized in the statement of the theorem. Then

$$W = \bigcap_{a \in S} \mathscr{N}(\pi(a))$$

is a nonzero subspace of X. Let $T = \{a \in A^+ : W \subset \mathscr{N}(\pi(a))\}$. By Lemma 3.2(3),

$$\bigcap_{a \in T} \mathscr{N}(\pi(a)) = W .$$

The set T is partially ordered in the usual ordering of positive elements of A. Furthermore, T is a directed set since if $a, b \in T$, then $a + b \in T$, $a + b \geqq a$, and $a + b \geqq b$. By Lemma 3.1, for each $a \in T$ there exists a projection E_a such that $\mathscr{R}(E_a) = \mathscr{N}(\pi(a))$ and $\mathscr{N}(E_a) = \overline{\mathscr{R}(\pi(a))}$. Let Z be the closure of $\bigcup_{a \in T} \mathscr{N}(E_a)$. We begin the proof of Theorem 4.1 with two lemmas.

LEMMA 4.2. *The net $\{E_a\}_{a \in T}$ converges in the strong operator topology to a projection $E \in \mathscr{B}(X)$ with $\mathscr{R}(E) = W$ and $\mathscr{N}(E) = Z$.*

Proof. The collection $\{E_a : a \in T\}$ is ordered in the usual ordering of projections, i.e., $E \geqq F$ if $EF = FE = F$. If $a, b \in T$, then $a + b \in T$, and by Lemma 3.2(2), we have $\overline{\mathscr{R}(\pi(a))} \subset \overline{\mathscr{R}(\pi(a + b))}$ and $\mathscr{N}(\pi(a + b)) \subset \mathscr{N}(\pi(a))$. It follows that $E_{a+b}(I - E_a) = 0$ and that $(I - E_a)E_{a+b} = 0$. Therefore, $E_a \geqq E_{a+b}$, and by symmetry, $E_b \geqq E_{a+b}$. This proves that when $c, d \in T$ and $d \geqq c$, then $E_c \geqq E_d$. Also by Lemma 3.1, there exists $M > 0$ such that $\| E_a \| \leqq M$ for all $a \in T$.

By the construction of $\{E_a\}_{a \in T}$, we have that $W = \bigcap_{a \in T} \mathscr{R}(E_a)$. Let $Y = \bigcup_{a \in T} \mathscr{N}(E_a)$. Note that Y is a subspace of X since when $a, b \in T$, then $\mathscr{N}(E_a) \cup \mathscr{N}(E_b) \subset \mathscr{N}(E_{a+b})$. Recall that Z is the closure of Y. Let $z \in Z$, and let $\varepsilon > 0$ be arbitrary. Choose $y \in Y$ such that $\| z - y \| < \varepsilon$. Choose $a \in T$ such that $E_a y = 0$. Then if $b \in T$ and $b \geqq a$, we have that $E_a \geqq E_b$, so that $E_b y = 0$. Thus for all $b \geqq a$, $\| E_b z \| = \| E_b(z - y) \| \leqq M\varepsilon$. This proves that the net $\{E_a z\}_{a \in T}$ converges to 0 whenever $z \in Z$. Also if $a \in T$ and $w \in W$, then $E_a(w) = w$. This proves that the net $\{E_a w\}_{a \in T}$ converges to w whenever $w \in W$. It follows that $Z \cap W = \{0\}$. Next we prove that $X = Z \oplus W$. Assuming this result, the previous argument implies that $\{E_a\}_{a \in T}$ converges strongly in $\mathscr{B}(X)$ to the projection E with $\mathscr{R}(E) = W$ and $\mathscr{N}(E) = Z$.

