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ON METRIZABILITY OF COMPLETE MOORE SPACES

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This paper is concerned with the relationships between certain 'strong' completeness properties in Moore spaces and with conditions under which Moore spaces satisfying these properties are metrizable.

In [3], Heath showed that each regular T_2-space which admits a strongly complete semi-metric is a complete Moore space. Furthermore, in [6], Heath established that each separable Moore space which admits a strongly complete semi-metric is metrizable. In [10], the author defined strong star-screenability, a property shared by separable spaces and metrizable spaces, and conjectured that separability could be replaced by strong star-screenability in Heath's result. In this paper, the author establishes relationships between different types of completeness in Moore spaces and gives two new metrization theorems for complete Moore spaces. From these results, it follows that each strongly star-screenable Moore space which admits a continuous, strongly complete semi-metric is metrizable.

An admissible semi-metric d for a T_2-space S is a distance function for S such that (1) if each of x and y is a point of S, then $d(x, y) = d(y, x) \geqq 0$, (2) $d(x, y) = 0$ if and only if $x = y$, and (3) the topology of S is invariant with respect to d. A semi-metric d for the space S is said to be strongly complete provided that if $M_1, M_2, \cdots$ is a decreasing sequence of closed sets such that for each i, $M_i \subset \{y \in S \mid d(x_i, y) < 1/i\}$ for some $x_i \in S$, then $\bigcap M_i \neq \emptyset$. A space which admits a strongly complete semi-metric is said to be strongly complete. A development for a space S is a sequence $G_1, G_2, \cdots$ of open coverings of S such that, for each n, G_{n+1} is a subcollection of G_n, and for each point p and each open set D containing p, there is an integer n such that every element of G_n containing p is a subset of D. A development $G_1, G_2, \cdots$ for the space S is said to be complete (sequentially complete) provided that if $M_1, M_2, \cdots$ is a monotonic sequence of closed sets such that for each i, $M_i \subset g_i$ for some $g_i \in G_i$ ($M_i \subset \mathrm{st}(x_i, G_i)$ for some $x_i \in S$), then $\bigcap M_i \neq \emptyset$. A regular space having a development is a Moore space [1]. A Moore space having a complete (sequentially complete) development is said to be complete (sequentially complete). The property of sequential completeness is due to Traylor in [11]. Although each of strong completeness and sequential completeness is stronger than completeness in Moore spaces ([8] and [11]), for pointwise paracompact Moore spaces, all three are equivalent ([4] and [11]). A space S is said to be star-screenable (strongly star-screenable) if and only if, for each open covering G of S, there exists a σ-pairwise disjoint (σ-discrete) open covering H of S which refines $\{\mathrm{st}(x, G) \mid x \in S\}$.

LEMMA 1. *Each sequentially complete Moore space S is strongly complete.*

Proof. Let $G_1, G_2, \cdots$ denote a sequentially complete development for S. Denote by d the "natural semi-metric" for S determined by this development, i.e., $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1/n$, where n is the first positive integer such that no element of G_n contains both x and y, if $x \neq y$. It follows that if $M_1, M_2, \cdots$ is a monotonic decreasing sequence of closed sets such that for each i, there exists $x_i \in S$ such that $M_i \subset \{y \in S \mid d(x_i, y) < 1/i\}$, then for each i, $M_i \subset \mathrm{st}(x_i, G_i)$ and $\bigcap M_i \neq \emptyset$. Thus S is strongly complete.

LEMMA 2. *Each Moore space S which admits a continuous, strongly complete semi-metric is sequentially complete.*

Proof. Let d denote a continuous, strongly complete semi-metric for S. For each $p \in S$ and each positive integer n, let $g_n(p)$ denote an open set containing p such that if $x \in g_n(p)$ and $y \in g_n(p)$, then $d(x, y) < 1/n$. Now, for each n, let $H_n = \{g_n(p) \mid p \in S\}$. It follows immediately that $G_1, G_2, \cdots$, where for each i, $G_i = \bigcup_{j=i}^{\infty} H_j$, is a development for S. It is also a sequentially complete development. For suppose that $M_1, M_2, \cdots$ is a monotonic decreasing sequence of closed sets such that for each i, there exists a point p_i such that $M_i \subset \mathrm{st}(p_i, G_i)$, then for each i, $M_i \subset \{x \in S \mid d(x, p_i) < 1/i\}$ and $\bigcap M_i \neq \emptyset$. Thus, S is sequentially complete.

