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VARIOUS TYPES OF LOCAL HOMOGENEITY

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In this paper some notions of local homogeneity for metric space are investigated. A theorem on convergence of a sequence of homeomorphisms to a homeomorphism is proved and applied i.e., to show that for every two countable and dense subsets A and B of the Hilbert space l_2 there exists a homeomorphism H of l_2 onto itself such that $H(A) = B$.

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of homeomorphisms of a space X onto itself and let

$$H_n = h_n \circ h_{n-1} \circ \cdots \circ h_1 \tag{1}$$

denote their composition.

In this note conditions—similar to those obtained by J. L. Paul in [4]—are given under which $H = \lim_{n\to\infty} H_n$ exists and is a homeomorphism of X onto itself. The obtained results are applied to investigate various types of local homogeneity and to prove that if $X = l_2$ is the Hilbert space then X has the following property: For every two countable and dense subsets A and B of X there exists a homeomorphism H of X onto itself such that $H(A) = B$.

A space X having this property was called by R. B. Bennett countable dense homogeneous (see [1]).

A proof that the Euclidean n-space E^n is countable dense homogeneous can be found in the book of W. Hurewicz and H. Wallmann "Dimension Theory" on p. 44.

In this paper the notion of countable dense homogeneity in the class of complete metric spaces is investigated and results—similar to those given by R. B. Bennett in [1] for locally compact spaces—are obtained.

In the sequel X denotes a metric space with metric ρ, $B(x, r)$—the open ball with radius r centered at x, $\delta(V)$—the diameter of a subset V of X and nbd stands for neighborhood. We also always assume that $\delta(X) \leqq 1$.

1. In this section conditions are given under which a sequence of homeomorphisms H_n of X onto itself converges to a homeomorphism H of X onto itself. Theorem 1 is a generalization of a similar theorem proved by J. L. Paul, (see [4]). We shall use the following two definitions:

DEFINITION 1. A sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X is said to be proper if

$$(2)\qquad U_i \cap U_j \neq \varnothing \Longrightarrow U_i \supset \bar{U}_j \quad \text{for} \quad i < j\ .$$

DEFINITION 2. A homeomorphism h of X onto itself will be called of type (ξ, r, U) where $0 < \xi < 1$, $1 \leqq r$ are constants and U is an open subset of X if:

$$(3)\qquad h|_{X\setminus V} \text{ is the identity mapping}$$

and

$$(4)\qquad \rho(h(x), h(y)) > \xi[\rho(x, y)]^r \quad \text{for all} \quad x \neq y \quad \text{in} \quad X\ .$$

THEOREM 1. *Let X be a complete metric space and $\{U_n\}_{n=1}^{\infty}$ a proper sequence. Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of homeomorphisms of X onto itself where h_n is of type (ξ_n, r_n, U_n). If*

$$(5)\qquad \delta(U_n) < \xi_{n-1} \cdot \xi_{n-2}^{r_{n-1}} \cdot \cdots \xi_1^{r_{n-1}\cdots r_2} \cdot \left(\frac{1}{2^n}\right)^{r_1 r_2 \cdots r_n}$$

then $H = \lim_{n\to\infty} H_n$ exists and is a homeomorphism of X onto itself.

Proof. The proof being similar to Theorem 1 in [4] we shall confine ourselves to showing only that H is continuous and one-to-one.

By (3), (4), and (5) $\rho(h_n(x), x) < 1/2^n$. Hence $\rho(H_n(x), H_{n-1}(x)) < 1/2^n$. The space X being complete it follows that $H(x) = \lim_{n\to\infty} H_n(x)$ exists for all x and that H is continuous. To show that H is one-to-one let $x \neq y$ be points of X. There are two cases:

Case 1. There exist integers $j(x)$ and $k(y)$ such that

$$H_l(x) = H_{j(x)}(x) \text{ for } l > j(x) \text{ and } H_m(y) = H_{k(y)}(y) \text{ for } m > k(y)\ .$$

