PACIFIC JOURNAL OF MATHEMATICS
Vol. 50, No. 2, 1974

FOURIER TRANSFORMS OF ODD AND EVEN TEMPERED DISTRIBUTIONS

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In this paper certain previous results of the author concerning Abelian theorems for the Fourier transform of distributions are generalized to two new distribution spaces, those of odd and even tempered distributions. These spaces arise in the consideration of Fourier sine and cosine transforms of distributions.

In [2] Abelian theorems concerning the Fourier transform of functions provided the initial motivation for similar results about the Fourier transform of distributions. They also contributed directly to these results through the representability of certain types of distributions by functions. It turns out that the analogous procedure is possible with Fourier sine and cosine transforms, leading to the space of even tempered distributions ($\mathscr{S}_e'$) and the space of odd tempered distributions ($\mathscr{S}_0'$).

The basic idea is to generalize the facts for the classical transform that the Fourier transform of an even function is actually a cosine transform and the Fourier transform of an odd function is actually a sine transform. Then as in [2] Abelian theorems can be obtained for these transforms of distributions which are representable in certain ways by functions. In § 5, results of this type are obtained for semiregular distributions, those which are regular over a subset of their respective supports.

With this approach, classical results for both the Fourier sine transform and Fourier cosine transform yield distributional results for these two transforms and can then be combined to yield results about the Fourier transform of a distribution itself. Thus, not only do we have a direct generalization of results for Fourier sine and cosine transforms of functions and hence an alternate approach to Abelian theorems about the Fourier transform, but we are dealing with the larger distribution spaces, $\mathscr{S}_e'$ and $\mathscr{S}_0'$.

2. **Notation and definitions.** The evaluation of a distribution T at a test function φ will be denoted by $\langle T, \varphi\rangle$. All integrals are Lebesgue integrals and $f \in BV(\Omega)$ will mean the function f is of bounded variation over the set Ω. $x \to \pm a$ is shorthand for the two statements $x \to a^-$ (approach from the left only) and $x \to a^+$ (approach from the right only). As usual, $f \sim Kg$ $(x \to a)$ for $K \neq 0$ will mean $f/g \to K$ as $x \to a$. If at any time the variable in a given expression is not

clear from the context, a subscript will be used as an indicator, such as T_t or $[\mathscr{S}_e]_\xi$.

A distribution is said to be regular if it is defined by a locally integrable function f, that is, if $\langle T, \varphi\rangle = \langle T_f, \varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$ for each test function φ. Then a distribution which is regular over a subset of its support will be called semiregular. A distribution which is not semiregular is said to be singular. We denote by $\mathscr{S}'$ the class of all tempered distributions [3, p. 188]; $\mathscr{S}$ is the corresponding test function space. Also $\mathscr{E}'$ is the class of all distributions with bounded support [3, p. 99]; $\mathscr{E}$ is the corresponding test function space.

DEFINITION 2.1. The space $\mathscr{S}_e(\mathscr{S}_0)$ is the subset of all even (odd) functions in $\mathscr{S}$. Thus $\varphi \in \mathscr{S}_e(\mathscr{S}_0)$ if $\varphi \in \mathscr{E}$, φ is an even (odd) function, and for any arbitrary integer $k \geqq 0$ there exist constants C_{km} such that

$$|x^k\varphi^{(m)}(x)| \leqq C_{km}\,(m = 0, 1, 2, \cdots) - \infty < x < \infty\ .$$

An example of an element of $\mathscr{S}_e$ is given by e^{-x^2}, since $e^{-x^2} \in \mathscr{S}$ and is even. Thus $xe^{-x^2} \in \mathscr{S}_0$, since if φ is an even differentiable function, then $K\varphi'$ is odd for any constant K.

DEFINITION 2.2. A sequence of functions $\varphi_j \in \mathscr{S}_e(\mathscr{S}_0)$ converges to zero in $\mathscr{S}_e(\mathscr{S}_0)$ as $j \to \infty$ $[\varphi_j \to 0$ in $\mathscr{S}_e(\mathscr{S}_0)]$ if for arbitrary integers $k, m \geqq 0$ the sequence $x^k\varphi_j^{(m)}(x) \to 0$ uniformly in $\boldsymbol{R}$.

