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A CLASS OF ABELIAN GROUPS CLOSED UNDER DIRECT LIMITS AND SUBGROUPS FORMATION

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This is a solution to the following problem: Which classes of Abelian groups are closed under taking subgroups and direct limits. (Problem 6(a), L. Fuchs, Infinite Abelian Groups, $I, A - P$.) Each such class is uniquely determined by its subclass of finitely generated Abelian groups, which in turn can be described by a set of numerical invariants.

In like manner, an analogue for modules over a Dedekind domain is also obtained.

Throughout this paper, we shall make no distinction between isomorphic copies of the same Abelian group. Thus when we say that an Abelian group belongs to a class we mean that the former is isomorphic to a member of the latter.

In many places, proofs of our results are omitted. This is because our discussion is of a constructive nature. As soon as a construction is carried out, the fact that it has the desired property becomes self-evident. Thus a formal proof is not necessary.

2. **A reduction of the problem.** Suppose Γ is a class of Abelian groups satisfying:

(I) $A \in \Gamma$, and B is a subgroup of A, implies $B \in \Gamma$.

(II) If $\{A_\alpha, \pi_{\alpha\beta}\}$ is a direct system with $A_\alpha \in \Gamma$, then $(\varinjlim A_\alpha) \in \Gamma$.

Define Γ_0 to be the subclass of finitely generated Abelian groups in Γ. Suppose Φ is a class of finitely generated Abelian groups satisfying (I), define $\hat{\Phi}$ as follows: $A \in \hat{\Phi}$ if there is a direct system $\{A_\alpha, \pi_{\alpha\beta}\}$ s.t.

$$A_\alpha \in \Phi$$
$$\pi_{\alpha\beta} \text{ are monomorphisms}$$
$$\varinjlim A_\alpha = A$$

(i.e., an Abelian group $A \in \hat{\Phi}$ if it is the union of a directed (by inclusions) family of finitely generated subgroups each of which belongs to Φ).

LEMMA 2.1. *If $\{A_\alpha, \pi_{\alpha\beta}\}$ is a direct system* s.t. $A_\alpha \in \Phi$, *then* $(\varinjlim A_\alpha) \in \hat{\Phi}$.

Proof. For a fixed α, among the subgroups $(\operatorname{Ker} \pi_{\alpha\beta})$ of A_α, $(\beta \geqq \alpha)$, there is a maximum one, (because A_α is finitely generated). Therefore, there is $\alpha' \geqq \alpha$ s.t. $(\operatorname{Ker} \pi_{\alpha\alpha'}) = (\operatorname{Ker} \pi_{\alpha\beta}) \forall \beta \geqq \alpha$. Let $A'_\alpha = A_\alpha/(\operatorname{Ker} \pi_{\alpha\alpha'})$, then for $\gamma \geqq \alpha$, $\pi_{\alpha\gamma}: A_\alpha \to A_\gamma$ induces a monomorphism $\pi'_{\alpha\gamma}: A'_\alpha \to A'_\gamma$. Obviously $\{A'_\alpha, \pi'_{\alpha\gamma}\}$ is a direct system and $(\varinjlim A'_\alpha) = (\varinjlim A_\alpha)$. Therefore, $(\varinjlim A_\alpha) \in \hat{\Phi}$.

THEOREM 2.2. *If Γ is a class of Abelian groups satisfying* (I), (II), *then*

(i) Γ_0 *satisfies* (I),

(ii) $(\hat{\Gamma}_0) = \Gamma$.

If Φ *is a class of finitely generated Abelian groups satisfying* (I), *then*

(iii) $\hat{\Phi}$ *satisfies* (I), (II),

(iv) $(\hat{\Phi})_0 = \Phi$.

Proof. (i) is obvious.

(ii) If $A \in \Gamma$, let $\{A_\alpha\}$ be the family of all finitely generated subgroups of A, then each $A_\alpha \in \Gamma_0$. Since $A = \bigcup A_\alpha$, $A \in (\hat{\Gamma}_0)$.

