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ON A "SQUARE" FUNCTIONAL EQUATION

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In a recent paper Stanton and Cowan have generalized the Pascal's triangle to a tableau. They have developed several expressions for these numbers, using combinatorial techniques. In the present paper we derive some of their results very simply, by using the calculus of finite differences. We further obtain the relations of these numbers to hypergeometric function and derive many relations among these numbers which are useful in constructing the tableau.

1. Introduction. The triangular array of binomial coefficients, well-known as Pascal's triangle, has been much studied. Basically, it depends on the recursion relation

$$f(n+1, r) = f(n, r) + f(n, r-1) . \tag{1}$$

In a recent paper Stanton and Cowan [5], have considered a generalization of this situation by defining a tableau by the recurrence relation

$$g(n+1, r+1) = g(n, r+1) + g(n+1, r) + g(n, r) . \tag{2}$$

This formula, together with the boundary conditions, $g(n, 0) = g(0, r) = 1$, uniquely determines $g(n, r)$. The lower half of the first portion of this tableau is presented in Table 1, the upper half can be obtained by symmetry in n and r (see § 2).

TABLE 1

$g(n, r)$							
1							
1	3						
1	5	13					
1	7	25	63.				
1	9	41	129	321			
1	11	61	231	681	1683		
1	13	85	377	1289	3653	8989	
1	15	113	575	2241	7183	19825	48639

Stanton and Cowan [5], have developed several expressions for these numbers $g(n, r)$, and indicated that they have a further combinatorial interpretation, namely, $g(n, r)$ is the volume of an r-sphere in n-space under the Lee metric. In this paper we obtain some of the results of Stanton and Cowan by using the calculus of finite differences and obtain some additional properties of the function $g(n, r)$.

2. Main results. In order to obtain an explicit formula for $g(n, r)$ the calculus of finite differences, [4], is efficacious. Let E denote the shift operator such that $EU(r) = U(r+1)$, and Δ denote the difference operator defined by $\Delta U(r) = U(r+1) - U(r) = (E-1)U(r)$. We use the convention, [1], that the binomial coefficient $\binom{n}{r}$ is difined only for r an integer, and that $\binom{n}{r}$ vanishes for $r < 0$, and for $r > n$.

With this notation we prove the following results.

LEMMA 1.

$$g(n, r) = \sum_\alpha \binom{n}{\alpha}\binom{r+\alpha}{n} = \sum_\alpha \binom{r+\alpha}{\alpha}\binom{r}{n-\alpha} . \tag{3}$$

Proof. From (2), we get

$$\Delta g(n+1, r) = (E+1) g(n, r) , \tag{4}$$

which yields,

$$\begin{aligned} g(n, r) &= (E+1)^n \Delta^{-n} g(0, r) = \sum_\alpha \binom{n}{\alpha} E^\alpha \Delta^{-n} g(0, r) \\ &= \sum_\alpha \binom{n}{\alpha} E^\alpha \binom{r}{n} , \quad \text{since} \quad \Delta^j \binom{r}{j} = 1 \ (\text{see } (3)) , \\ &= \sum_\alpha \binom{n}{\alpha}\binom{r+\alpha}{n} = \sum_\alpha \binom{r+\alpha}{\alpha}\binom{r}{n-\alpha} . \end{aligned}$$

LEMMA 2.

$$g(n, r) = \sum_\alpha 2^\alpha \binom{n}{\alpha}\binom{r}{\alpha} . \tag{5}$$

Proof. From (4), we have

$$g(n+1, r) = \left(\frac{E+1}{E-1}\right) g(n, r) ,$$

which can be written as,

$$g(n, r) = (1 + 2\Delta^{-1})^n g(0, r)$$
$$= \sum_\alpha \binom{n}{\alpha} 2^\alpha \Delta^{-\alpha} g(0, r) = \sum_{\alpha} 2^\alpha \binom{n}{\alpha}\binom{r}{\alpha} .$$

It is obvious that $g(n, r)$ is symmetric in n and r. The ffective upper limit of the summations in (5) is at $K = \min. (n, r)$.

In view of the fact that the numbers $g(n, r)$ possess some applications, in addition to their intrinsic interest, it may be of interest to discuss some additional properties, and their relation to some well-known functions.

LEMMA 3.

$$g(n, r + s) = \sum_k g(k, r)\{g(n - k, s) - g(n - k - 1, s)\} . \tag{6}$$

Proof. It is known, [5], that $g(n, r)$ is the coefficient of x^r in the expansion of $(1 + x)^n/(1 - x)^{n+1}$. Or, since $g(n, r) = g(r, n)$, we might use the coefficient of x^n in $(1 + x)^r/(1 - x)^{r+1}$. Let $f_r(x) = (1 + x)^r/(1 - x)^{r+1}$. Then it is easy to verify that

$$(1 - x)f_r(x)f_s(x) = f_{r+s}(x) . \tag{7}$$

Now equate the coefficients of x^n in the expansion of (7) and we get the desired result.