If f is a complex valued function absolutely integrable over $(-\infty, \infty)$, then the Fourier transform of f is the function of the real variable σ defined by

$$\hat{f}(\sigma) = \mathscr{F}[f(x); \sigma] = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)e^{-i\sigma x}dx\ .$$

Under the same conditions, the Fourier sine transform will be denoted by $\hat{f}_s(\sigma) = \mathscr{F}_s[f(x); \sigma]$ and the Fourier cosine transform by $\hat{f}_c(\sigma) = \mathscr{F}_c[f(x); \sigma]$. Also, for the one-sided Fourier sine transform we use the symbols $\hat{f}_s^+(\sigma) = \mathscr{F}_s{}^+[f(x); \sigma]$ and for the one-sided Fourier cosine transform $\hat{f}_c^+(\sigma) = \mathscr{F}_c{}^+[f(x); \sigma]$. At times certain comments will apply to either $\hat{f}_c$ or $\hat{f}_s$, so in such a situation the symbols $\tilde{f}$ and $\mathscr{F}_t$ will be used. Finally, if $T \in S'$ then the Fourier transform of T is the distribution $\mathscr{F}[T]$ or $\hat{T}$ defined by $\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle$ for any test function $\varphi \in \mathscr{S}$.

3. Classical preliminaries. The development of Fourier sine and cosine transforms of distributions is carried out along lines that are standard for the Fourier transform. However, the only result of that theory which is needed explicitly is that $\mathscr{F}(\mathscr{F}^{-1}\varphi) = \mathscr{F}^{-1}(\mathscr{F}\varphi) = \varphi$ for any $\varphi \in \mathscr{S}$. Therefore, some basic properties of the Fourier sine and cosine transforms of functions need mention.

THEOREM 3.1. *If $f \in L^1$ then $\tilde{f}$ exists, is uniformly continuous and bounded, and satisfies:*

(i) $(2\pi)^{1/2}|\tilde{f}(\xi)| \leqq \|f\|_{L^1}$

(ii) $|\tilde{f}(\xi)| \to 0$ *as* $|\xi| \to \infty$

(iii) $\mathscr{F}_t[f(kx); \xi] = (1/|k|)\mathscr{F}_t[f(x); \xi/k]$, $k \in \boldsymbol{R}$, $k \neq 0$.

If f is m-times differentiable, then

$$\text{(iv)}\quad \mathscr{F}_c[f^{(m)}(x); \xi] = \begin{cases} -\xi^m \mathscr{F}_c[f(x); \xi] & m \text{ even} \\ \xi^m \mathscr{F}_s[f(x); \xi] & m \text{ odd} \end{cases}$$

and

$$\text{(v)}\quad \mathscr{F}_s[f^{(m)}(x); \xi] = \begin{cases} \xi^m \mathscr{F}_s[f(x); \xi] & m \text{ even} \\ -\xi^m \mathscr{F}_c[f(x); \xi] & m \text{ odd.} \end{cases}$$

Also

(vi) $|\xi|^m|\tilde{f}(\xi)| \leqq (2\pi)^{-1/2}\|f^{(m)}\|_{L^1}$.

If $x^m f \in L^1$ and $f \in L^1$, then $\tilde{f}(\xi)$ is m-times differentiable and

$$\text{(vii)}\quad \hat{f}_c^{(m)}(\xi) = \begin{cases} (-1)^{(m+1)/2}\mathscr{F}_s[x^m f(x); \xi] & m \text{ odd} \\ (-1)^{m/2}\mathscr{F}_c[x^m f(x); \xi] & m \text{ even} \end{cases}$$

$$\text{(viii)}\quad \hat{f}_s^{(m)}(\xi) = \begin{cases} (-1)^{(m-1)/2}\mathscr{F}_c[x^m f(x); \xi] & m \text{ odd} \\ (-1)^{m/2}\mathscr{F}_s[x^m f(x); \xi] & m \text{ even.} \end{cases}$$

Hence

(ix) $|\tilde{f}^{(m)}(\xi)| \leqq (2\pi)^{-1/2}\|x^m f(x)\|_{L^1}$.

The proof follows easily by standard techniques.