Then for $g = \max\{j(x), k(y)\}$ one has

$$H(x) = H_g(x) \neq H_g(y) = H(y)\ .$$

Case 2. For at least one of the points x or y, (say x), there exists a sequence $m_1 < m_2 < \cdots$ such that $H_{m_{i-1}}(x) \in U_{m_i}$. Let $p = m_i$ be an integer of this sequence such that

$$(6)\qquad \left(\frac{1}{2^p}\right)^{r_1 r_2 \cdots r_{p-1}} < [\rho(x, y)]^{r_1 r_2 \cdots r_{p-1}}\ .$$

By (2) and (3) the points $H_{p-1}(x)$ and $H(x)$ belong to U_p.
By (4) we have

$$\rho(H_{p-1}(x), H_{p-1}(y)) > \xi_{p-1}\xi_{p-2}^{r_{p-1}} \cdots \xi_1^{r_2 r_3 \cdots r_{p-1}}[\rho(x, y)]^{r_1 r_2 \cdots r_{p-1}}\ .$$

Now by (5) and (6) $\delta(U_p) < \rho(H_{p-1}(x), H_{p-1}(y))$.

Therefore, $H_{p-1}(y) \notin U_{p-1}$, and thus also $H(y) \notin U_{p-1}$. Since $H(x) \in U_{p-1}$ it follows that $H(x) \neq H(y)$

REMARK 1. Theorem 1 remains true if the requirement that X is complete is replaced by requiring the relative compactness of the U_i. The special case where $r_n = 1$ $(n = 1, 2 \cdots)$ is Theorem 1 of [4] and the proof is the same.

The following theorem shows a relation between the types (see Def. 2) of the homeomorphisms $\{h_n\}_{n=1}^{\infty}$ and of the limit $H = \lim_{n\to\infty} H_n$ of their superposition.

THEOREM 2. *Let* $\{U_n\}$ *be a sequence of open subsets of* X *such that* $U_n \supset \bar{U}_{n+1}(n = 1, 2 \cdots)$ *and let* $\{h_n\}_{n=1}^{\infty}$ *be a sequence of homeomorphisms of* X *onto itself. If* h_n *is of type* $(1/2, 1, U_n)$ *and*

$$\delta(U_n) < \frac{1}{4^n} . \tag{7}$$

Then $H = \lim_{n\to\infty} H_n$ *is a homeomorphism of* X *onto itself of type* $(1/4, 2, U_1)$.

Proof. By Theorem 1, H is a homeomorphism of X onto itself and it suffices to show that

$$\rho(H(x), H(y)) > \frac{1}{4}[\rho(x, y)]^2 \quad \text{for all} \quad x \neq y . \tag{8}$$

Since by assumption $\delta(X) \leqq 1$ there exists an integer $n_0 \geqq 1$ such that $(1/2)^{n_0} < \rho(x, y) = d \leqq (1/2)^{n_0-1}$. By (4) used for $\xi_n = 1/2$, $r_n = 1$ we have

$$\rho(H_{n_0}(x), H_{n_0}(y)) > \left(\frac{1}{2}\right)^{n_0} d .$$

Since by (7) $\delta(U_{n_0+1}) < 1/4^{n_0+1}$, the points $H_{n_0}(x)$ and $H_{n_0}(y)$ are not both in U_{n_0+1}.

If both points are outside U_{n_0+1} then by $U_n \supset \bar{U}_{n+1}$ and by (3) we have $H(x) = H_{n_0}(x)$ and $H(y) = H_{n_0}(y)$. Hence

$$\rho(H(x), H(y)) > \frac{1}{2^{n_0}} \cdot d > \frac{1}{4}[\rho(x, y)]^2 .$$

Thus we may assume that $H_{n_0}(x) \in U_{n_0+1}$ and $H_{n_0}(y) \notin U_{n_0+1}$. By (3) $\rho(H_{n_0}(x), H(x)) < \delta(U_{n_0+1}) < 1/4^{n_0+1}$, whence

$$\begin{aligned}\rho(H(x), H(y)) = \rho(H(x), H_{n_0}(y)) &\geqq \rho(H_{n_0}(x), H_{n_0}(y)) - \rho(H_{n_0}(x), H(x)) \\ &> \left(\frac{1}{2}\right)^{n_0} \cdot d - \left(\frac{1}{4}\right)^{n_0+1} > \frac{1}{4}d^2 .\end{aligned}$$

REMARK 2. Clearly the types $(1/2, 1, U_n)$ and $(1/4, 2, U_1)$ can be replaced by other appropriate combinations.

