GLOBAL PROPERTIES OF RATIONAL AND LOGARITHMICO-RATIONAL MINIMAL SURFACES

E. F. BECKENBACH, F. H. ENG, AND R. E. TAFEL

In this paper, rational and logarithmico-rational minimal surfaces are defined and some of their properties are investigated. In particular, it is shown—perhaps somewhat surprisingly—that the fundamental theorem of algebra, suitably formulated, holds for these surfaces.

1. **Introduction.** Let $f(w)$ be a rational complex function, of degree m, of the complex variable w; that is, let $f(w)$ admit a representation of the form

$$f(w) = \frac{p(w)}{q(w)} ,$$

where $p(w)$ and $q(w)$ are relatively prime complex polynomial functions:

$$[p(w), q(w)] = 1 ;$$

and let

$$\deg f(w) = \max [\deg p(w), \deg q(w)] = m .$$

It is well known (see, for instance, [1, p. 31]) that, with multiple values suitably counted, for any such rational function of degree m, $m \geq 1$, the equation $\zeta = f(w)$ maps the closed single-sheeted w-plane onto the closed m-sheeted ζ-plane. That is, each ζ-value, including ∞, is taken on exactly m times (counting multiplicities) as w ranges once over the closed complex plane. This is a rather immediate consequence of the fundamental theorem of algebra.

Now for $n \geq 3$, a (two-dimensional) minimal surface S in n-dimensional Euclidean space E_n has n-dimensional measure 0, and accordingly the foregoing result concerning complex rational functions $f(w)$ could hardly be expected to extend directly to minimal-surface theory. It is nevertheless one purpose of this paper to show (see §10) that the result, *when suitably formulated*, does indeed extend precisely to minimal surfaces.

Since, as we shall see in §4, the map $\zeta = f(w)$ given by a rational complex function $f(w)$ is a special rational minimal surface, it therefore follows in particular that the fundamental theorem of algebra concerning *plane* maps can be given a formulation relative to the points of any containing n-dimensional Euclidean space!

2. Meromorphic minimal surfaces. A surface S in n-dimensional Euclidean space E_n, $n \geqq 2$, is said to be given in terms of *isothermal parameters* u, v provided the representation

$$S: x_j = x_j(u, v) , \qquad\qquad j = 1, 2, \cdots, n ,$$

or

$$(1) \qquad\qquad S: \boldsymbol{x} = \boldsymbol{x}(u, v) ,$$

for $(u, v) \in D$, where D is some finite domain of definition, is such that

$$(2) \qquad\qquad E = G = \lambda(u, v) , \quad F = 0 ,$$

where

$$E = \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial u}, \ F = \frac{\partial \boldsymbol{x}}{\partial u} \cdot \frac{\partial \boldsymbol{x}}{\partial v}, \ G = \frac{\partial \boldsymbol{x}}{\partial v} \cdot \frac{\partial \boldsymbol{x}}{\partial v} .$$

Such an isothermal representation is conformal, or angle-preserving, except at points where $\lambda(u, v) = 0$.

According to a theorem of Weierstrass [16, p. 27], a necessary and sufficient condition that the surface (1), given in terms of isothermal parameters, be a minimal surface is that the vector function $\boldsymbol{x}(u, v)$ be harmonic:

$$(3) \qquad\qquad \varDelta \boldsymbol{x}(u, v) \equiv \frac{\partial^2 \boldsymbol{x}}{\partial u^2} + \frac{\partial^2 \boldsymbol{x}}{\partial v^2} = 0 .$$

Then in any simply connected part of D, the functions $x_j(u, v)$ are the real parts of analytic functions of a complex variable,

$$x_j(u, v) = \Re f_j(w) , \qquad\qquad w = u + iv ,$$

and (2) is equivalent to

$$(4) \qquad\qquad \sum_{j=1}^{n} \left[\frac{df_j(w)}{dw} \right]^2 = 0 .$$

If an isothermal representation (1) of the minimal surface S is such that all the coordinate functions except two are identically zero, say $x_j(u, v) \equiv 0$, $j = 3, 4, \cdots, n$, then either $x_1(u, v) + ix_2(u, v)$ or $x_2(u, v) + ix_1(u, v)$ is an analytic function of the complex variable $w = u + iv$, and $x_1(u, v)$ and $x_2(u, v)$ are said to be a *couple of conjugate harmonic functions*. By analogy, the coordinate functions of any minimal surface S in E_n, given in isothermal representation, are called an *n-tuple of conjugate harmonic functions* [8].

For a given (u_0, v_0), let

$$u - u_0 = r \cos \theta , \quad v - v_0 = r \sin \theta ,$$

and suppose that for some R_1, R_2 the minimal surface S is given in isothermal representation by (1) in the annulus

$$(5) \qquad R_1 < r < R_2 .$$

Since the vector function $\boldsymbol{x}(u, v)$ is harmonic in (5), it can be represented [14, p. 692] there by a series of the form

$$(6) \qquad \boldsymbol{x}(u, v) = \boldsymbol{c} \log r + \sum_{k=-\infty}^{\infty} r^k (\boldsymbol{a}_k \cos k\theta + \boldsymbol{b}_k \sin k\theta) .$$

The constant vector $\boldsymbol{b}_0$ is arbitrary; we shall henceforth assume that

$$(7) \qquad \boldsymbol{b}_0 = \boldsymbol{0} .$$

The other coefficients, including the *logarithmic coefficient* $\boldsymbol{c}$, are uniquely determined by $\boldsymbol{x}(u, v)$.

The harmonic vector function $\boldsymbol{x}(u, v)$ given by (6) also satisfies the condition (2), since S is given in isothermal representation. For the representation (6), the condition (2) are equivalent [4, 5, 6, 7] to

$$(8) \quad \begin{cases} k\boldsymbol{b}_k \cdot \boldsymbol{c} + \sum_{l=-\infty}^{\infty} l(k-l)\boldsymbol{a}_l \cdot \boldsymbol{b}_{k-l} = 0, \; k = 0, \pm 1, \pm 2, \cdots, \\ 2k\boldsymbol{a}_k \cdot \boldsymbol{c} + \sum_{l=-\infty}^{\infty} l(k-l)(\boldsymbol{a}_l \cdot \boldsymbol{a}_{k-l} - \boldsymbol{b}_l \cdot \boldsymbol{b}_{k-l}) = 0, \; k = \pm 1, \pm 2, \cdots, \\ \boldsymbol{c} \cdot \boldsymbol{c} - 2 \sum_{l=1}^{\infty} l^2(\boldsymbol{a}_l \cdot \boldsymbol{a}_{-l} - \boldsymbol{b}_l \cdot \boldsymbol{b}_{-l}) = 0 . \end{cases}$$

If $R_1 = 0$ in (5), we say that S has an *isolated singularity* at (u_0, v_0). We then call

$$(9) \qquad \mathscr{P}_0[\boldsymbol{x}(u, v)] = \sum_{k=-\infty}^{-1} r^k (\boldsymbol{a}_k \cos k\theta + \boldsymbol{b}_k \sin k\theta)$$

the *principal part* (cf. [11, pp. 212, 213]), and

$$(10) \qquad \mathscr{L}_0[\boldsymbol{x}(u, v)] = \boldsymbol{c} \log r = \boldsymbol{c} \log |w - w_0|, \; w_0 = u_0 + iv_0 ,$$

the *logarithmic part*, of (6).