(iii) $\hat{\Phi}$ satisfies (I): Suppose $A \in \hat{\Phi}$, and B is a subgroup of A. We have a direct family $\{A_\alpha\}$ of subgroups of A s.t.

$$A_\alpha \in \Phi ,$$
$$A = \bigcup A_\alpha .$$

Let $B_\alpha = B \cap A_\alpha$, then

$$B_\alpha \in \Phi ,$$
$$B = \bigcup B_\alpha .$$

Hence $B \in \hat{\Phi}$.

$\hat{\Phi}$ satisfies (II): Suppose we have a direct system $\{A_\alpha, \pi_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ s.t.

$$A_\alpha \in \hat{\Phi} ,$$
$$A = \varinjlim A_\alpha .$$

For each α, we have a directed family $\{A_{\alpha,a}\}_{a\in\Lambda_\alpha}$ of finitely generated subgroups of A_α s.t.

$$A_{\alpha,a} \in \Phi ,$$
$$A_\alpha = \bigcup A_{\alpha,a} .$$

Let $\hat{\Lambda} = \{(\alpha, a) \mid a \in \Lambda_\alpha\}$ and define $(\alpha, a) \leqq (\beta, b)$ if

$$\alpha \leqq \beta ,$$
$$\pi_{\alpha\beta}(A_{\alpha,a}) \subseteqq A_{\beta,b} .$$

We claim that $\tilde{\Lambda}$ is a directed set: Given $(\alpha, a), (\beta, b) \in \tilde{\Lambda}$, there is $\gamma \in \Lambda$ s.t. $\alpha, \beta \leqq \gamma$. Since $A_{\alpha,a}$ finitely generated and $A_\gamma = \bigcup A_{\gamma,c}$, there is $c' \in \Lambda_\gamma$ s.t. $\pi_{\alpha\gamma}(A_{\alpha,a}) \subseteqq A_{\gamma,c'}$. Likewise there is $c'' \in \Lambda_\gamma$ s.t. $\pi_{\beta\gamma}(A_{\beta,b}) \subseteqq A_{\gamma,c''}$. Choose $c \in \Lambda_\gamma$ s.t. $c', c'' \leqq c$, then $(\alpha, a), (\beta, b) \leqq (\gamma, c)$.

For $(\alpha, a) \leqq (\beta, b)$, define

$$\pi_{(\alpha,a),(\beta,b)}: A_{\alpha,a} \longrightarrow A_{\beta,b}$$

by $\pi_{(\alpha,a),(\beta,b)}(x) = \pi_{\alpha\beta}(x) \forall x \in A_{\alpha,a}$, then

$$\{A_{\alpha,a}, \pi_{(\alpha,a),(\beta,b)}\}$$

is a direct system with

$$\varinjlim A_{\alpha,a} = A .$$

Therefore, $A \in \hat{\Phi}$.

(iv) If $A \in (\hat{\Phi})_0$, then there is a directed family $\{A_\alpha\}$ of subgroups of A s.t.

$$A_\alpha \in \Phi ,$$
$$A = \bigcup A_\alpha .$$

Since A is finitely generated, $A = A_\alpha$ for some α. Therefore $A \in \Phi$.

3. **A further reduction.** Let Σ be the class of all finitely generated Abelian groups. For $A, B \in \Sigma$, $A \leqq B$ means A is (isomorphic to) a subgroup of A. Clearly this is a partial ordering on Σ. A subclass Θ of Σ is called an ideal if it satisfies (I) and

(III) $A, B \in \Theta$ implies there is $C \in \Theta$ s.t. $A, B \leqq C$.

LEMMA 3.1. *The union of a directed (by inclusions) family of ideals is an ideal. In particular the union of a linearly ordered family of ideals is an ideal.*

The proof is obvious.

Suppose Φ is a class of finitely generated Abelian groups satisfying (I), we use Φ_m to denote the family of all maximum ideals in Φ, (i.e., those which are maximum among ideals contained in Φ). Suppose Δ is a family of ideals. We say that Δ is irredundant if none of its members is contained in another member, (i.e., each member is maximum in the family). We say that Δ is closed if given an ideal $\Theta' \subseteqq \bigcup_{\Theta \in \Delta} \Theta$ there is $\Theta'' \in \Delta$ s.t. $\Theta' \subseteqq \Theta''$. We define $\phi(\Delta) = \bigcup_{\Theta \in \Delta} \Theta$.