Let x be a vector in X. The net $\{E_a x\}_{a \in T}$ is bounded in X. Since X is reflexive, it follows from [6, Corollary 8, p. 425] that there exists a vector $v \in X$ and a subnet of $\{E_a x\}_{a \in T}$ that converges weakly to v. Thus there exists a directed set $(Q, \geqq)$ and a map $m: Q \rightarrow T$ such that $\{E_{m(q)} x\}_{q \in Q}$ converges weakly to v, and such that for each $a \in T$, there exists $q \in Q$ such that when $p \in Q$ and $p \geqq q$, then $m(p) \geqq a$. If $b \in T$, let Q_b be the cofinal subset of Q defined by $Q_b = \{q \in Q: m(q) \geqq b\}$. Then the net $\{E_{m(q)} x\}_{q \in Q_b}$ converges weakly to v, and

$$\begin{aligned} E_b v &= E_b(\text{wk.} \lim_{a \in Q_b} E_{m(q)} x) \\ &= \text{wk.} \lim_{q \in Q_b} (E_b E_{m(q)} x) \\ &= \text{wk.} \lim_{q \in Q_b} (E_{m(q)} x) \\ &= v \,. \end{aligned}$$

Thus $v \in W$. Let α be an arbitrary functional in $Z^\perp$, the annihilator of Z. Then since $x - E_{m(q)} x \in Z$ for all $q \in Q$, we have $\lim_{q \in Q} \alpha(x - E_{m(q)} x) = 0$. Thus $\alpha(x - v) = 0$ for all $\alpha \in Z^\perp$. Therefore, $x - v \in Z$, so that $x = (x - v) + v \in Z + W$. As we noted previously, the fact that $X = Z \oplus W$ implies that the bounded net $\{E_a\}_{a \in T}$ converges strongly to the projection E with range W and null space Z.

LEMMA 4.3. *Assume that $x \in W$, $x \neq 0$. If $y \in Z$ and $\varepsilon > 0$, then there exists an invertible element $u \in A$ such that $\| y - \pi(u)x \| < \varepsilon$.*

Proof. It suffices to prove the lemma when $y \in \mathscr{N}(E_a)$ for some $a \in T$. In this case $x \in \mathscr{N}(\pi(a))$, $y \in \overline{\mathscr{R}(\pi(a))}$, and $\mathscr{N}(\pi(a)) \oplus \overline{\mathscr{R}(\pi(a))} = X$ (Lemma 3.1). Let $\varepsilon > 0$ be arbitrary. There exists $z \in X$ such that $\| y - \pi(a)z \| < \varepsilon$. Since $\pi(A)$ acts irreducibly on X, there exists $b \in A$ such that

$$\| \pi(ab)x - \pi(a)z \| < \varepsilon .$$

Then $\| y - \pi(ab)x \| < 2\varepsilon$, and since $\pi(b^*a)x = 0$,

$$\| y - \pi(ab + b^*a)x \| < 2\varepsilon .$$

Since $ab + b^*a$ is self-adjoint, we can choose a number λ with $|\lambda| \| x \| < \varepsilon$ and such that $u = \lambda + (ab + b^*a)$ is invertible in A. Then $\| y - \pi(u)x \| < 3\varepsilon$. This proves the lemma.

Now we complete the proof of Theorem 4.1. By hypothesis W is finite dimensional, say m dimensional. Fix $x \in W$, $x \neq 0$. Assume $y \in Z$. By Lemma 4.3, there exists a sequence of invertible elements $\{u_n\} \subset A$ such that $\lim_n \pi(u_n)x = y$. For each $n \geqq 1$, let

$$W_n = \pi(u_n)(W) = \bigcap_{a \in T} \mathscr{N}(\pi(au_n^{-1})) .$$

Set $y_n = \pi(u_n)x$. Then W_n is a m dimensional subspace of X and $y_n \in W_n$ for $n \geqq 1$. Let $T_n = \{a \in A^+ \colon W_n \subset \mathscr{N}(\pi(a))\}$. For each $a \in T_n$ there exists a projection E_a with $\mathscr{R}(E_a) = \mathscr{N}(\pi(a))$ and $\mathscr{N}(E_a) = \overline{\mathscr{R}(\pi(a))}$. By Lemma 4.2 there exists a projection E_n with $\mathscr{R}(E_n) = W_n$, $\| E_n \| \leqq \| \pi \|$, and such that

$$\text{(1)} \qquad \text{st. op.} \lim_{a \in T_n} E_a = E_n .$$

By Lemma 3.3 we may assume (by passing to a subsequence if necessary) that there is an operator F with finite dimensional range such that