Similarly, if $R_2 = \infty$ in (5) we say that S has an isolated singularity at $w = \infty$. The transformation

$$W = U + iV = \frac{1}{w - w_0} = \frac{1}{(u - u_0) + i(v - v_0)}$$

$$= \frac{(u - u_0) - i(v - v_0)}{(u - u_0)^2 + (v - v_0)^2} = \frac{1}{r} \cos\theta - \frac{i}{r} \sin\theta$$

then maps (5) conformally onto

$$0 < |W| < 1/R_1 \, ,$$

and the isolated singularity at $w = \infty$ is treated by discussing it as an isolated singularity at $W = 0$. The principal part of (6) now becomes

$$(11) \qquad \mathscr{P}_{\infty}[\boldsymbol{x}(u, v); w_0] = \sum_{k=1}^{\infty} r^k(\boldsymbol{a}_k \cos k\theta + \boldsymbol{b}_k \sin k\theta) \, ,$$

and the logarithmic part becomes

$$(12) \qquad \mathscr{L}_{\infty}[\boldsymbol{x}(u, v); w_0] = \boldsymbol{c} \log r = \boldsymbol{c} \log |w - w_0| \, .$$

The sum of $\boldsymbol{x}(u_0, v_0)$ and the principal part $\mathscr{P}_{\infty}[\boldsymbol{x}(u, v); w_0]$, and likewise the logarithmic coefficient $\boldsymbol{c}$ in the logarithmic part $\mathscr{L}_{\infty}[\boldsymbol{x}(u, v); w_0]$, are independent of (u_0, v_0).

The equations (8) are more tractable than they might at first glance appear to be.

For $n = 2$, the equations (8) reduce simply to

$$(13) \qquad b_{1,k} = \pm a_{2,k}, \ b_{2,k} = \mp a_{1,k}, \ c_1 = c_2 = 0, \ k = \pm 1, \pm 2, \cdots .$$

Thus if $n = 2$ then the series for $x_2(u, v)$ in (6) can be written, to within sign and the arbitrary additive constant $a_{2,0}$, directly from the series for $x_1(u, v)$; and neither of the two series can contain a logarithmic term.

Let the minimal surface S in E_n be given in isothermal representation by (6) for $0 < r < R$, so that S has an isolated singularity at (u_0, v_0). We distinguish four cases, (a), (b), (c′), and (c″):

(a) *Essential singularity.* If for an infinitude of negative indices l we have

$$(14) \qquad \boldsymbol{a}_l \cdot \boldsymbol{a}_l + \boldsymbol{b}_l \cdot \boldsymbol{b}_l \neq 0 \, ,$$

then we say that the isolated singularity of S at (u_0, v_0) is *essential*; otherwise, we say that the isolated singularity is *nonessential*.

(b) *Pole.* If S has a nonessential isolated singularity at (u_0, v_0), and the lowest index $l = t$ for which (14) holds is negative, then we say that S has a *pole of order* $|t|$ at (u_0, v_0). By definition, then, the poles of S are isolated.

If S has a pole of order $-t > 0$ at (u_0, v_0), then (9) reduces to

$$(15) \qquad \begin{aligned} \mathscr{P}_0[\boldsymbol{x}(u, v)] &= \sum_{k=t}^{-1} r^k(\boldsymbol{a}_k \cos k\theta + \boldsymbol{b}_k \sin k\theta) \\ &= \Re \sum_{k=t}^{-1} (\boldsymbol{a}_k - i\boldsymbol{b}_k)(w - w_0)^k \\ &= \Re \frac{\boldsymbol{p}(w)}{q(w)} \, , \end{aligned}$$

where $\boldsymbol{p}(w)$ is a complex polynomial vector function of w, of degree

less than $|t|$; $q(w) = (w - w_0)^{|t|}$ is a complex polynomial scalar function of w; $p(w)$ and $q(w)$ are relatively prime:

$$[p(w), \, q(w)] = 1;$$

and

$$\deg \frac{p(w)}{q(w)} = \max \, [\deg p(w), \, \deg q(w)] = |t| \, .$$

If (14) does not hold for any $l < 0$, so that

$$\mathscr{P}_0[x(u, \, v)] \equiv 0 \, ,$$

then it follows from the third equation in (8) that

(16) $$c \cdot c = 0 \, ,$$

and therefore that

$$\mathscr{L}_0[x(u, \, v)] \equiv 0 \, .$$

Hence *a minimal surface given in isothermal representation by* (1) *cannot have an isolated singularity that is merely logarithmic.* [But see example (e) in §5.]

(c) *Removable singularity.* If (14) does not hold for any $l < 0$, then we say that S has a *removable singularity* at $(u_0, \, v_0)$. In this case, we adjoin to S the point a_0 corresponding to $(u_0, \, v_0)$, if indeed this correspondence was not already given in the definition of S. Then the vector function (6) gives an isothermal map of $|w| < R$ onto the (extended) surface, which we again denote by S. We then say that S is *regular* at $(u_0, \, v_0)$.

If S is regular at $(u_0, \, v_0)$, then either

(c′) $x(u, \, v) \equiv a_0$ and S reduces to a point, or

(c″)   there is a lowest positive index $l = t$ for which (14) holds. In the former case (c′), we say that S is a *constant minimal surface.* In the latter case (c″), we say that S has an a_0-*point of order* t at $(u_0, \, v_0)$; in particular, if $a_0 = 0$ then we say that S has a *zero of order* t at $(u_0, \, v_0)$.

If S has a pole of order $-t > 0$ or an a_0-point of order $t > 0$ at $(u_0, \, v_0)$, then for $k = 2t$ the first two equations in (8) reduce respectively to

$$t^2 a_t \cdot b_t = 0 \quad \text{and} \quad t^2(a_t \cdot a_t - b_t \cdot b_t) = 0 \, ,$$

whence

(17) $$a_t \cdot b_t = 0 \quad \text{and} \quad a_t \cdot a_t = b_t \cdot b_t \neq 0 \, .$$

If S has a pole of order $-t > 0$ at $(u_0, \, v_0)$, then from (6) and

(17) we obtain

(18) $$x(u, v) \cdot x(u, v) = r^{2t} a_t \cdot a_t + o(r^{2t})$$

as $r \to 0$. Similarly, if S has an a_0-point of order $t > 0$ at (u_0, v_0), then

(19) $$[x(u, v) - a_0] \cdot [x(u, v) - a_0] = r^{2t} a_t \cdot a_t + o(r^{2t}) \ .$$

By (18) and (19) we thus see that if S does not reduce to a point, then *not only the poles but also the finite a-points of S are isolated* [6].

Again if S has a pole of order $-t > 0$ or an a_0-point of order $t > 0$ at (u_0, v_0), then from (2), (6), and (17) we obtain

(20) $$\lambda(u, v) = t^2 r^{2t-2} a_t \cdot a_t + O(r^{2t-1})$$

as $r \to 0$. Therefore, if $t < 0$ then $\lambda(u, v)$ has an isolated infinite value at (u_0, v_0); if $t = 1$ then $\lambda(u, v)$ has a finite nonzero value there; and if $t > 1$ then $\lambda(u, v)$ has an isolated zero at (u_0, v_0). Thus if S is a nonconstant minimal surface, and S has no singularities other than poles, then *the zeros and infinities of* $\lambda(u, v)$ *are isolated*.

For the present theory, we extend Euclidean n-space by postulating a single ideal *point at* ∞. In this space, the isothermal transformation

$$x^* = \frac{x}{x \cdot x}$$

effects an inversion [10] in the unit hypersphere with center at the origin. If S has a pole, or ∞-point, of order $-t > 0$ at (u_0, v_0), then the surface

$$S^*: x = x^*(u, v) = \frac{x(u, v)}{x(u, v) \cdot x(u, v)}$$

has a zero of order $|t|$ at (u_0, v_0). The surface S^* will not ordinarily be a minimal surface, but since the transformation is isothermal we say that the measure of the angle between arcs from the origin on S^* is the measure of the angle between corresponding arcs from the point at ∞ on S.

If S has a pole of order $-t > 0$ at (u_0, v_0), then (17) and (18) show, through a cosine evaluation, that measures of angles between arcs from (u_0, v_0) are multiplied by $|t|$ in the map onto S^* and therefore, by definition, in the map onto S.

Similarly, if S has an a_0-point of order $t > 0$ at (u_0, v_0) then again (17) and (18) show, through a cosine evaluation, that measures of angles between arcs from (u_0, v_0) are multiplied by t in the map onto S.

If, except for poles, S is a regular minimal surface given in

isothermal representation by (1) for (u, v) in a domain D, then we say that S is a *meromorphic minimal surface* for (u, v) in D. In particular, if D is the entire finite plane then we simply say that S is a *meromorphic minimal surface*. In the latter case, if S has no poles in the finite plane then we say that S is an *entire minimal surface*.