THEOREM 1. *Each normal, sequentially complete, star-screenable Moore space S is metrizable.*

Proof. Denote by $G_1, G_2, \cdots$ a sequentially complete development for S. Each normal star-screenable Moore space is strongly star-screenable [10]. Thus for each i, let $H_i = \bigcup_j H_{ij}$ denote an open cover of S which refines $\{\mathrm{st}(x, G_i) \mid x \in S\}$ such that H_{ij} is discrete for each j. Since S is normal and each open set in S is the union of countably many closed sets, for each i and j, let $H_{ij}^* = \bigcup_k H_{ijk}$ such that for each k, H_{ijk} is open in S and $\mathrm{CL}(H_{ijk}) \subset H_{ij}^*$. For each i, j, and k, let $F_{ijk} = \{H_{ijk} \cap h \mid h \in H_{ij}\}$ and note that if $f \in F_{ijk}$, then $\mathrm{CL}(f) \subset \mathrm{st}(x, G_i)$ for some $x \in S$. Now, for each n, let F_n denote a σ-discrete collection of open sets covering S such that if $f \in F_n$, then $\mathrm{CL}(f) \subset \mathrm{st}(x, G_n)$ for some $x \in S$. Let $B_1 = F_1$ and for each $i > 1$, let B_i denote the σ-discrete collection $\{f \cap b \mid f \in F_i$ and $b \in B_{i-1}\}$. Finally, let $B = \bigcup B_i$ and note that B is a σ-discrete collection of open sets covering S. However, B is also a basis for S. For let $p \in S$ and let D be an open set containing p. Then by the construction of B there exists a sequence of open sets $g_1(p), g_2(p), \cdots$ such that for each i, $p \in g_i(p)$, $g_i(p) \in B_i$, $g_{i+1}(p) \subset g_i(p)$, and $\mathrm{CL}\,(g_i(p)) \subset \mathrm{st}\,(x_i, G_i)$ for some $x_i \in S$. Suppose that for each i, $g_i(p) \cap (S - D) \neq \varnothing$. Then $(\mathrm{CL}\,(g_1(p)) - D)$, $(\mathrm{CL}\,(g_2(p)) - D), \cdots$ is a monotonic decreasing sequence of closed sets such that for each i, $(\mathrm{CL}\,(g_i(p)) - D) \subset \mathrm{st}\,(x_i, G_i)$ for some $x_i \in S$. Since $G_1, G_2, \cdots$ is a sequentially complete development for S, $\bigcap\,(\mathrm{CL}\,(g_i(p)) - D) \neq \varnothing$. Thus, let $x \in \bigcap\,(\mathrm{CL}\,(g_i(p)) - D)$ and note that for each i, there exist intersecting elements g_1 and g_2 of G_i which contain x and p respectively. But this contradicts the fact that $G_1, G_2, \cdots$ is a development for S. Thus, for some n, $g_n(p) \subset D$ and B is a σ-discrete basis for S. Therefore, S is strongly screenable, hence metrizable [1].

Heath in [7] defines a Moore space S with the three link property to be one with a development $G_1, G_2, \cdots$ having the three link property, i.e., for each two points p and q of S, there exists an n such that if g_1, g_2, and g_3 are elements of G_n, g_1 contains p and intersects g_2, and g_2 intersects g_3, then g_3 does not contain q. Zenor has shown in [12] that A Moore space has the three link property if and only if it has a regular G_δ-diagonal.