COROLLARY 3.2. *If $\varphi \in \mathscr{S}_e$, then*

$$\left.\begin{array}{ll} \text{(i)} & \mathscr{F}_c[\varphi^{(2m)}; \xi] = -\xi^{2m}\mathscr{F}_c[\varphi; \xi] \\ \text{(ii)} & \mathscr{F}_c[\varphi^{(2m-1)}; \xi] = 0 \end{array}\right. \quad (m = 1, 2, 3, \cdots)$$

and if $\varphi \in \mathscr{S}_0$, then

$$\left.\begin{array}{ll} \text{(iii)} & \mathscr{F}_s[\varphi^{(2m)}; \xi] = -\xi^{2m}\mathscr{F}_s[\varphi; \xi] \\ \text{(iv)} & \mathscr{F}_s[\varphi^{(2m-1)}; \xi] = 0 \end{array}\right. \quad (m = 1, 2, 3, \cdots)\,.$$

For odd and even functions in L^1 the use of the Fourier and inverse Fourier transforms reduces to the use of sine and cosine transforms, respectively.

THEOREM 3.3. *If f is an even (odd) function in L^1, then*

$$\mathscr{F}[f(x); \xi] = \mathscr{F}_c[f(x); \xi] \qquad (-i\mathscr{F}_s[f(x); \xi])\,.$$

Also

$$\mathscr{F}^{-1}[f(x); \xi] = \mathscr{F}_c[f(x); \xi] \qquad (i\mathscr{F}_s[f(x); \xi]) .$$

The proof is immediate from the definitions of the quantities involved and the results of multiplying different combinations of odd and even functions.

For the purposes of the applications in § 5, we need to mention a classical Abelian theorem covering Fourier sine and cosine transforms. This result will later be generalized to odd and even tempered distributions.

THEOREM 3.4. *Let* $f \in L^1(0, \infty)$ *such that*

$$f(x) = \begin{cases} x^{-\alpha}g(x) & 0 < x < a \\ x^{-\beta}h(x) & a < x < \infty \end{cases}$$

where $a \geqq 0$, $g \in BV[0, a]$, *and* $h \in BV[a, \infty]$, *then*

(i) *if* $0 < \alpha < 1$ *and* $\beta \geqq 0$

$$\hat{f}_c^+(x) \sim g(0^+)(2/\pi)^{1/2}\Gamma(1-\alpha)\sin(\pi\alpha/2)|x|^{\alpha-1} \qquad (x \longrightarrow \pm\infty) ,$$

and

$$\hat{f}_s^+(x) \sim \operatorname{sgn}(x)g(0^+)(2/\pi)^{1/2}\Gamma(1-\alpha)\cos(\pi\alpha/2)|x|^{\alpha-1} \qquad (x \longrightarrow \pm\infty) .$$

(ii) *if* $\alpha < 1$ *and* $0 < \beta < 1$

$$\hat{f}_c^+(x) \sim h(\infty)(2/\pi)^{1/2}\Gamma(1-\beta)\sin(\pi\beta/2)|x|^{\beta-1} \qquad (x \longrightarrow 0) ,$$

and

$$\hat{f}_s^+(x) \sim \operatorname{sgn}(x)h(\infty)(2/\pi)^{1/2}\Gamma(1-\beta)\cos(\pi\beta/2)|x|^{\beta-1} \qquad (x \longrightarrow \pm 0) .$$

Proof. See [2, p. 164].

4. Fourier sine and cosine transforms of distributions. Just as the fact that $\varphi \in \mathscr{S}$ implies $\hat{\varphi} \in \mathscr{S}$ is important in the development of the Fourier transform of an element of $\mathscr{S}'$, the analogous facts for $\mathscr{S}_e$ and $\mathscr{S}_0$ are of equal importance in the development of the Fourier sine and cosine transforms.

THEOREM 4.1. *If* $\varphi \in \mathscr{S}_e(\mathscr{S}_0)$, *then* $\mathscr{F}_c\varphi \in \mathscr{S}_e(\mathscr{F}_s\varphi \in \mathscr{S}_0)$ *and* $\mathscr{F}_c[\mathscr{F}_c\varphi] = \varphi$ $(\mathscr{F}_s[\mathscr{F}_s\varphi] = \varphi)$. *Moreover,* $\varphi_n \to 0$ *in* $\mathscr{S}_e(\mathscr{S}_0)$ *implies that* $\mathscr{F}_c\varphi_n \to 0$ *in* $\mathscr{S}_e(\mathscr{F}_s\varphi_n \to 0$ *in* $\mathscr{S}_0)$. *Thus the mapping* $\mathscr{F}_c(\mathscr{F}_s)$ *is a continuous isomorphism of* $\mathscr{S}_e(\mathscr{S}_0)$ *onto* $\mathscr{S}_e(\mathscr{S}_0)$.