3. Rational and logarithmico-rational minimal surfaces. Let the meromorphic minimal surface S in E_n, given in isothermal representation by (1), have at most a finite number of poles, at the points $w_1, w_2, \cdots, w_l, l \geqq 0$, in the finite w-plane, $w = u + iv$; let the orders of these poles be $n_1, n_2, \cdots, n_l$, respectively; and let the corresponding logarithmic coefficients be $c_1, c_2, \cdots, c_l$.

Then, by (15), the principal part of $x(u, v)$ at w_j, $j = 1, 2, \cdots, l$, is

$$(21) \qquad \mathscr{P}_j[x(u, v)] = \Re \frac{p_j(w)}{q_j(w)} ,$$

where $p_j(w)$ is a polynomial vector function of degree less than n_j, $q_j(w) = (w - w_j)^{n_j}$, and $p_j(w)$ and $q_j(w)$ are relatively prime. The logarithmic part of $x(u, v)$ at w_j is

$$(22) \qquad \mathscr{L}_j[x(u, v)] = c_j \log |w - w_j| .$$

Now the vector function

$$(23) \qquad y(u, v) = x(u, v) - \sum_{j=1}^{l} \mathscr{P}_j[x(u, v)] - \sum_{j=1}^{l} \mathscr{L}_j[x(u, v)]$$

is harmonic in the entire finite plane, since each term in the right-hand member is harmonic except at the points w_j, and since the singularities at the w_j exactly cancel out. Therefore, $y(u, v)$ can be represented in the finite plane by a series of the form

$$(24) \qquad \begin{aligned} y(u, v) &= y(r \cos \theta, r \sin \theta) \\ &= \sum_{k=0}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) . \end{aligned}$$

Thus from (23) and (24) we have

$$(25) \qquad \begin{aligned} x(u, v) &= \sum_{j=1}^{l} \mathscr{P}_j[x(u, v)] + \sum_{j=1}^{l} \mathscr{L}_j[x(u, v)] \\ &\quad + \sum_{k=0}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) . \end{aligned}$$

In particular, let the minimal surface S in E_n, given in isothermal representation by (1), be meromorphic in the *closed* w-plane, $w = u + iv$.

Then, since poles are isolated, S can have at most a finite number of poles in all, and accordingly the vector function $x(u, v)$ has

a representation of the form (25).

But now $x(u, v)$ has at most a pole at $w = \infty$, and therefore, for some integer $n_\infty \geqq 0$,

$$
\begin{aligned}
x(u, v) &= a_0 + \sum_{j=1}^{l} \mathscr{P}_j[x(u, v)] + \sum_{k=1}^{n_\infty} r^k(a_k \cos k\theta + b_k \sin k\theta) \\
&\quad + \sum_{j=1}^{l} \mathscr{L}_j[x(u, v)] \\
&= a_0 + \sum_{j=1}^{l} \mathscr{P}_j[x(u, v)] + \mathscr{P}_\infty[x(u, v); 0] + \sum_{j=1}^{l} \mathscr{L}_j[x(u, v)] .
\end{aligned}
$$

(26)

Since, in (26), $\mathscr{P}_\infty[x(u, v); 0]$ is the 0 polynomial vector function if $n_\infty = 0$, and is a polynomial vector function of degree n_∞ if $n_\infty > 0$, it follows from (21) that

$$
(27) \quad a_0 + \sum_{j=1}^{l} \mathscr{P}_j[x(u, v)] + \mathscr{P}_\infty[x(u, v); 0] = \mathscr{P}[x(u, v)] = \Re\frac{p(w)}{q(w)} ,
$$

where $p(w)$ is a polynomial vector function;

$$
q(w) = \prod_{j=1}^{l} (w - w_j)^{n_j} ;
$$

$p(w)$ and $q(w)$ are relatively prime; and

$$
(28) \quad \deg \frac{p(w)}{q(w)} = \max \left[\deg p(w), \deg q(w) \right] = n_\infty + \sum_{j=1}^{l} n_j .
$$

By (22),

$$
\begin{aligned}
\sum_{j=1}^{l} \mathscr{L}_j[x(u, v)] &= \mathscr{L}[x(u, v)] \\
&= \sum_{j=1}^{l} c_j \log |w - w_j| = \log \prod_{j=1}^{l} |w - w_j|^{c_j} .
\end{aligned}
$$

(29)

It follows from (27) and (29) that (26) can be written as

$$
\begin{aligned}
x(u, v) &= \mathscr{P}[x(u, v)] + \mathscr{L}[x(u, v)] \\
&= \Re\frac{p(w)}{q(w)} + \log \prod_{j=1}^{l} |w - w_j|^{c_j} .
\end{aligned}
$$

(30)

If $c_j = 0$ for all j, $j = 1, 2, \cdots, l$, then (30) reduces to

$$
(31) \quad x(u, v) = \mathscr{P}[x(u, v)] = \Re\frac{p(w)}{q(w)} ,
$$

and we say that S is a *rational minimal surface of degree*

$$
(32) \quad m = n_\infty + \sum_{j=1}^{l} n_j
$$

[or is the *zero rational minimal surface* if $x(u, v) = 0$].

If the c_j, $j = 1, 2, \cdots, l$, are not all 0, then we say that S, given by (30), is a *logarithmico-rational minimal surface of degree m*, where m is given by (32).

In accordance with these definitions, *if S is a rational minimal surface, or a logarithmico-rational minimal surface, of degree m, then S has exactly m poles, counting multiplicities, in the closed w-plane.*

We have thus shown that if a minimal surface S, given in isothermal representation, is meromorphic in the closed plane, then S is either a rational minimal surface or a logarithmico-rational minimal surface.

Conversely, if a minimal surface S, given in isothermal representation, is either a rational minimal surface or a logarithmico-rational minimal surface, then S clearly is meromorphic in the closed plane.

We accordingly have the following result.

THEOREM 1. *A minimal surface S, given in isothermal representation, is meromorphic in the closed plane if and only if S is either a rational minimal surface or a logarithmico-rational minimal surface.*

4. Remarks. (a) For a logarithmico-rational minimal surface S, given in isothermal representation by (1), the vector c_j in (30) might be called the *order* of the logarithmic part of $x(u, v)$ at the singular point $w = w_j$. Since

$$\lim_{w \to \infty} \left| \frac{w - w_j}{w} \right| = 1 ,$$

by (12) and (30) we have

$$\mathscr{L}_\infty[x(u, v); 0] = \sum_{j=1}^{l} c_j \log |w|$$

$$= -\sum_{j=1}^{l} c_j \log |W|, \qquad W = \frac{1}{w} .$$

We therefore take

(33) $$c_\infty = -\sum_{j=1}^{l} c_j$$

as the order of the logarithmic part of $x(u, v)$ at $w = \infty$.

Then *for any logarithmico-rational minimal surface S, given in isothermal representation by (1), the sum of the orders of the logarithmic part of $x(u, v)$ at all the singular points of S is 0,* since

$$\sum_{j=1}^{l} c_j + c_\infty = \sum_{j=1}^{l} c_j - \sum_{j=1}^{l} c_j = 0 .$$

(b) By (13), in E_2 there are no logarithmico-rational minimal surfaces. [But see example (e) in §5.]

(c) Let S be a rational or logarithmico-rational minimal surface in E_n, given in isothermal representation by (1), for which the function $q(w)$ in (30) satisfies $q(w) \equiv 1$. Then (14) does not hold for any $l < 0$, and therefore, by the third equation in (8), c satisfies (16). Thus S must be a rational minimal surface. Since also $q(w) \equiv 1$, we say that S is a *polynomial*, or *entire rational*, *minimal surface*. There are no logarithmico-polynomial minimal surfaces.

(d) If S is a rational (or polynomial) minimal surface of degree m, given in isothermal representation by (1), and $x_j(u, v) \equiv 0$, $j = 3$, $4, \cdots, n$, then either $x_1(u, v) + ix_2(u, v)$ or $x_2(u, v) + ix_1(u, v)$ is a rational (or polynomial) complex function, of degree m, of the complex variable $w = u + iv$. Thus the study of rational (or polynomial) minimal surfaces of degree m subsumes the study of rational (or polynomial) complex functions of degree m.