2. In this section various notions of local homogeneity are defined. It is remarked that the Hilbert space l_2 is locally homogeneous of Type 1. It is then proved that for separable and complete metric spaces local homogeneity of a variable type implies countable dense homogeneity. Some problems and examples are also given.

DEFINITION 3. Let $r \geqq 1$. A metric space X will be called locally homogeneous of type r if for every X and every nbd. U of x there exists a nbd. V of x such that $V \subset U$ and such that for each two points y_1 and y_2 of V there exists a homeomorphism h of X onto itself of type (ξ, r, U) satisfying $h(y_1) = y_2$.

Let us note that if X is locally homogeneous of type r then X is strongly locally homogeneous according to [1]. Furthermore, if X is locally homogeneous of type r then clearly X is locally homogeneous of type s for $s > r$ (we assume $\delta(X) \leqq 1$). The following example shows the converse does not hold.

EXAMPLE 1. Let X be the subset of E^2 defined as follows: For any real number $r \geqq 1$ denote by L_n the segment in E^2 whose endpoints are $(1/n, 0)$ and $(1/n, (1/n)^{1/r})$, and by M_n the segment in E^2 whose endpoints are $(1/n, (1/n)^{1/r})$ and $(1/(n+1), 0)$, where $n = 1, 2 \cdots$. Define

$$X = \bigcup_{n=1}^{\infty} (L_n \cup M_n) \cup \{(x, 0) \,|\, x \leqq 0 \quad \text{or} \quad x \geqq 1\} .$$

Let $\tilde{d}$ be the metric on X induced by $\tilde{d}(x, y) = \min\{d(x, y), 1\}$ where d is the usual Euclidean metric in E^2. Then $(X, \tilde{d})$ is *not* locally homogeneous of type r but *is* locally homogeneous of type s for every $s > r$.

DEFINITION 4. A space X will be called locally homogeneous of variable type if for every $x \in X$ and every nbd U of x there exists a nbd. V of x such that $V \subset U$ and such that for each two points y_1 and y_2 of V there exists a homeomorphism h of X onto itself of type $(\xi, r(x, U), U)$ satisfying $h(y_1) = y_2$. (r depends on x and U.) A space which is locally homogeneous of strictly variable type is a space which is locally homogeneous of variable type but not of any fixed type r.

Clearly a space which is locally homogeneous of type r is locally homogeneous of variable type. For an example of a space which is locally homogeneous of strictly variable type we give

EXAMPLE 2. By modifying Example 1 we obtain an example of a locally homogeneous space of strictly variable type. Define

$$X = \bigcup_{n,m=1}^{\infty} (L_{n,m} \cup M_{n,m}) \cup \{(x, 0) \mid x \leqq 0\}$$

where $L_{n,m}$ is the segment in E^2 whose endpoints are $(1/n + m - 1, 0)$ and $(1/n + m - 1, (1/n + m - 1)^{1/m})$ and $M_{n,m}$ is the segment in E^2 whose endpoints are $(1/n + m - 1, (1/n + m - 1)^{1/m})$ and $(1/(n + 1) + m - 1, 0)$.

If X is topologized by $\tilde{d}$ of Example 1 then $(X, \tilde{d})$ is locally homogeneous of strictly variable type.

DEFINITION 5. A separable metric space X will be called (see [1]) countable dense homogeneous if for every two countable and dense subsets A and B of X there exists a homeomorphism of (X, A) onto (X, B). The proof of the following theorem is not difficult and will be omitted.

THEOREM 3. *The Euclidean n-space E^n, the Hilbert space l_2 and the Cantor set C are locally homogeneous of Type* 1.

We now prove

THEOREM 4. *A complete separable metric space X which is locally homogeneous of variable type is countable dense homogeneous.*

Proof. Let $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$ be two countable dense subsets of X. We shall construct by induction a sequence $\{h_n\}_{n=1}^{\infty}$ of homeomorphisms of X onto itself such that $H = \lim_{n\to\infty} H_n$ is a homeomorphism of X onto itself satisfying $H(A) = B$.