Proof. $\varphi \in \mathscr{S}_e \Rightarrow x^m\varphi \in L^1$ for any integer $m \geqq 0$, thus by Theorem 3.1 $\hat{\varphi}_c \in \mathscr{E}$. Also by Theorem 3.1 [(vii) and (vi)]

$$|\xi|^m|\hat{\varphi}_c(\xi)| \leqq (2\pi)^{-1/2}\|\{x^l\varphi(x)\}^{(m)}\|_{L^1}$$

for arbitrary nonnegative integers m and l, that is $\hat{\varphi}_c \to 0$ as $|\xi| \to \infty$ faster than any power of $1/|\xi|$. Also $\hat{\varphi}_c(-\xi) = \hat{\varphi}_c(\xi)$, thus $\hat{\varphi}_c \in \mathscr{S}_e$. Similarly for $\hat{\varphi}_s \in \mathscr{S}_0$. If $\varphi \in \mathscr{S}_e$, then the fact that $\mathscr{F}_c[\mathscr{F}_c\varphi] = \varphi$ follows from Theorem 3.3 and the fact that if $\varphi \in \mathscr{S}$ then $\mathscr{F}[\mathscr{F}^{-1}\varphi] = \mathscr{F}^{-1}(\mathscr{F}\varphi) = \varphi$ [3, p. 192]. Similarly for $\varphi \in \mathscr{S}_0$ and $\mathscr{F}_s$. Now if $\varphi_n \to 0$ in $\mathscr{S}_e$, then as above

$$|\xi^m\hat{\varphi}^{(l)}_{n_c}(x)| \leqq \|\{x^l\varphi_n(x)\}^{(m)}\|_{L^1} .$$

By Definition 2.2 the right hand side of this inequality approaches zero as n approaches infinity and hence $\xi^m\hat{\varphi}_{n_c} \to 0$ uniformly, that is $\hat{\varphi}_{n_c} \to 0$ in $\mathscr{S}_e$. Similarly for $\hat{\varphi}_s$.

It is not at all surprising that the motivation for the definition of Fourier sine and cosine transforms of distributions is supplied by a classical result.

THEOREM 4.2. *If $f \in L^1$ and $\varphi \in \mathscr{S}_e(\mathscr{S}_0)$, then*

$$\langle \mathscr{F}_c f, \varphi \rangle = \langle f, \mathscr{F}_c\varphi \rangle \qquad (\langle \mathscr{F}_s f, \varphi \rangle = \langle f, \mathscr{F}_s\varphi \rangle) .$$

Proof. This is a special case of the well-known result (related to the Parseval formula) that if f and g belong to L^1, then

$$\int_{-\infty}^{\infty} \mathscr{F}[f(\xi); x]g(x)dx = \int_{-\infty}^{\infty} f(x)\mathscr{F}[g(\xi); x]dx .$$

See, for example, Bochner and Chandrasekharan [1, p. 2] or Titchmarsh [4, pp. 50-54].

DEFINITION 4.3. An even (odd) tempered distribution is a continuous linear functional on the vector space $\mathscr{S}_e(\mathscr{S}_0)$.

THEOREM 4.4. *$\mathscr{S}_e' \supset \mathscr{S}'$ and $\mathscr{S}_0' \supset \mathscr{S}'$. We denote this space by $\mathscr{S}_e'(\mathscr{S}_0')$.*

Proof. $\varphi \in \mathscr{S}_e(\mathscr{S}_0)$ implies $\varphi \in \mathscr{S}$ and $\varphi_j \to 0$ in $\mathscr{S}_e(\mathscr{S}_0)$ implies $\varphi_j \to 0$ in $\mathscr{S}$. Hence a continuous linear functional on $\mathscr{S}$ is also a continuous linear functional on $\mathscr{S}_e(\mathscr{S}_0)$. Thus $\mathscr{S}_e'(\mathscr{S}_0') \supset \mathscr{S}'$.

Now that all of the necessary machinery has been developed, it is possible to define the desired transforms. This is a direct analogue of the definition of the Fourier transform of an element of $\mathscr{S}'$.