THOREM 3.2. *If Φ is a class of finitely generated Abelian groups satisfying* (I), *then*

(i) $\Phi = \Phi(\Phi_m)$,

(ii) Φ_m *is irredundant,*
(iii) Φ_m *is closed.*
If Δ *is a closed irredundant family of ideals, then*
(iv) $\Phi(\Delta)$ *satisfies* (I),
(v) $(\Phi(\Delta))_m = \Delta$.

Proof. (i) and (iii) are obtained through a routine use of Zorn's Lemma.

(ii) and (iv) are obvious.

(v) is a consequence of the definition of closeness.

THEOREM 3.3. *If* Δ *is a closed, irredundant family of ideals, and* $\Phi = \Phi(\Delta)$, *then*

$$\hat{\Phi} = \bigcup_{\Theta \in \Delta} \hat{\Theta} .$$

Proof. Clearly $\hat{\Phi} \supseteqq \bigcup_{\Theta \in \Delta} \hat{\Theta}$.

If $A \in \hat{\Phi}$, then there is a directed family $\{A_\alpha\}$ of finitely generated subgroups of A s.t.

$$A_\alpha \in \Phi ,$$
$$A = \bigcup A_\alpha .$$

Define Φ_A as follows: $B \in \Phi_A$ if $B \leqq A_\alpha$ for some α. Clearly Φ_A is an ideal contained in Φ. Since Δ is closed, there is $\Theta \in \Delta$ s.t. $\Phi_A \subseteqq \Theta$. Clearly $A \in \hat{\Theta}$.

Therefore, $\hat{\Phi} = \bigcup_{\Theta \in \Delta} \hat{\Theta}$.

4. **Arithmetization.** Let

$\mathscr{P}$ = the set consisting of 0 and all positive integral powers of every prime number,

$\mathscr{N}$ = the set of all nonnegative integers,

$\hat{\mathscr{N}} = \mathscr{N} \cup \{\infty\}$,

Ω = the set of all mappings $\mu: \mathscr{P} \to \hat{\mathscr{N}}$ satisfying the condition: $\mu(p^m) \geqq \mu(p^n)$ whenever $m \leqq n$, ($\hat{\mathscr{N}}$ is ordered in the obvious manner),

Ω_0 = the set of all $\mu \in \Omega$ satisfying the conditions: (i) $\mu(x) \neq \infty \forall x \in \mathscr{P}$; (ii) $\mu(x) = 0$ for almost all $x \in \mathscr{P}$.

For $\lambda, \mu \in \Omega$, $\lambda \leqq \mu$ means $\lambda(x) \leqq \mu(x) \forall x \in \mathscr{P}$. Obviously this is a partial ordering on Ω. Define $\chi: \Sigma \to \Omega_0$ as follows: For $A \in \Sigma$, $A = A_1 \oplus \cdots \oplus A_s$, where each A_j is isomorphic with Z/xZ for some $x \in \mathscr{P}$. Set

$$\chi(A)(x) = \text{the number of } A_j \cong Z/xZ .$$

Clearly χ is an order isomorphism. Suppose that Θ is an ideal in Σ.

Define $\chi(\Theta) \in \Omega$ as follows:

$$\chi(\Theta)(x) = \text{l.u.b.}\,\{\chi(A)(x) \mid A \in \Phi\} .$$

For $\mu \in \Omega$, define

$$\Phi(\mu) = \{A \in \Sigma \mid \chi(A) \leqq \mu\} .$$

Suppose that Δ' is a subset of Ω. We say that Δ' is irredundant if every element of Δ' is maximum in Δ'. We say that Δ' is closed if the following condition is satisfied: Given $\omega \in \Omega$ with the property that for each $\mu \in \Omega_0$, s.t. $\mu \leqq \omega$, there is $\delta' \in \Delta'$ s.t. $\mu \leqq \delta'$, then there is $\delta \in \Delta'$ s.t. $\omega \leqq \delta$.