(a) Let $U = U_1 = B(a_1, 1)$ and $V = V_1 \subset U_1$ be the subset of U given by Definition 4. Let b_{i_0} be an arbitrary point of $V_1 \cap B$. By hypothesis there exists a homeomorphism h_1 of X onto itself of type (ξ_1, r_1, U_1) such that $h(a_1) = b_{i_0}$. Put $a_1^* = a_1$ and $b_1^* = b_{i_0}$.

Since $A \cup B$ is countable we can assume that $(A \cup B) \cap \mathrm{Bd}\,(U_1) = \varnothing$.

(b) Define $b_2^* = b_j$ where j is the first integer such that $b_j \in B \backslash \{b_1^*\}$. Let $\varepsilon_2 = \min\{\rho(b_2^*, a_1^*), \rho(b_2^* b_1^*), \rho(b_2^*, \mathrm{Bd}\, U_1), \xi_1(1/2^2)^{r_1}\}$.

Put $U = U_2 = B(b_2^*, \varepsilon_2/2)$ and let $V = V_2 \subset U_2$ be the subset of U_2 given by Definition 4. Let $h_1(a_2^*)$ be an arbitrary point of $h_1(A) \cap V_2$ (evidently $a_2^* \varepsilon A \backslash \{a_1^*\}$).

By hypothesis there exists a homeomorphism h_2 of type (ξ_2, r_2, U_2) of X onto itself such that $h_2 \circ h_1(a_2^*) = b_2^*$. Again we can assume that $(h_1(A) \cup B) \cap \mathrm{Bd}\,(U_2) = \varnothing$.

(c) Define $a_3^* = a_i$ where i is the first integer such that $a_i \in A \backslash \{a_1^*, a_2^*\}$ and let

$$\varepsilon_3 = \min_{i \leq 2} \left\{ \rho(h_2 \circ h_1(a_3^*), h_2 \circ h_1(a_i^*)), \rho(h_2 \circ h_1(a_3^*), \mathrm{Bd}\,(U_i)), \right.$$
$$\left. \rho(h_2 \circ h_1(a_3^*), b_i^*), \xi_1 \xi_2 \left(\frac{1}{2^3}\right)^{r_1 r_2} \right\} .$$

Put $U_3 = B(h_2 \circ h_1(a_3^*), \varepsilon_3/2)$ and let $V_3 \subset U_3$ be the subset of U_3 given by Definition 4. We can assume that $(h_2(h_1(A)) \cup B) \cap \mathrm{Bd}\,(U_3) = \varnothing$. Let b_3^* be an arbitrary point of $B \cap V_3$ (evidently $b_3^* \in B \backslash \{b_1^*, b_2^*\}$). Again by hypothesis there exists a homeomorphism h_3 of X onto itself of type (ξ_3, r_3, U_3) such that $h_3 \circ h_2 \circ h_1(a_3^*) = b_3^*$.

(d) It should be clear how steps (b) and (c) can be repeated to obtain a sequence $\{h_n\}_{n=1}^*$ of homeomorphisms of X onto itself satisfying the assumptions of Theorem 1. By Theorem 1 $H = \lim_{n\to\infty} H_n$ is a homeomorphism of X onto itself and clearly $H(A) = B$. Theorem 3 and Theorem 4 imply the following

COROLLARY 1. *The Hilbert space* l_2 *is countable dense homogeneous. This result does not follow from* [1].

The above concepts now lead to the following problems:

Problem 1. Let $r > s \geqq 1$ be fixed real numbers. Does there exist a metric space (X, ρ) which is locally homogeneous of type r, and if ρ_1 is any metric on X equivalent to ρ then the space (X, ρ_1) is not locally homogeneous of type S?

Problem 2. Does there exist a metric space (X, ρ) which is locally homogeneous of strictly variable type for each metric ρ_1 equivalent to ρ?

Problem 3. Determine conditions which imply that a strongly locally homogeneous space is locally homogeneous of variable type.

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