(e) We note that the expression $\Re[p(w)/q(w)]$ in (27) can be written as a rational harmonic vector function of the real variables u, v, since

(34) $$\Re\frac{p(w)}{q(w)} = \Re\frac{p(w)\overline{q(w)}}{q(w)\overline{q(w)}} = \frac{\Re p(w)\overline{q(w)}}{q(w)\overline{q(w)}} = \frac{P(u, v)}{Q(u, v)} ,$$

and $P(u, v)$ and $Q(u, v)$ are polynomial functions of u, v, as desired.

Although $P(u, v)/Q(u, v)$ is a harmonic vector function of u, v, the functions $P(u, v)$ and $Q(u, v)$ themselves ordinarily are not harmonic.

By (28), (32), and (34), the degree m of S is given by

$$m = \max [\deg P(u, v), \deg Q(u, v)] - \frac{1}{2} \deg Q(u, v) .$$

(f) We note also, conversely, that if $x(u, v)$ is a rational harmonic vector function,

$$x(u, v) = \frac{P(u, v)}{Q(u, v)} ,$$

where $P(u, v)$ and $Q(u, v)$ are polynomial functions of u, v, and we let

$$x(w, v) = \Re f(w) , \qquad\qquad w = u + iv ,$$

where $f(w)$ is an analytic vector function of w, then $f(w)$ is a rational vector function of w, since, in a neighborhood of any point (u_0, v_0), or $w_0 = u_0 + iv_0$, at which $Q(u, v) \neq 0$, $f(w)$ is given (see, for instance, [9, p. 127]), to within an arbitrary additive pure imaginary constant vector, by

$$f(w) = 2x\left(\frac{w + \bar{w}_0}{2}, \frac{w - \bar{w}_0}{2i}\right) - x(u_0, v_0)$$

$$= 2\frac{P\left(\dfrac{w + \bar{w}_0}{2}, \dfrac{w - \bar{w}_0}{2i}\right)}{Q\left(\dfrac{w + \bar{w}_0}{2}, \dfrac{w - \bar{w}_0}{2i}\right)} - \frac{P(u_0, v_0)}{Q(u_0, v_0)} = \frac{p(w)}{q(w)},$$

and $p(w)$ and $q(w)$ are polynomial functions of w. By analytic continuation, $f(w)$ is therefore a rational vector function of w throughout its domain of analyticity.

5. **Examples.** (a) The minimal surface S of Enneper [13, p. 221] is given in isothermal representation by

$$x_1(u, v) = 3u - u^3 + 3uv^2 = \Re(3w - w^3) = 3r\cos\theta - r^3\cos 3\theta ,$$

$$x_2(u, v) = -3v + v^3 - 3u^2v = \Re(3iw + iw^3) = -3r\sin\theta - r^3\sin 3\theta ,$$

$$x_3(u, v) = 3u^2 - 3v^2 = \Re(3w^2) = 3r^2\cos 2\theta .$$

This surface is a polynomial, or entire rational, minimal surface of degree 3. In the closed w-plane, S has just one zero; this is at $w = 0$ and is of order 1. The surface has a pole of order 3 at $w = \infty$.

(b) The functions

$$x_1(u, v) = \frac{(1 + u^2 + v^2)u}{u^2 + v^2} = \Re\left(\frac{1}{w} + w\right) = \left(\frac{1}{r} + r\right)\cos\theta ,$$

$$x_2(u, v) = \frac{(1 + u^2 + v^2)v}{u^2 + v^2} = \Re\left(\frac{i}{w} - iw\right) = \left(\frac{1}{r} + r\right)\sin\theta ,$$

$$x_3(u, v) = \log (u^2 + v^2) = \Re(2\log w) = 2\log r$$

are the coordinate functions of a logarithmico-rational minimal surface (actually a catenoid) in isothermal representation. The surface is of degree 2, with poles of order 1 at $w = 0$ and at $w = \infty$; it has no zero.

(c) The rational minimal surfaces of Eng, given in isothermal Weierstrass-formula [16, pp. 28, 29] representation by

$$x_1(u, v) = \Re f_1(w) = \Re\int^w (1 - \zeta^2)\left[\frac{a}{(1 - \zeta)^4} + \frac{b}{(i - \zeta)^4} + \frac{c}{\zeta^4}\right]d\zeta ,$$

$$x_2(u, v) = \Re f_2(w) = \Re\int^w i(1 + \zeta^2)\left[\frac{a}{(1 - \zeta)^4} + \frac{b}{(i - \zeta)^4} + \frac{c}{\zeta^4}\right]d\zeta ,$$

$$x_3(u, v) = \Re f_3(w) = \Re\int^w 2\zeta\left[\frac{a}{(1 - \zeta)^4} + \frac{b}{(i - \zeta)^4} + \frac{c}{\zeta^4}\right]d\zeta ,$$

where a, b, and c are arbitrary nonzero real constants, are of degree 9, with poles of order 3 at $w = 0$, $w = 1$, and $w = i$. Although the standard representation (30) of each of these surfaces is

$$x_1(u, v) = \Re f_1(w) = \Re\frac{(1 - w)p_1(w)}{q(w)} ,$$

$$x_2(u, v) = \Re f_2(w) = \Re\frac{(i - w)p_2(w)}{q(w)} ,$$

$$x_3(u, v) = \Re f_3(w) = \Re\frac{wp_3(w)}{q(w)} ,$$

where $q(w) = (1 - w)^3(i - w)^3 w^3$ is of degree 9, each of the functions $f_j(w)$ actually is of degree only 8:

$$f_1(w) = \frac{p_1(w)}{q_1(w)} = \frac{p_1(w)}{(1 - w)^2(i - w)^3 w^3} ,$$

$$f_2(w) = \frac{p_2(w)}{q_2(w)} = \frac{p_2(w)}{(1 - w)^3(i - w)^2 w^3} ,$$

$$f_3(w) = \frac{p_3(w)}{q_3(w)} = \frac{p_3(w)}{(1 - w)^3(i - w)^3 w^2} ,$$

with

$$[p_j(w), q_j(w)] = 1, \deg p_j(w) \leqq 7, \deg q_j(w) = 8, j = 1, 2, 3 .$$

(d) Generalizations [3] of the Weierstrass formulas for the coordinate functions of a minimal surface in isothermal representation can be applied to obtain examples of rational and logarithmico-rational minimal surfaces in E_n, $n > 3$. Thus, with

$$(u, v) = w = (r \cos \theta, r \sin \theta) ,$$

the formulas

$$x_1(u, v) = 12r \cos \theta - 3r^4 \cos 4\theta ,$$

$$x_2(u, v) = - 12r \sin \theta - 3r^4 \sin 4\theta ,$$

$$x_3(u, v) = 6r^2 \cos 2\theta + 4r^3 \cos 3\theta ,$$

$$x_4(u, v) = - 6r^2 \sin 2\theta + 4r^3 \sin 3\theta$$

represent an entire rational minimal surface, of degree 4, in Euclidean 4-space. This surface is not contained in any 3-dimensional Euclidean subspace. The surface has a pole of order 4 at $w = \infty$; it single zero is at $w = 0$ and is of order 1.

(e) The relations

$$x_1 = \Re\left(\log w - \frac{1}{2}w^2\right) = \log r - \frac{1}{2}r^2 \cos 2\theta ,$$

(35) $$x_2 = \Re\left(i \log w + \frac{i}{2}w^2\right) = - \theta - \frac{1}{2}r^2 \sin 2\theta ,$$

$$x_3 = \Re(2w) = 2r \cos \theta , \quad w = r(\cos \theta + i \sin \theta) ,$$

give an isothermal representation of a minimal surface with a singularity of a different sort at the origin. The second of the relations (35) is not a (single-valued) function of $w = u + iv$, however, so that this surface is not included in the class of surfaces (1) presently under consideration.