LEMMA 4.1. *If Θ is an ideal in Σ and $A \in \Sigma$ s.t. $\chi(A) \leqq \chi(\Theta)$, then $A \in \Theta$.*

Proof. Since $A \in \Sigma$, there are $x_1, \cdots, x_s \in \mathscr{P}$ s.t. $\chi(A)(x) = 0$ except $x = x_1, \cdots, x_s$. Since $\chi(A) \leqq \chi(\Theta)$, for each x_j, there is $A_j \in \Theta$ s.t.

$$\chi(A)(x_j) \leqq \chi(A_j)(x_j) .$$

Since Θ is an ideal, there is $A' \in \Theta$ s.t.

$$A_1, A_2, \cdots \leqq A' .$$

Obviously $\chi(A) \leqq \chi(A')$, i.e., $A \leqq A'$, and hence $A \in \Theta$.

COROLLARY 4.2. *Given two ideals Θ, Θ', $\chi(\Theta) = \chi(\Theta')$ iff $\Theta = \Theta'$.*

LEMMA 4.3. *For $\mu \in \Omega$,*
(i) *$\Phi(\mu)$ is an ideal,*
(ii) *$\chi(\Phi(\mu)) = \mu$.*

Proof. (i) $\Phi(\mu)$ satisfies (I) is obvious.
$\Phi(\mu)$ satisfies (III): Suppose $A, B \in \Phi(\mu)$. Define $\lambda: \mathscr{P} \to \mathscr{N}$ by

$$\lambda(x) = \max\{\chi(A)(x), \chi(B)(x)\}$$

$\forall x \in \mathscr{P}$. Obviously $\lambda \in \Omega_0$, and $\lambda \leqq \mu$. Since χ is an isomorphism, there is $C \in \Sigma$ s.t. $\chi(C) = \lambda$. Obviously $A, B \leqq C$, and $C \in \Phi(\mu)$.

(ii) According to the definition, for $x \in \mathscr{P}$, $A \in \Phi(\mu)$, we have $\chi(A)(x) \leqq \mu(x)$. Therefore, $\chi(\Phi(\mu)) \leqq \mu$.

If $\chi(\Phi(\mu)) \neq \mu$, then there is $x \in \mathscr{P}$ s.t. $\chi(\Phi(\mu))(x) < \mu(x)$. Suppose $\chi(\Phi(\mu))(x) = n$. (We cannot have $\chi(\Phi(\mu))(x) = \infty$ because $\infty < \mu(x)$ does not hold.) Let $A = (Z/xZ) \oplus \cdots \oplus (Z/xZ)$, ($n+1$ copies), then $\chi(A)(x) = n + 1 \leqq \mu(x)$, $\chi(A)(z) = 0 \leqq \mu(z) \forall z \neq x$. Therefore, $A \in \Phi(\mu)$. This contradicts the assumption that $\chi(\Phi(\mu))(x) = n$. Hence $\chi(\Phi(\mu)) = \mu$.

THEOREM 4.4. *If Δ is a closed irredundant family of ideals in Σ, then $\{\chi(\Theta) \mid \Theta \in \Delta\}$ is*

(i) *closed,*

(ii) *irredundant.*

If Δ' is a closed irredundant subset of Ω, then $\{\Phi(\mu) \mid \mu \in \Delta'\}$ is

(iii) *closed,*

(iv) *irredundant.*

The proof is obvious.

Combining all the earlier results together we have

THEOREM 4.5. (i) *If Δ' is a closed, irredundant subset of Ω, then*

$$\bigcup_{\mu \in \Delta'} \widehat{\Phi(\mu)}$$

is a class of Abelian group satisfying (I) *and* (II).

(ii) *If Γ is a class of Abelian groups satisfying* (I) *and* (II), *then*

$$\Gamma = \bigcup_{\mu \in \Delta'} \widehat{\Phi(\mu)}$$

where $\Delta' = \{\chi(\Theta) \mid \Theta \in (\Gamma_0)_m\}$.

5. **An explicit construction.** We shall adopt the following notations:

$T(A)$, A is an Abelian group: The torsion part of A.