If we consider the identity (7) as

$$f_r(x)f_s(x) = (1 - x)^{-1}f_{r+s}(x) \tag{8}$$

and equate the coefficients of x^n, we get

$$\sum_{k=0}^{n} g(k, r + s) = \sum_{k=0}^{n} g(k, r)g(n - k, s) .$$

In the following lemma we prove that $g(n, r)$ is a special case of the hypergeometric function ${}_2F_1$ defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{x^k}{k!} ,$$

where,

$$(a)_k = a(a + 1) \cdots (a + k - 1) .$$

LEMMA 4.

$$g(n, r) = {}_2F_1(-n, -r; 1; 2) . \tag{9}$$

Proof.

$$g(n, r) = \sum_\alpha 2^\alpha \binom{n}{\alpha}\binom{r}{\alpha} = \sum_\alpha \frac{2^\alpha n!\, r!}{\alpha!\, \alpha!\, (n-\alpha)!\, (r-\alpha)!}$$
$$= \sum_\alpha \frac{2^\alpha}{\alpha!} \frac{(-n)_\alpha(-r)_\alpha}{(1)_\alpha} = {}_2F_1(-n, -r; 1; 2) .$$

Note that the diagonal sums become, $g(n, n) = {}_2F_1(-n, -n; 1; 2)$. In the case of binomial coefficients such sums provide the Fibonacci numbers.

LEMMA 5.

$$g(n, r) = \frac{1}{n-r}\{ng(n-1, r) - rg(n, r-1)\} , \qquad n \neq r . \tag{10}$$

Proof. We have,

$$ng(n-1, r) - rg(n, r-1) = n \sum_\alpha 2^\alpha \binom{n-1}{\alpha}\binom{r}{\alpha} - r \sum_\alpha 2^\alpha \binom{n}{\alpha}\binom{r-1}{\alpha}$$
$$= \sum_\alpha 2^\alpha \binom{n}{\alpha}\binom{r}{\alpha}(n-r) = (n-r)g(n, r) .$$

For actual computations of $g(n, r)$ the result (10) is as easy as the basic definition (2).

The result of Lemma 5, can also be obtained from the corresponding identity for hypergeometric functions, $(b-a)\,{}_2F_1(a, b; c; x) + a\,{}_2F_1(a+1, b; c; x) - b\,{}_2F_1(a, b+1; c; x) = 0$ where now $a = -n$, $b = -r$, $c = -1$, and $x = 2$.

LEMMA 6.

$$g(n, r) = \frac{2r+1}{n} g(n-1, r) + \frac{n-1}{n} g(n-2, r) \tag{11}$$

$$= \frac{2n+1}{r} g(n, r-1) + \frac{r-1}{r} \cdot g(n, r-2) . \tag{12}$$

Proof. The result (11) follows from the following identity for hypergeometric functions,

$$(c-a)_2F_1(a-1, b; c; x) + (2a - c - ax + bx)\,{}_2F_1(a, b; c; x)$$
$$+ a(x-1)\,{}_2F_1(a+1, b; c; x) = 0$$

where now $a = -n$, $b = -r$, $c = 1$, and $x = 2$. Similarly the result (12) can be obtained from the symmetry or from the corresponding identity for hypergeometric functions, viz,

$$(c-b)_2F_1(a, b-1; c; x) + (2b - c - bx + ax)\,{}_2F_1(a, b; c; x)$$
$$+ b(x-1)\,{}_2F_1(a, b+1; c; x) = 0 .$$

Lemma 6 is useful to compute the numbers $g(n, r)$ in any one column or row of the tableau and for this purpose has the advantage over the formulae (2) or (10) of Lemma 5.

Some more relations between the numbers $g(n, r)$, which can be easily obtained from the corresponding identities for hypergeometric functions, are the following.

$$(n + r + 1)g(n, r) + ng(n - 1, r) - (r + 1)g(n, r + 1) = 0 ,$$

or equivalently,

$$(n + r + 1)g(n, r) + rg(n, r - 1) - (n + 1)g(n + 1, r) = 0 ,$$

and,

$$(n + 1)g(n + 1, r) - (r - n)g(n, r) - (r + 1)g(n, r + 1) = 0 .$$

3. Additional remarks.

It may be noted that the combinatorial interpretation of $g(n, r)$; the value $\sum_\alpha 2^\alpha \binom{n}{\alpha}\binom{r}{\alpha}$, is the volume (i.e., the number of lattice points in Euclidean n-space within a regular cross-polytope) of a sphere of radius r in n-dimensions (or a sphere of radius n in r-dimensions) using the Lee metric as deduced by Golomb and Welch [3]. Golomb [2], has also derived the generating function for $g(n, r)$,

$$\frac{1}{1 - x - y - xy} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} g(n, r)x^n y^r$$

which may be used to evaluate $g(n, r)$ either explicitly or asymptotically and is simpler than our results. Applications of these numbers in sphere packing, coding metrics, and chess puzzles are also described by Golomb [2]. Thus the numbers $g(n, r)$ have many applications in addition to their intrinsic interest. The relationship with the hypergeometric functions further illustrates their usefulness.

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