The same remark might be made, of course, concerning the familiar relations

$$x_1 = \log r \; ,$$
$$x_2 = \theta \; ,$$

which give an isothermal representation of a minimal surface in E_2, and for which $x_1 + ix_2 = \log w$.

6. **The first fundamental theorem for meromorphic minimal surfaces.** Nevanlinna's first fundamental theorem concerning meromorphic functions of a complex variable [15; 12, p. 212] can be generalized [7] to give an analogous result concerning meromorphic minimal surfaces. The concepts involved in the generalization are central to the discussion in the rest of the present paper.

In this section, some definitions are given and the first fundamental theorem and its generalization are stated without proof; but see §8 for the formulas from which these theorems result.

Let $f(w)$ be a nonconstant meromorphic function, and let S be a nonconstant meromorphic minimal surface given in isothermal representation by (1) in the entire finite w-plane, $w = u + iv$, $|w| < \infty$.

In the classical Nevanlinna theory, for $0 < r < \infty$ and for a in the closed complex plane, the *proximity function* $m(r, a; f)$ is defined by

$$m(r, a; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta \; , \qquad a \neq \infty \; ,$$

$$m(r, \infty; f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \; .$$

By analogy, for $\boldsymbol{a}$ in the closed n-dimensional space, the *proximity function* $m(r, \boldsymbol{a}; S)$ is defined by

(36)
$$m(r, \boldsymbol{a}; S) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|\boldsymbol{x}(re^{i\theta}) - \boldsymbol{a}|} d\theta \; , \qquad \boldsymbol{a} \neq \infty \; ,$$

$$m(r, \infty; S) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\boldsymbol{x}(re^{i\theta})| d\theta \; ,$$

where, for simplicity, we have introduced the notation

$$|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$$

and

$$\boldsymbol{x}(re^{i\theta}) = \boldsymbol{x}(r\cos\theta,\, r\sin\theta) \; .$$

In the Ahlfors-Shimizu version [2, 17; 12, p. 216] of the Nevanlinna theory, the closed complex $\zeta = f(w)$ plane is projected stereographically onto the sphere of radius 1/2 having center at

$$(x_1,\, x_2,\, x_3) = (0,\, 0,\, 1/2) \; .$$

Distances between points in the ζ-plane are replaced with the chordal distances between corresponding points on the sphere, and the *spherical proximity function* $m^\circ(r,\, a;\, f)$ is accordingly defined by

$$m^\circ(r,\, a;\, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{[1 + |f(re^{i\theta})|^2]^{1/2}(1 + |a|^2)^{1/2}}{|f(re^{i\theta}) - a|} d\theta \;, \quad a \neq \infty \;,$$

$$m^\circ(r,\, \infty;\, f) = \frac{1}{2\pi}\int_0^{2\pi} \log\,[1 + |f(re^{i\theta})|^2]^{1/2} d\theta \;.$$

For S, similarly, the closed n-dimensional space is projected stereographically onto the $(n+1)$-dimensional hypersphere $\mathscr{S}$ of radius 1/2 having center at

$$(x_1,\, x_2,\, \cdots,\, x_n,\, x_{n+1}) = (0,\, 0,\, \cdots,\, 0,\, 1/2) \; .$$

Distances between points in the n-dimensional space are replaced with the chordal distances between corresponding points on the hypersphere, and the *hyperspherical proximity function* $m^\circ(r,\, \boldsymbol{a};\, S)$ is accordingly defined by

$$
\begin{aligned}
m^\circ(r,\, \boldsymbol{a};\, S) &= \frac{1}{2\pi}\int_0^{2\pi} \log \frac{[1 + |\boldsymbol{x}(re^{i\theta})|^2]^{1/2}(1 + |\boldsymbol{a}|^2)^{1/2}}{|\boldsymbol{x}(re^{i\theta}) - \boldsymbol{a}|} d\theta \;, \quad \boldsymbol{a} \neq \infty \;, \\
m^\circ(r,\, \infty;\, S) &= \frac{1}{2\pi}\int_0^{2\pi} \log\,[1 + |\boldsymbol{x}(re^{i\theta})|^2]^{1/2} d\theta \;.
\end{aligned}
$$

(37)

For $0 \leqq \rho < \infty$ and for a in the closed complex plane, let $n(\rho,\, a;\, f)$ denote the number of a-points of $f(w)$ in $|w| \leqq \rho$. Then the *enumerative function* $N(r,\, a;\, f)$ is defined, for $0 < r < \infty$, by

$$N(r,\, a;\, f) = \int_0^r \frac{n(\rho,\, a;\, f) - n(0,\, a;\, f)}{\rho} d\rho + n(0,\, a;\, f)\log r \;,$$

for a finite or infinite.

Similarly, for $\boldsymbol{a}$ in the closed n-dimensional space, and with an analogous definition of $n(\rho,\, \boldsymbol{a};\, S)$ as the number of $\boldsymbol{a}$-points of S in $|w| \leqq \rho$, the *enumerative function* $N(r,\, \boldsymbol{a};\, S)$ is defined, for $0 < r < \infty$, by

$$(38) \quad N(r, \boldsymbol{a}; S) = \int_0^r \frac{n(\rho, \boldsymbol{a}; S) - n(0, \boldsymbol{a}; S)}{\rho} d\rho + n(0, \boldsymbol{a}; S) \log r \,,$$

for $\boldsymbol{a}$ finite or infinite.

For the extended theory, a function that does not occur in the classical theory is needed.

For any finite a, we have

$$\varDelta \log |f(w) - a| = 0$$

except at the isolated a-points and poles of $f(w)$, since

$$\log |f(w) - a| = \Re \log [f(w) - a] \,.$$

By contrast, for any finite $\boldsymbol{a}$, a computation [8] using the fact that the $x_j(u, v)$ form an n-tuple of conjugate harmonic functions shows that

$$(39)$$
$$\varDelta \log |\boldsymbol{x} - \boldsymbol{a}|$$
$$= \frac{2\lambda\{|\boldsymbol{x} - \boldsymbol{a}|^2 - [(\boldsymbol{x} - \boldsymbol{a}) \cdot (\partial\boldsymbol{x}/\partial u)/\lambda^{1/2}]^2 - [(\boldsymbol{x} - \boldsymbol{a}) \cdot (\partial\boldsymbol{x}/\partial v)/\lambda^{1/2}]^2\}}{|\boldsymbol{x} - \boldsymbol{a}|^4}$$

except at the isolated $\boldsymbol{a}$-points and poles of S and the isolated zeros and infinities of λ.

Since $|\boldsymbol{x}(u, v) - \boldsymbol{a}|^2$ is the square of the length of the geometric vector $\boldsymbol{x}(u, v) - \boldsymbol{a}$, and

$$\left\{[\boldsymbol{x}(u, v) - \boldsymbol{a}] \cdot \frac{\partial\boldsymbol{x}(u, v)/\partial u}{[\lambda(u, v)]^{1/2}}\right\}^2 + \left\{[\boldsymbol{x}(u, v) - \boldsymbol{a}] \cdot \frac{\partial\boldsymbol{x}(u, v)/\partial v}{[\lambda(u, v)]^{1/2}}\right\}^2$$

is the square of the length of the projection of $\boldsymbol{x}(u, v) - \boldsymbol{a}$ on the plane tangent to S at (u, v), it follows from (39) that

$$(40) \qquad\qquad \varDelta \log |\boldsymbol{x}(u, v) - \boldsymbol{a}| \geqq 0$$

at the nonexceptional points, ordinarily with the strict inequality for $n > 2$.

The function

$$(41) \qquad\qquad \varDelta \log |\boldsymbol{x}(re^{i\theta}) - \boldsymbol{a}|$$

is continuous except at the isolated $\boldsymbol{a}$-points and poles of S. A computation shows that, as $r \to 0$, (41) is $O(1)$ at the $\boldsymbol{a}$-points and poles except at poles of order 1 with nonvanishing logarithmic part, where it is $O[(\log r)^2]$. Thus (41) can become infinite, but not rapidly enough to render it nonintegrable.