THEOREM 4.5. *If T is an even (odd) tempered distribution then we can define the Fourier cosine (sine) transform $\hat{T}_c = \mathscr{F}_c T$ ($\hat{T}_s = \mathscr{F}_s T$) of T by*

$$(4.1) \qquad \langle \mathscr{F}_c T, \varphi \rangle = \langle T, \mathscr{F}_c \varphi \rangle \qquad (\langle \mathscr{F}_s T, \varphi \rangle = \langle T, \mathscr{F}_s \varphi \rangle)$$

for arbitrary $\varphi \in \mathscr{S}_e(\mathscr{S}_0)$. *Thus* $\mathscr{F}_c(\mathscr{F}_s)$ *is a continuous mapping of* $\mathscr{S}_e'(\mathscr{S}_0')$ *onto* $\mathscr{S}_e'(\mathscr{S}_0')$.

Proof. If $\varphi \in \mathscr{S}_e$, then $\hat{\varphi}_c \in \mathscr{S}_e$ by Theorem 4.1, that is if $T \in \mathscr{S}_e'$, then $\langle T, \hat{\varphi}_c \rangle$ is meaningful and $\langle \mathscr{F}_c T, \varphi \rangle = \langle T, \mathscr{F}_c \varphi \rangle$ defines a linear functional on $\mathscr{S}_e$. If $\varphi_j \to 0$ in $\mathscr{S}_e$ as $j \to \infty$ then $\hat{\varphi}_{j_c} \to 0$ in $\mathscr{S}_e$ as $j \to \infty$ by Theorem 4.1. Thus $\langle T, \hat{\varphi}_{j_c} \rangle \to 0$ and hence $\langle \mathscr{F}_c T, \varphi_j \rangle \to 0$ as $j \to \infty$. That is, equation (4.1) defines a continuous linear functional on $\mathscr{S}_e$, and hence an element of $\mathscr{S}_e'$.

Finally, if $T_j \to T$ in $\mathscr{S}_e'$ then $\langle T_j, \psi \rangle \to \langle T, \psi \rangle$ for arbitrary $\psi \in \mathscr{S}_e$. Thus $\langle \mathscr{F}_c T_j, \varphi \rangle = \langle T_j, \mathscr{F}_c \varphi \rangle \to \langle T, \mathscr{F}_c \varphi \rangle = \langle \mathscr{F}_c T, \varphi \rangle$ for arbitrary $\varphi \in \mathscr{S}_e$, that is $\mathscr{F}_c T_j \to \mathscr{F}_c T$ in $\mathscr{S}_e'$. Similarly for $\mathscr{F}_s T$.

At this stage we have established the fact that $\mathscr{S}_e'$ and $\mathscr{S}_0'$ both contain $\mathscr{S}'$ but have not indicated whether or not the containment is proper. The following examples clarify the relationships of the three spaces. Define the expression $O(f)$ by

$$\langle O(f), \varphi \rangle = \lim_{X \to \infty} \int_{-X}^{X} f(x)\varphi(x)dx ,$$

for functions f and φ defined on the whole real line. Also, let g be given by

$$g(x) = \begin{cases} e^{x^2} & |x| \geqq a \\ h(x) & |x| < a \end{cases}$$

where h is any function integrable over the interval $(-a, a)$. Then $O(g)$ is not an element of $\mathscr{S}'$, as can be seen by the fact that it is not defined for the test function given by $\varphi(x) = e^{-x^2}$. This same test function is also an element of $\mathscr{S}_e$, which indicates that $O(g)$ is not in $\mathscr{S}_e'$ either. But given an arbitrary element φ of $\mathscr{S}_0$,

$$\begin{aligned} \langle O(g), \varphi \rangle &= \lim_{X \to \infty} \int_{-X}^{X} g(x)\varphi(x)dx \\ &= \lim_{X \to \infty} \left[\int_{-X}^{-a} g(x)\varphi(x)dx + \int_{-a}^{a} g(x)\varphi(x)dx + \int_{a}^{X} g(x)\varphi(x)dx \right] \\ &= \int_{-a}^{a} g(x)\varphi(x)dx , \end{aligned}$$

which exists since g is integrable over $(-a, a)$ and φ is continuous. This functional is clearly linear. To show continuity, consider $\varphi_j \to 0$ in $\mathscr{S}_0$. Then $|\langle O(g), \varphi_j \rangle| = \left| \int_{-a}^{a} g(x)\varphi_j(x)\,dx \right| \leqq \max_{x \in [-a,a]} |\varphi_j(x)| \cdot \int_{-a}^{a} g(x)dx$ which tends to 0 as j tends to infinity. Thus $O(g)$ is a continuous linear functional on $\mathscr{S}_0$ and hence an element of $\mathscr{S}_0'$. A similar argument with $0(g)$ where g is given by