THEOREM 2. *Each sequentially complete, strongly star-screenable Moore space S with the three link property is metrizable.*

Proof. Without loss of generality, let $G_1, G_2, \cdots$ denote a sequentially complete development for S with the three link property. Now, by a construction similar to the one used in the proof of Theorem 1, let $B = \bigcup B_i$ denote a σ-discrete open covering of S such that for each i, if $b \in B_i$, then $b \subset \mathrm{st}\,(x, G_i)$ for some $x \in S$. (Note that without normality, we cannot require $\mathrm{CL}\,(b) \subset \mathrm{st}\,(x, G_i)$.) However, B still forms a basis for S. For suppose that $p \in S$ and D is an open set containing p. Then there exists a sequence of open sets $g_1(p), g_2(p), \cdots$ such that for each i, $p \in g_i(p)$, $g_i(p) \in B_i$, $g_{i+1}(p) \subset g_i(p)$, and $g_i(p) \subset \mathrm{st}\,(x_i, G_i)$ for some $x_i \in S$. Suppose that for each i, $g_i(p) \cap (S - D) \neq \varnothing$. Thus, for each i, let $p_i \in g_i(p) \cap (S - D)$. Consider $\{p_1, p_2, \cdots\}$. Suppose this set has no limit point. Then for each i, $M_i = \{p_i, p_{i+1}, \cdots\}$ is a closed set such that $M_i \subset \mathrm{st}\,(x_i, G_i)$ for some $x_i \in S$. And since $G_1, G_2, \cdots$ is a sequentially complete development for S, $\bigcap M_i \neq \varnothing$. But if $\bigcap M_i \neq \varnothing$, as in the proof of Theorem 2, we contradict the fact that $G_1, G_2, \cdots$ is a development for S. However, if x is a limit point of $\{p_1, p_2, \cdots\}$, then for each i, there exists an element g_1 of G_i

which contains both x and p_j for some $j > i$. But $p_j \in g_j(p)$ and $g_j(p) \subset \mathrm{st}(x_j, G_j)$ for some $x_j \in S$. Thus, there exist intersecting elements g_2 and g_3 of G_j, hence of G_i, which contain p and p_j respectively. Therefore, for each i, there exist elements g_1, g_2, and g_3 of G_i such that g_1 contains x and intersects g_2, g_2 intersects g_3, and $p \in g_3$. But this contradicts the fact that $G_1, G_2, \cdots$ has the three link property. Thus, for some n, $g_n(p) \subset D$ and it follows that B is a σ-discrete basis for S. Again, by [1], S is metrizable.

THEOREM 3. *Each strongly star-screenable Moore space S which admits a continuous, strongly complete semi-metric is metrizable.*

Proof. By Lemma 2, S is sequentially complete. And from ([2], Theorem 8) it follows that S has the three link property. Thus, by Theorem 2, S is metrizable.

The next two theorems show that Theorem 3 is a reasonable partial answer to question (5) in [10].

THEOREM 4. *There exists a strongly star-screenable Moore space which admits a continuous semi-metric that is not metrizable.*

Proof. In [2], Cook gave an example of a separable, nonmetrizable Moore space which admits a continuous semi-metric. Since each separable space is strongly star-screenable, that example has the desired properties.

THEOREM 5. *There exists a Moore space S which admits a continuous, strongly complete semi-metric which is not metrizable.*

Proof. Consider the following Moore space S given by Heath in [5]. The points of S are all points of the plane on or above the x-axis. For each positive integer n, define H_n as follows: (1) for p above the x-axis, $\{p\} \in H_n$; (2) for each rational number r on the x-axis, $\{(r, y) \mid o \leqq y \leqq 1/n\} \in H_n$; (3) for each irrational number x on the x-axis, $\{(t, y) \mid t = x + y, o \leqq y \leqq 1/n\} \in H_n$. Then, $\bigcup H_n$ forms a basis for S and the sequence $G_1, G_2, \cdots$, where for each i, $G_i = \bigcup_{j=i}^{\infty} H_j$, is a development for S. It is easily seen that the "natural semi-metric" for S with respect to this development has the required properties.

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