For $0 \leqq \rho < \infty$ and for any finite $\boldsymbol{a}$, the function $h(\rho, \boldsymbol{a}; S)$ is defined by

$$(42) \qquad h(\rho, \boldsymbol{a}; S) = \frac{1}{2\pi} \iint_{|w| \leqq \rho} \varDelta \log |\boldsymbol{x}(w) - \boldsymbol{a}| \, dA_w \,.$$

In particular, we have

(43) $h(0, \boldsymbol{a}; S) = 0$.

The function $h(\rho, \boldsymbol{a}; S)$ given by (42) might be considered to furnish a measure of the visibility of the map of $|w| \leqq \rho$ on S, as viewed from $\boldsymbol{a}$.

Since

$$\lim_{a \to \infty} h(\rho, \boldsymbol{a}; S) = 0 ,$$

we define $h(\rho, \infty; S)$ by

(44) $h(\rho, \infty; S) = 0$.

To the proximity function $m(r, \boldsymbol{a}; S)$ or $m°(r, \boldsymbol{a}; S)$ and the enumerative function $N(r, \boldsymbol{a}; S)$, we now adjoin a *visibility function* $H(r, \boldsymbol{a}; S)$, defined, for $0 < r < \infty$, by

(45) $$H(r, \boldsymbol{a}; S) = \int_0^r \frac{h(\rho, \boldsymbol{a}; S)}{\rho} d\rho ,$$

for $\boldsymbol{a}$ finite or infinite. In particular, then, by (44), we have

(46) $H(r, \infty; S) = 0$.

By (43) and (44), the definition (45) of $H(r, \boldsymbol{a}; S)$ is quite analogous to the definition (38) of $N(r, \boldsymbol{a}; S)$, with $h(\rho, \boldsymbol{a}; S)$ in place of $n(\rho, \boldsymbol{a}; S)$.

In the classical Nevanlinna theory, the *total affinity* $\mathfrak{A}(r, a; f)$ of $f(w)$ to a, or the *affinity function* of $f(w)$, is defined by

$$\mathfrak{A}(r, a; f) = m(r, a; f) + N(r, a; f)$$

for a finite or infinite. In particular, $\mathfrak{A}(r, \infty; f)$ is called the *Nevanlinna characteristic function* of $f(w)$ and is denoted by $T(r; f)$, so that

$$T(r; f) = \mathfrak{A}(r, \infty; f) = m(r, \infty; f) + N(r, \infty; f) .$$

The *first fundamental theorem of Nevanlinna* is the statement that, *for each finite a, $T(r; f)$ differs from $\mathfrak{A}(r, a; f)$ by a bounded function of r*:

(47) $T(r; f) = \mathfrak{A}(r, a; f) + C(r, a; f)$.

In the extended theory, the *total affinity* $\mathfrak{A}(r, \boldsymbol{a}; S)$ of S to $\boldsymbol{a}$, or the *affinity function* of S, is defined by

(48) $\mathfrak{A}(r, \boldsymbol{a}; S) = m(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S)$

for $\boldsymbol{a}$ finite or infinite. In particular, $\mathfrak{A}(r, \infty; S)$ is called the *Nevanlinna characteristic function* of S and is denoted by $T(r; S)$, so that

$$\begin{aligned}(49) \quad T(r;S) = \mathfrak{A}(r,\infty;S) &= m(r,\infty;S) + N(r,\infty;S) + H(r,\infty;S) \\ &= m(r,\infty;S) + N(r,\infty;S)\,.\end{aligned}$$

The extended first fundamental theorem is then the statement that, *for each finite **a**, $T(r;S)$ differs from $\mathfrak{A}(r,\boldsymbol{a};S)$ by a bounded function of r*:

$$(50) \qquad\qquad T(r;S) = \mathfrak{A}(r,\boldsymbol{a};S) + C(r,\boldsymbol{a};S)\,.$$

In the Ahlfors-Shimizu version of the classical theory, the proximity function is replaced by the spherical proximity function. The correction term then reduces to a constant for each finite a. It is customary in this case to add a constant to both sides of the equation so that each approaches 0 as $r \to 0$. Thus the *spherical affinity function* $\mathfrak{A}(r,a;f)$ is defined by

$$\mathfrak{A}^\circ(r,a;f) = m^\circ(r,a;f) + N(r,a;f) + C(a;f)\,,$$

for a finite or infinite, with the constant $C(a;f)$ chosen so that

$$\lim_{r\to 0} \mathfrak{A}^\circ(r,a;f) = 0\,;$$

the *spherical characteristic functions* $T^\circ(r;f)$ is defined by

$$\begin{aligned} T^\circ(r;f) &= \mathfrak{A}^\circ(r,\infty;f) \\ &= m^\circ(r,\infty;f) + N(r,\infty;f) + C(\infty;f)\,; \end{aligned}$$

and the *spherical form of the first fundamental theorem* is the statement that, *for each finite a*,

$$(51) \qquad\qquad T^\circ(r;f) = \mathfrak{A}^\circ(r,a;f)\,.$$

Similarly, in the extension of the Ahlfors-Shimizu version, the *hyperspherical affinity function* $\mathfrak{A}^\circ(r,\boldsymbol{a};S)$ is defined by

$$(52) \quad \mathfrak{A}^\circ(r,\boldsymbol{a};S) = m^\circ(r,\boldsymbol{a};S) + N(r,\boldsymbol{a};S) + H(r,\boldsymbol{a};S) + C(\boldsymbol{a};S)\,,$$

for $\boldsymbol{a}$ finite or infinite, with the constant $C(\boldsymbol{a};S)$ chosen so that

$$(53) \qquad\qquad \lim_{r\to 0} \mathfrak{A}^\circ(r,\boldsymbol{a};S) = 0\,;$$

the *hyperspherical characteristic function* $T^\circ(r;S)$ is defined by

$$\begin{aligned} (54) \quad T^\circ(r;S) &= \mathfrak{A}^\circ(r,\infty;S) \\ &= m^\circ(r,\infty;S) + N(r,\infty;S) + H(r,\infty;S) + C(\infty;S) \\ &= m^\circ(r,\infty;S) + N(r,\infty;S) + C(\infty;S); \end{aligned}$$

and the *hyperspherical form of the first fundamental theorem* is the statement that, *for each finite **a**,*

(55) $$T^\circ(r; S) = \mathfrak{A}^\circ(r, \boldsymbol{a}; S) \, .$$

7. Basic growth properties. Proofs of the following convexity properties can be found in [4, 7].

Like the function $N(r, a; f)$ in the complex-variable case, *the function $N(r, \boldsymbol{a}; S)$ for a meromorphic minimal surface S is a non-decreasing, piecewise linear, convex function of $\log r$.*

The function $H(r, \boldsymbol{a}; S)$ vanishes identically if $\boldsymbol{a}$ is infinite or if S is a plane surface and $\boldsymbol{a}$ lies in the plane. Otherwise, for $r > 0$, $H(r, \boldsymbol{a}; S)$ is a positive, increasing, strictly convex function of $\log r$.

The function $T^\circ(r; S)$ can be given a quasi-geometric representation by integrating both members of (55) over the hyperspherical surface $\mathscr{S}$ and dividing by the content $\mathscr{V}$ of $\mathscr{S}$. After a simplification, this yields

(56) $$T^\circ(r; S) = \int_0^r \mathscr{H}(\rho; S)\frac{d\rho}{\rho} \, ,$$

where

(57) $$\mathscr{H}(\rho; S) = \frac{1}{\mathscr{V}} \int \cdots \int_{\mathscr{S}} h(\rho, \boldsymbol{a}; S) d\,\mathscr{V}_a \, .$$

From (40), (42), (56), and (57), it follows that *the function $T^\circ(r; S)$ is positive for $r > 0$ and is an increasing, strictly convex function of $\log r$.*

From (56), we therefore obtain

(58) $$T^\circ(r; S) > \mathscr{H}(1; S) \log r$$

for $r > 1$, and accordingly

(59) $$\liminf_{r \to \infty} \frac{T^\circ(r; S)}{\log r} > 0 \, .$$

In §§9 and 10 of this paper, we shall be concerned with meromorphic minimal surfaces S for which the left-hand member of (59) has a finite value.