$$g(x) = \begin{cases} xe^{x^2} & |x| \geqq a \\ h(x) & |x| < a \end{cases}$$

with h any function integrable over $(-a, a)$ shows that there is at least one element of $\mathscr{S}_e'$ which is neither in $\mathscr{S}'$ nor $\mathscr{S}_0'$ and thus completes the argument that no two of the three spaces $\mathscr{S}'$, $\mathscr{S}_e'$, and $\mathscr{S}_0'$ are equal. Thus $\mathscr{S}_e'$ and $\mathscr{S}_0'$ are in fact bigger spaces than $\mathscr{S}'$.

Since $\mathscr{S}_e'$ and $\mathscr{S}_0'$ both properly contain $\mathscr{S}'$ it is trivial that $\mathscr{S}_e' \cap \mathscr{S}_0' \supseteqq \mathscr{S}'$. But it is interesting to note that $\mathscr{S}_e' \cap \mathscr{S}_0'$ is exactly $\mathscr{S}'$. To show this we first need a lemma.

LEMMA 4.6. *Let φ_j be a sequence of elements of $\mathscr{S}$ converging to 0 as $j \to \infty$. Writing each φ_j as $\varphi_{j_e} + \varphi_{j_0}$, where $\varphi_{j_e} + \varphi_{j_0}$ is the usual decomposition of a function into even and odd parts, $\varphi_j \to 0$ in $\mathscr{S}$ implies $\varphi_{j_e} \to 0$ in $\mathscr{S}_e$ and $\varphi_{j_0} \to 0$ in $\mathscr{S}_0$.*

The proof follows from the fact that $\varphi_e(x) = (1/2)[\varphi(x) + \varphi(-x)]$ and $\varphi_0(x) = (1/2)[\varphi(x) - \varphi(-x)]$.

THEOREM 4.7. $\mathscr{S}_e' \cap \mathscr{S}_0' = \mathscr{S}'$.

Proof. As mentioned above $\mathscr{S}_e' \cap \mathscr{S}_0' \supseteqq \mathscr{S}'$. To show $\mathscr{S}' \supseteqq \mathscr{S}_e' \cap \mathscr{S}_0'$ consider an element T of $\mathscr{S}_e' \cap \mathscr{S}_0'$. For an arbitrary element φ of $\mathscr{S}$,

$$\langle T, \varphi\rangle = \langle T, \varphi_e\rangle + \langle T, \varphi_0\rangle$$

and each term on the right hand side of this equality is well defined, hence T is a linear functional on $\mathscr{S}'$. To prove continuity consider a sequence of test functions φ_j in $\mathscr{S}$ such that $\varphi_j \to 0$. Each $\varphi_j = \varphi_{j_e} + \varphi_{j_0}$ and by Lemma 4.6 $\varphi_j \to 0$ as $j \to \infty$ implies $\varphi_{j_e} \to 0$ and $\varphi_{j_0} \to 0$ in their respective function spaces as $j \to \infty$. Thus

$$\langle T, \varphi_j\rangle = \langle T, \varphi_{j_e}\rangle + \langle T, \varphi_{j_0}\rangle \to 0 \quad \text{as} \quad j \to \infty$$

and T is a continuous linear functional on $\mathscr{S}$.

5. Applications. In this section we obtain Abelian theorems for the distributional Fourier sine and cosine transforms defined above. As mentioned in the introduction, these results generalize known Abelian theorem for the Fourier sine and cosine transforms of functions to the distributional setting. These results can then be combined to yield an alternate approach to Abelian theorems about the Fourier transform in $\mathscr{S}'$. Thus we are not only working with larger spaces than $\mathscr{S}'$, but can combine results of the same type to get results about $\mathscr{S}'$.

The following theorem is a necessary preliminary to investigating the sine and cosine transforms. It is a direct analogue of a result for the Fourier transform and can be proved by a very slight modification of that proof.

THEOREM 5.1. *The Fourier cosine transform of a distribution* $T \in \mathscr{E}'$ *is (the distribution defined by) the function*

$$v(\xi) = (2\pi)^{-1/2}\langle T_x, \cos x\xi\rangle \ . \tag{5.1}$$

The right hand side is also defined for every complex number ξ *and is an entire function of* ξ. *Similarly the Fourier sine transform of* T *is*

$$w(\xi) = (2\pi)^{-1/2}\langle T_x, \sin x\xi\rangle \ , \tag{5.2}$$

which is also entire.