$$\text{(2)} \qquad \text{wk. op.} \lim_n E_n = F ,$$

and $y \in \mathscr{R}(F)$. The subspace $\pi(A)x$ is dense in X. Then since F has finite dimensional range, $y \in F(\pi(A)x)$. Choose $b \in A$ such that $y = F\pi(b)x$. Fix an integer $n \geqq \| b \|$, and let $A_n = \{a \in A \colon \| a \| \leqq n\}$. We now prove that y is in the weak closure of $\pi(A_n)x$. Let U be a weak neighborhood of 0 in X, and choose V a weak neighborhood of 0 in X such that $V + V + V \subset U$. By (2) we can choose an integer k so large that

$$E_k\pi(b)x - y \in V .$$

By (1) we can choose an element $a \in T_k$ such that

$$E_a\pi(b)x - E_k\pi(b)x \in V .$$

By Lemma 3.1 we can choose an element $c \in A$ such that $\|c\| \leqq 1$ and

$$\pi(cb)x - E_a\pi(b)x \in V .$$

Then $\pi(cb)x - y \in U$ and $\|cb\| \leqq n$. Therefore, y is in the weak closure of $\pi(A_n)x$, as asserted.

Now let E be the projection with $\mathscr{R}(E) = W$ as in the statement of Lemma 4.2. Assume $y \in W$. Since E has finite dimensional range and $\pi(A)x$ is dense in X, then there exists $b \in A$ such that $y = E\pi(b)x$. By Lemma 4.2,

$$\text{st. op.} \lim_{a \in T} E_a = E .$$

Let $\varepsilon > 0$ be arbitrary. Choose $a \in T$ such that $\|y - E_a\pi(b)x\| < \varepsilon$. By Lemma 3.1, there exists a sequence $\{a_n\} \subset A$ such that $\|a_n\| = 1$ and

$$\text{st. op.} \lim_{n} \pi(a_n) = E_a .$$

Now choose k so large that $\|y - \pi(a_k b)x\| < 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that y is in the norm closure of $\pi(A_n)x$ where n is any integer such that $n \geqq \|b\|$.

By Lemma 4.2, $X = Z \oplus W$. The arguments given above imply that given any $y \in X$, there exists a positive integer n sufficiently large such that y is in the weak closure of $\pi(A_n)x$. Therefore, the theorem follows from Proposition 2.5.

Added in proof. We have found a proof of Theorem 4.1 which is considerably simpler than the one presented here. Using a modification of an argument due to P. G. Spain in [J. London Math. Soc. (2), 7 (1973), p. 385], we can show that a continuous representation π of a B^*-algebra A on a reflexive Banach space X can be extended to a continuous representation $\tilde{\pi}$ of the second dual of A on X. Also, $\tilde{\pi}$ is continuous with respect to the appropriate weak topologies. Then combining this result with arguments similar to those appearing in § 2, it is not difficult to derive Theorem 4.1.

REFERENCES

1. W. Bade and P. C. Curtis, Jr., *Embedding theorems for commutative Banach algebras*, Pacific J. Math., **18** (1966), 391–409.

2. B. A. Barnes, *Irreducible algebras of operators which contain a minimal idempotent*, Proc. Amer. Math. Soc., **30** (1971), 337–342.

3. ———, *Strictly irreducible *-representations of Banach*-algebras*, Trans. Amer. Math. Soc., **170** (1972), 459–469.

4. J. Bunce, *Representations of strongly ammenable C*-algebras*, Proc. Amer. Math. Soc., **32** (1972), 241–246.

5. J. Dixmier, *Les C*-algebres et leurs representations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964.

6. N. Dunford and J. Schwartz, *Linear Operators Part I*, Interscience, New York, 1958.
7. R. V. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math., **77** (1955), 600-620.
8. I. Kaplansky, *The structure of certain operator algebras*, Trans, Amer. Math. Soc., **70** (1951), 219-255.
9. C. E. Rickart, *Banach Algebras*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960.
10. M. Takesaki, *On the conjugate space of an operator algebra*, Tôhoku Math. J., **10** (1958), 194-203.

Received September 13, 1972 and in revised form December 27, 1972. This research was partially supported by NSF Grant GP-28250.

UNIVERSITY OF OREGON