$T_0(A)$: $A/T(A)$.

$T_p(A)$, p is a prime number: The p-primary component of $T(A)$.

rank (A): the number of summands in the direct sum decomposition of the injective envelope of A into indecomposable subgroups.

THEOREM 5.1. *For $\mu \in \Omega$, $\widehat{\Phi(\mu)}$ consists of Abelian groups A subject to the following conditions:*

(i) *If $\mu(0) = \infty$, then $T_0(A)$ can be arbitrary.*

If $\mu(0) \neq \infty$, *then* rank $(T_0(A)) \leqq \mu(0)$.

(ii) *For each $p^k \in \mathscr{P}$, if $\mu(p^k) = \infty$, then $p^{k-1}T_p(A)$ can be arbitrary.*

If $\mu(p^k) \neq \infty$, then rank $(p^{k-1}T_p(A)) \leqq \mu(p^k)$.

Proof. This is obvious. (Observe that these conditions are preserved under taking subgroups $T_0(\ )$, $T_p(\ )$ and direct limits.)

REMARK 5.2. In view of Theorem 4.5, there can be classes of Abelian groups satisfying (I) and (II) whose structures are extremely complicated. However, almost all known examples take the simplest possible form, viz., they are $\widehat{\Phi(\mu)}$ for certain $\mu \in \Omega$, e.g.,

(1) The class of all (p-primary) co-cyclic groups is given by

$$\mu(x) = \begin{cases} 1, & x = p^k, k = 1, 2, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

(2) The class of all locally cyclic groups is givin by

$$\hat{\Phi}(\mu) \cup \hat{\Phi}(\nu),\ \nu(x) = \begin{cases} 1, & x = 0\,, \\ 0, & x \neq 0\,, \end{cases} \qquad \nu(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0\,. \end{cases}$$

(3) The class of all torsion groups is given by

$$\mu(x) = \begin{cases} 0, & x = 0\,, \\ \infty, & x = 0\,. \end{cases}$$

(4) The class of all torsion-free groups is given by

$$\mu(x) = \begin{cases} \infty, & x = 0\,, \\ 0, & x \neq 0\,. \end{cases}$$

(5) The class of all groups is given by

$$\mu(x) = \infty \quad \forall x \in \mathscr{P}\,.$$

(6) The class of all groups annihilated by n is given by

$$\mu(x) = \begin{cases} \infty, & x \mid n\,, \\ 0, & x \nmid n\,. \end{cases}$$

REMARK 5.3. $\widehat{\Phi(\mu)}$ generalizes the class of all subgroups in a given Abelian group. In fact, for a fixed cardinal number $\aleph$, the members of $\widehat{\Phi(\mu)}$ with cardinality $\leqq \aleph$ are exactly the subgroups in the direct sum constructed as follows:

(a) For each prime number p, if

$$\begin{aligned} &\mu(p) = \cdots = \mu(p^{k_1}) = \lambda_1 \\ &< \mu(p^{k_1+1}) = \cdots = \mu(p^{k_2}) = \lambda_2 \\ &\qquad \vdots \\ &< \mu(p^{k_r+1}) = \cdots = \lambda_{r+1} \end{aligned}$$

where $k_1 < k_2 < \cdots < k_r$, (including $r = 0$ to mean $\mu(p) = \cdots = \lambda_1$), we put in the following summands:

λ_{r+1} copies of $Z(p^\infty)$,

$(\lambda_j - \lambda_{j+1})$ copies of $Z/p^{k_j}Z$, $j = 2, \cdots, r$,

$$\begin{cases} \lambda_1 - \lambda_2 \text{ copies of } Z/p^{k_1}Z,\ (\lambda_1 < \infty), \text{ or} \\ \aleph \text{ copies } Z/p^{k_1}Z,\ (\lambda_1 = \infty)\,, \end{cases}$$

(in case $r = 0$, $\lambda_1 = \infty$ these are $\aleph$ copies of $Z(p^\infty)$).