We mention in passing, however, that just as the *order* $\rho(f)$ of a meromorphic function $f(w)$ is defined by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T^\circ(r; f)}{\log r} \, ,$$

so is the *order of a meromorphic minimal surface S*, given in isothermal representation by (1), defined by

(60) $$\rho(S) = \limsup_{r \to \infty} \frac{\log T^\circ(r; S)}{\log r} \, .$$

Since for any two nonnegative numbers a and b, we have

$$(61) \qquad \log^+ (a + b) \leqq \log^+ a + \log^+ b + \log 2 ,$$

it follows from (36) and (37) that

$$(62) \qquad 0 \leqq m(r, \infty; S) \leqq m°(r, \infty; S) \leqq m(r, \infty; S) + \log 2^{1/2} .$$

Therefore, by (48) and (54), *the difference* $T°(r; S) - T(r; S)$ *is a bounded function of* r. For this reason, by (59), $T°(r; S)$ *and* $T(r; S)$ *are interchangeable in the study of growth properties as* $r \to \infty$.

Actually, it can be shown that, like $T°(r; S)$, *the function* $T(r; S)$ *is an increasing, strictly convex function of* $\log r$.

8. **The Poisson-Jensen-Nevanlinna formula.** The first fundamental theorem of Nevanlinna, (47) or (51), results from the following Jensen-Nevanlinna formula [15].

Let $f(w)$ be meromorphic in $|w| \leqq R$, and in the neighborhood of the origin let

$$f(w) = \alpha_t w^t + \alpha_{t+1} w^{t+1} + \cdots ,$$

with $\alpha_t \neq 0$, so that at the origin $f(w)$ has a zero of order t if $t > 0$ or a pole of order $- t$ if $t < 0$, and $f(0) \neq 0$ or ∞ if $t = 0$. Further, let the zeros and poles of $f(w)$ in $0 < |w| \leqq R$ be at

$$z_1, z_2, \cdots, z_\mu \quad \text{and} \quad p_1, p_2, \cdots, p_\nu ,$$

respectively, with repetitions for multiple values and with

$$0 < |z_1| \leqq \cdots \leqq |z_\mu| < R \text{ and } 0 < |p_1| \leqq \cdots \leqq |p_\nu| < R .$$

Then the *Jensen-Nevanlinna formula* is

$$(63) \qquad \begin{aligned} \log |\alpha_t| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta \\ - \sum_{j=1}^{\mu} \log \frac{R}{|z_j|} + \sum_{k=1}^{\nu} \log \frac{R}{|p_k|} - t \log R . \end{aligned}$$

The extension, (50) or (55), of the first fundamental theorem to meromorphic minimal surfaces results similarly from an extension [7] of the Jensen-Nevanlinna formula.

Namely, for a meromorphic minimal surface S, given in isothermal representation by (1) for $|w| \leqq R$, $w = u + iv$, let the z_j and p_k have the same meaning as above but this time relative to S, and in the neighborhood of the origin let

$$(64) \qquad \boldsymbol{x}(u, v) = \boldsymbol{c} \log r + \sum_{k=t}^{\infty} r^k (\boldsymbol{a}_k \cos k\theta + \boldsymbol{b}_k \sin k\theta) ,$$

with $|a_t|^2 + |b_t|^2 \neq 0$.

Then the *extension of the Jensen-Nevanlinna formula* is

$$
\log |a_t| = \frac{1}{2\pi} \int_0^{2\pi} \log |x(Re^{i\theta})| \, d\theta
$$

(65)
$$
- \sum_{j=1}^{\mu} \log \frac{R}{|z_j|} + \sum_{k=1}^{\nu} \log \frac{R}{|p_k|} - t \log R
$$

$$
- \frac{1}{2\pi} \int_0^R \left[\frac{1}{\rho} \iint_{|w| \leq \rho} \varDelta \log |x(w)| \, dA_w \right] d\rho \ .
$$

An important tool in the further investigation of meromorphic functions $f(w)$ is obtained by mapping the circular disc $|w| \leq R$ conformally onto itself, with a given point

(66)
$$
w_0 = u_0 + iv_0 = re^{i\varphi} \ , \qquad\qquad r < R \ ,
$$

mapped onto the origin. If $f(w_0) \neq 0$ or ∞, and if the symbols z_j and p_k now represent all the zeros and poles in the disc $|w| \leq R$, not merely those in the punctured disc $0 < |w| \leq R$, then an application of the Jensen-Nevanlinna formula (63) in the transformed disc yields the *Poisson-Jensen-Nevanlinna formula*,

$$
\log |f(w_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} d\theta
$$

(67)
$$
- \sum_{j=1}^{\mu} \log \left| \frac{R^2 - \bar{z}_j w_0}{R(w_0 - a_j)} \right| + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_0}{R(w_0 - p_k)} \right| \ .
$$

In the same way, an application of the extension (65) of the Jensen-Nevanlinna formula in the transformed disc yields an extension of the Poisson-Jensen-Nevanlinna formula to meromorphic minimal surfaces S. Namely, with the same notation as above, but now relative to S, the *extension of the Poisson-Jensen-Nevanlinna formula* is

$$
\log |x(w_0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |x(Re^{i\theta})| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} d\theta
$$

(68)
$$
- \sum_{j=1}^{\mu} \log \left| \frac{R^2 - \bar{z}_j w_0}{R(w_0 - z_j)} \right| + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_0}{R(w_0 - p_k)} \right|
$$

$$
- \frac{1}{2\pi} \int_0^R \left[\frac{1}{\rho} \iint_{B(\rho, w_0)} \varDelta \log |x(w)| \, dA_w \right] d\rho \ ,
$$

where

(69)
$$
B(\rho, w_0) = \left\{ w : \left| \frac{R^2(w - w_0)}{R^2 - \bar{w}_0 w} \right| \leq \rho \right\} \ .
$$

[Like the Jensen-Nevanlinna formula (63) and its extension (65), the Poisson-Jensen-Nevanlinna formula (67) and its extension (68) can

readily be adjusted to include the case in which there is a zero or pole at w_0.]

Notice in (68) that

$$(70) \qquad \sum_{j=1}^{\mu} \log \left| \frac{R^2 - \bar{z}_j w_0}{R(w_0 z_j)} \right| \geqq 0 \quad \text{and} \quad \sum_{k=1}^{\nu} \log \left| \frac{R^2 - p_k w_0}{R(w_0 - p_k)} \right| \geqq 0 \; ,$$

and that also, by (40),

$$(71) \qquad \frac{1}{2\pi} \int_0^R \left[\frac{1}{\rho} \iint_{B(\rho, w_0)} \varDelta \log |\boldsymbol{x}(w)| \, dA_w \right] d\rho \geqq 0 \; .$$

Substituting from (71) and the first inequality in (70) into (68), we obtain the useful inequality

$$(72) \qquad \begin{aligned} \log |\boldsymbol{x}(w_0)| &\leqq \frac{1}{2\pi} \int_0^{2\pi} \log |\boldsymbol{x}(Re^{i\theta})| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} d\theta \\ &\quad + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_0}{R(w_0 - p_k)} \right| . \end{aligned}$$

A fortiori, we therefore have

$$(73) \qquad \begin{aligned} \log^+ |\boldsymbol{x}(w_0)| &\leqq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\boldsymbol{x}(Re^{i\theta})| \frac{R^2 - r^2}{R^2 - 2Rr \cos(\theta - \varphi) + r^2} d\theta \\ &\quad + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_0}{R(w_0 - p_k)} \right| , \end{aligned}$$

and accordingly, by (36),

$$(74) \qquad \log^+ |\boldsymbol{x}(w_0)| \leqq \frac{R + r}{R - r} m(R, \infty; S) + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_0}{R(w_0 - p_k)} \right| .$$

In particular, letting

$$(75) \qquad M(r; |\boldsymbol{x}|) = \sup_{0 \leqq \varphi < 2\pi} |\boldsymbol{x}(re^{i\varphi})| \; ,$$

and choosing $w_r^* = re^{i\varphi*}$ so that

$$(76) \qquad |\boldsymbol{x}(w_r^*)| = M(r; |\boldsymbol{x}|) \; ,$$

from (74) we obtain

$$(77) \qquad \log^+ M(r; |\boldsymbol{x}|) \leqq \frac{R + r}{R - r} m(R, \infty; S) + \sum_{k=1}^{\nu} \log \left| \frac{R^2 - \bar{p}_k w_r^*}{R(w_r^* - p_k)} \right| .$$

9. An alternative characterization of rational and logarithmico-rational minimal surfaces. We shall now apply (77) in the study of meromorphic minimal surfaces for which the left-hand member of (59) has a finite value.