Proof. Modifying the proof by Schwartz [3, pp. 189–190] yields the result as follows. Since $\sin x\xi \in \mathscr{E}_x$ and $\mathscr{E}_\xi$, equation 5.2 makes sense. Also $w(\xi)$ is an infinitely differentiable function of ξ [3, Chapter III, Theorem 1]. Both statements are true if ξ is complex, hence $w(\xi)$ is entire. Now, if $\varphi \in [\mathscr{S}_0]_\xi$

$$\begin{aligned}\langle \mathscr{F}_s T_x, \varphi\rangle &= \langle T_x, \mathscr{F}_s\varphi\rangle = \left\langle T_x, (2\pi)^{-1/2}\int_{-\infty}^{\infty}\varphi(\xi)\sin x\xi d\xi\right\rangle \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\langle T_x, \sin x\xi\varphi(\xi)\rangle d\xi \quad [3,\ \mathrm{III},\ \mathrm{I};\ 8] \\ &= \int_{-\infty}^{\infty} w(\xi)\varphi(\xi)d\xi = \langle T_{w(\xi)}, \varphi(\xi)\rangle \ ,\end{aligned}$$

hence $\mathscr{F}_s T = w$. Similarly for $\mathscr{F}_c T$.

THEOREM 5.2. *Let* $T \in \mathscr{S}_e'$ *with support in* $(-\infty, \infty)$ *such that* T *equals the distribution corresponding to a function* f *over* $[a, \infty)$ *and corresponding to a function* g *over* $(-\infty, b]$ *for real numbers* $b < a$. *If* $f \in L^1(a, \infty)$ *and* $f(x) = x^{-\alpha}p(x)$ *over* $[c, \infty)$, *and if* $g \in L^1(-\infty, b)$ *and* $g(x) = (-x)^{-\beta}q(-x)$ *over* $(-\infty, d]$, *where* $c > a$, $c > 0$, $d < b$, $d < 0$, $0 < \alpha < 1$, $p \in BV\ [c, \infty)$, *and* $q \in BV\ [-d, \infty)$, *then* $\hat{T}_c$ *is a regular even tempered distribution defined by a function* Φ *such that*

$$\begin{aligned}\Phi \sim\ & (2\pi)^{-1/2}p(\infty)\Gamma(1-\alpha)\sin(\pi\alpha/2)|x|^{\alpha-1} \\ &+ (2\pi)^{-1/2}q(\infty)\Gamma(1-\beta)\sin(\pi\beta/2)|x|^{\beta-1} \qquad (x \longrightarrow \pm 0) \ .\end{aligned}$$

Also, if $T \in \mathscr{S}_0'$ *and satisfies the above conditions, then* $\hat{T}_s$ *is a regular* *odd tempered distribution defined by a function Φ and*

$$\Phi(x) \sim \operatorname{sgn}(x)(2\pi)^{-1/2}p(\infty)\Gamma(1-\alpha)\cos(\pi\alpha/2)|x|^{\alpha-1} + \operatorname{sgn}(x)(2\pi)^{-1/2}q(\infty)\Gamma(1-\beta)\cos(\pi\beta/2)|x|^{\beta-1} \qquad (x \longrightarrow \pm 0)\ .$$

Proof. For the first part of the theorem T can be written as $T = T_g + S + T_f$ where the supports of T_g, S, T_f are $(-\infty, b-\varepsilon]$, $[b-\varepsilon, a+\delta]$, and $[a+\delta, \infty)$, respectively, for positive δ and ε. S has compact support, hence $\hat{S}_c$ is regular defined by $u(x) = (2\pi)^{-1/2}\langle T_t, \cos xt\rangle$ as in Theorem 5.1.