(b) We also put in the following summands:

$$\begin{cases} \mu(0) \text{ copies of } Q, (\mu(0) < \infty), \text{ or} \\ \aleph \text{ copies of } Q, (\mu(0) = \infty) . \end{cases}$$

REMARK 5.4. One may wish that a class satisfying (I), (II) can be expressed either as a union of a finite number of $\widehat{\Phi(\mu)}$ or as a family of such which are mutually disjoined. The following example shows that this is not always possible.

Let $p_1, p_2, \cdots$ be a set of prime numbers and $\mu_1, \mu_2, \cdots \in \Omega$ be given by

$$\mu_i(p_j^k) = \begin{cases} 1, & j = i \quad \text{or} \quad k = 1 , \\ 0, & j \neq i, k > 1 . \end{cases}$$

Obviously $\Gamma = \bigcup \widehat{\Phi(\mu_i)}$ satisfies (I), (II), and cannot be decomposed into finite or disjoined union in the above mentioned way.

REMARK 5.5. The definition of closed irredundant subsets of Ω is closely related to the concept of closed sets in a topological space. For an irredundant subset Δ of Ω, let

$$\Delta' = \{\omega \in \Omega \mid \omega \leqq \delta \quad \text{for some} \quad \delta \in \Delta\} ,$$

then Δ is closed iff the least upper bound of every net in Δ' is still in Δ'.

6. **A generalization.** The results in the previous sections can be extended to modules over a Dedekind domain. (For basic properties of a Dedekind domain we refer to [2].) This is carried out in the following. Proofs are omitted because they are essentially the same as the case of Abelian groups. We adopt the following notations:

R: A Dedekind domain.

$\mathscr{P}$: The set of all primary ideals in R, (i.e., 0, and powers of nonzero prime ideals).

$\mathscr{N}$: The set of all nonnegative integers.

$\hat{\mathscr{N}}$: $\mathscr{N} \cup \{\infty\}$.

Ω: The set of all mappings $\mu: \mathscr{P} \to \mathscr{N}$ satisfying $\mu(x) \geqq \mu(y)$ whenever $x \mid y$, $x, y \neq 0$.

Ω_0: The set of $\mu \in \Omega$ satisfying: (i) $\mu(x) \neq \infty$, (ii) $\mu(x) = 0$ for almost all $x \in \mathscr{P}$.

$T(A)$, (A is an R-module): The torsion part of A.

$T_0(A)$: $A/T(A)$.

$T_p(A)$, (p is a prime ideal of R): The p-primary component of

$T(A)$.

rank (A): The number of summands in the direct sum decomposition of the injective envelope of A into indecomposable R-modules.

For $\lambda, \mu \in \Omega, \lambda \leqq \mu$ means that $\lambda(x) \leqq \mu(x) \forall x \in \mathscr{P}$. A subset Δ' of Ω is irredundant if every element of Δ' is maximum in Δ'. It is closed if the following condition is satisfied: Given $\omega \in \Omega$ with the property that for $\mu \in \Omega_0$, $\mu \leqq \omega$ implies there is $\delta' \in \Delta'$ s.t. $\mu \leqq \delta'$, then there is $\delta \in \Delta'$ s.t. $\omega \leqq \delta$.

For $\mu \in \Omega$, define $\widehat{\Phi(\mu)}$ as the class of all R-modules subject to the following conditions:

(i) If $\mu(0) = \infty$, $T_0(A)$ can be arbitrary.

If $\mu(0) \neq \infty$, rank $(T_0(A)) \leqq \mu(0)$.

(ii) If $\mu(p^k) = \infty$, (p is a prime ideal of R), $p^{k-1}T_p(A)$ can be arbitrary.

If $\mu(p^k) \neq \infty$, rank $(p^{k-1}T_p(A)) \leqq \mu(p^k)$.

THEOREM 6.1. (i) *If Δ' is a closed, irredundant subset of Ω, then*

$$\bigcup_{\mu \in \Delta'} (\widehat{\Phi(\mu)})$$

is a class of R-modules closed under direct limits and submodules formation.

(ii) *Every class of R-modules closed under direct limit and submodule formation can be obtained in this manner. Δ' is uniquely determined by the class.*

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