Then for T_f and T_g we have $\hat{T}_{f_c} = T_{\hat{f}_c}$ and $\hat{T}_{g_c} = T_{\hat{g}_c}$. But

$$\begin{aligned}\hat{f}_c(x) &= (2\pi)^{-1/2}\int_{a+\delta}^{c} f(t)\cos xt dt + (2\pi)^{-1/2}\int_c^\infty f(t)\cos xt dt \\ &= v(x) + (2\pi)^{-1/2}\int_c^\infty f(t)\cos xt dt\ ,\end{aligned}$$

and

$$\begin{aligned}\hat{g}_c(x) &= (2\pi)^{-1/2}\int_{-\infty}^{d} g(t)\cos xt dt + (2\pi)^{-1/2}\int_d^{b-\varepsilon} g(t)\cos xt dt \\ &= (2\pi)^{-1/2}\int_{-\infty}^{d} g(t)\cos xt dt + w(x)\ ,\end{aligned}$$

where v and w are functions bounded above and below. Also,

$$\mathscr{F}_c[T] = \mathscr{F}_c[T_g + S + T_f] = \hat{T}_{g_c} + \hat{S}_c + \hat{T}_{f_c} = T_{\hat{g}_c} + T_u + T_{\hat{f}_c} = T_\Phi$$

where

$$\Phi(x) = u(x) + v(x) + w(x) + (2\pi)^{-1/2}\int_{-\infty}^{d} g(t)\cos xt dt + (2\pi)^{-1/2}\int_c^\infty f(t)\cos xt dt\ .$$

The first three terms of Φ are bounded as $x \rightarrow \pm 0$, and the behavior of the last two is given by Theorem 3.4 with the function g appearing there being 0. The behavior of the last term is a direct application of the theorem and the behavior of the fourth term is obtained by applying the theorem to $g(-x)$. The proof of the second part of the theorem is similar.

In a result such as this, one of the terms in the sum will dominate, depending upon the relative sizes of the powers of $|x|$.

We can now combine these results to yield an alternate proof to a result previously obtained for the Fourier transform [2, Theorem 3.4, p. 167].

COROLLARY 5.3. *Let $T \in \mathscr{S}'$ satisfy the conditions of Theorem*

5.2. *Then $\hat{T}$ is a regular distribution defined by a function Φ and*

$$\Phi \sim (2\pi)^{-1/2} q(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(1-\beta)/2}|x|^{\beta-1} + (2\pi)^{-1/2} p(\infty)\Gamma(1-\alpha)e^{\pi i \operatorname{sgn}(x)(\alpha-1)/2}|x|^{\alpha-1} \qquad (x \longrightarrow \pm 0) .$$

Proof. For any $\varphi \in \mathscr{S}$,

$$\begin{aligned} \langle \mathscr{F} T, \varphi\rangle &= \langle \mathscr{F} T, \varphi_e + \varphi_0\rangle = \langle \mathscr{F} T, \varphi_e\rangle + \langle \mathscr{F} T, \varphi_0\rangle \\ &= \langle T, \mathscr{F}\varphi_e\rangle + \langle T, \mathscr{F}\varphi_0\rangle = \langle T, \mathscr{F}_c\varphi_e\rangle + \langle T, -i\mathscr{F}_s\varphi_0\rangle \end{aligned}$$

by Theorem 3.3. But this equals

$$\langle \mathscr{F}_c T, \varphi_e\rangle - i\langle \mathscr{F}_s T, \varphi_0\rangle \tag{5.3}$$

since $T \in \mathscr{S}' = \mathscr{S}_0' \cap \mathscr{S}_e'$. Then by Theorem 5.2 the distributions $\mathscr{F}_c T$ and $\mathscr{F}_s T$ are regular and correspond to functions, say X and Ψ, having the given asymptotic behavior. But expression 5.3 equals

$$\langle T_X + T_\Psi, \varphi_e + \varphi_0\rangle - \langle \mathscr{F}_c T, \varphi_0\rangle - \langle \mathscr{F}_s T, \varphi_e\rangle = \langle T_{X+\Psi}, \varphi\rangle$$

since

$$\langle \mathscr{F}_c T, \varphi_0\rangle = \langle \mathscr{F}_s T, \varphi_e\rangle = 0 .$$

Thus if $\Phi = X + \Psi$, then by Theorem 5.2 Φ behaves as stated.

For a discussion of Abelian theorems of this type with $x \to \pm\infty$, see [2, p. 167].

References

1. S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Annals of Mathematical Studies, No. 19, Princeton University Press, Princeton, N. J., 1949.
2. E. O. Milton, *Asymptotic behavior of transforms of distributions*, Trans. Amer. Math. Soc., **172** (1972), 161-176.
3. L. Schwartz, *Mathematics for the Physical Sciences*, rev. ed., Addison-Wesley Publishing Company, Reading, Mass., 1966.
4. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, London and New York, 1948.

Received October 16, 1972.

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