Theorem 2. *For a nonconstant meromorphic minimal surface S given in isothermal representation by* (1), $T^\circ(r; S)$ *satisfies*

$$\liminf_{r \to \infty} \frac{T^\circ(r; S)}{\log r} = c < \infty \tag{78}$$

if and only if c is a positive integer and S is a rational or logarithmico-rational minimal surface of degree c.

Proof. Suppose first that (78) is satisfied. It follows then from (54) that

$$\liminf_{r \to \infty} \frac{N(r, \infty; S)}{\log r} \leqq c , \tag{79}$$

since $C(\infty; S)$ is a constant and since, by (37), $m^\circ(r, \infty; S)$ is positive for $r > 0$.

Therefore, by (38) and (79), S can have at most a finite number of poles, and accordingly the vector function $\boldsymbol{x}(u, v)$ in (1) has a representation of the form (25).

We shall see now that (78) implies further that the representation (25) reduces to (26).

Let r_0 be great enough that all the poles of S in the finite plane are in $|w| < r_0$. Then for $r > r_0$ and $R = \delta r$, with $\delta > 1$ fixed, and with the notation of (25), (77) gives

$$
\begin{aligned}
\log^+ |M(r; |\boldsymbol{x}|)| &\leqq \frac{\delta + 1}{\delta - 1} m(\delta r, \infty; S) + \sum_{j=1}^{l} \log \left| \frac{\delta^2 r^2 - \bar{w}_j w_r^*}{\delta r(w_r^* - w_j)} \right|^{n_j} \\
&\leqq \frac{\delta + 1}{\delta - 1} m(\delta r, \infty; S) + O(1)
\end{aligned}
\tag{80}
$$

as $r \to \infty$, since the logarithmic sum is a bounded function of r.

From (80), we obtain

$$
\begin{aligned}
\liminf_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} &\leqq \frac{\delta + 1}{\delta - 1} \liminf_{r \to \infty} \left[\frac{m(\delta r, \infty; S)}{\log \delta r} \cdot \frac{\log \delta + \log r}{\log r} \right] \\
&= \frac{\delta + 1}{\delta - 1} \liminf_{r \to \infty} \frac{m(r, \infty; S)}{\log r} .
\end{aligned}
\tag{81}
$$

Letting $\delta \to \infty$ now yields

$$\liminf_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} \leqq \liminf_{r \to \infty} \frac{m(r, \infty; S)}{\log r} . \tag{82}$$

Therefore, by (54), (62), and (78),

$$\liminf_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} \leqq c , \tag{83}$$

since $C(\infty; S)$ is a constant and since, by (38), $N(r, \infty; S)$ is nonnegative for $r \geqq 1$.

By (21), (22), and (23), we have

$$M(r; |\boldsymbol{y}|) \leqq M(r; |\boldsymbol{x}|) + O(\log r)$$

as $r \to \infty$, and therefore

$$\liminf_{r \to \infty} \frac{M(r; |\boldsymbol{y}|)}{\log r} \leqq \liminf_{r \to \infty} \frac{\log [M(r; |\boldsymbol{x}|) + O(\log r)]}{\log r}$$

$$= \liminf \frac{\log M(r; |\boldsymbol{x}|)}{\log r} \leqq c .$$

Hence for every $\varepsilon > 0$ there is a sequence $\{r_k\}$, with $r_k \to \infty$, for which

(84) $$\lim_{k \to \infty} M(r_k; |\boldsymbol{y}|) = L < r^{c+\varepsilon} .$$

But from (24) we obtain

(85) $$\frac{1}{2\pi} \int_0^{2\pi} |\boldsymbol{y}(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} (|\boldsymbol{a}_k|^2 + |\boldsymbol{b}_k|^2) r^{2k} .$$

Now (84) and (85) imply that there is a $k_0 < \infty$ such that for all $k > k_0$ we have

$$|\boldsymbol{a}_k|^2 + |\boldsymbol{b}_k|^2 = 0 .$$

Thus $\boldsymbol{y}(u, v)$ is a polynomial and therefore (25) reduces to the form (26); that is, the nonconstant meromorphic minimal surface S is a rational or logarithmico-rational minimal surface of degree $m > 0$.

We shall now show that the constant c in (78) is a positive integer by showing that $c = m$, and shall at the same time prove the second part of the theorem. We shall in fact prove more:

If S is a rational or logarithmico-rational minimal surface of degree $m > 0$, then

$$\lim_{r \to \infty} \frac{T^\circ(r; S)}{\log r}$$

exists, and

(86) $$\lim_{r \to \infty} \frac{T^\circ(r; S)}{\log r} = m .$$

With S in the representation (26), and with m given by (32), by (38) we have

(87) $$\lim_{r \to \infty} \frac{N(r, \infty; S)}{\log r} = \sum_{j=1}^{l} n_j .$$

We now need an improvement of (82). Since, by (36) and (75), we have

(88) $$\log^+ M(r; |\boldsymbol{x}|) \geqq m(r, \infty; S) ,$$

it follows that (82) holds also with the inequality sign reversed, and therefore that

(89) $$\liminf_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} = \liminf_{r \to \infty} \frac{m(r, \infty; S)}{\log r} .$$

In the same way, with

$$\limsup_{r \to \infty} \quad \text{in place of} \quad \liminf_{r \to \infty}$$

in (81) and (82), from (88) we obtain

(90) $$\limsup_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} = \limsup_{r \to \infty} \frac{m(r, \infty; S)}{\log r} .$$

By (26), however, we have

(91) $$\lim_{r \to \infty} \frac{\log^+ M(r; |\boldsymbol{x}|)}{\log r} = n_\infty .$$

Therefore, by (89), (90), and (91),

(92) $$\lim_{r \to \infty} \frac{m(r, \infty; S)}{\log r} = n_\infty .$$

Thus, from (49), (87), (92), and a remark near the end of §7, we have

$$\lim_{r \to \infty} \frac{T^\circ(r; S)}{\log r} = \lim_{r \to \infty} \frac{T(r; S)}{\log r}$$

$$= \lim_{r \to \infty} \frac{m(r, \infty; S)}{\log r} + \lim_{r \to \infty} \frac{N(r, \infty; S)}{\log r}$$

$$= n_\infty + \sum_{j=1}^{l} n_j = m ,$$

as desired.

10. The fundamental theorem of algebra. If S is a rational or logarithmico-rational minimal surface of degree $m > 0$, given in the representation (26), then the vector function $\boldsymbol{x}(u, v)$ maps every point of the closed w, or $u + iv$, plane onto a unique point $\boldsymbol{a}$, finite or infinite.

Let $n_a \geqq 0$ be the order of the $\boldsymbol{a}$-point of S at $w = \infty$ (of course, $n_a = 0$ for all points $\boldsymbol{a}$ except one), and let

(93) $$n(\boldsymbol{a}; S) = n_a + \lim_{r \to \infty} n(r, \boldsymbol{a}; S) ,$$

(94) $$h(\boldsymbol{a}; S) = \lim_{r \to \infty} h(r, \boldsymbol{a}; S) \, .$$

Then $n(\boldsymbol{a}; S)$ is the total number of $\boldsymbol{a}$-points, counting multiplicities, of S in the closed w-plane, and $h(\boldsymbol{a}; S)$ is a measure of the visibility of the entire surface S when viewed from $\boldsymbol{a}$.

In particular, for $\boldsymbol{a} = \infty$, by (26) and (32) we have

$$n(\infty; S) = n_\infty + \sum_{j=1}^{l} n_j = m \, ,$$

and by (44) we have

$$h(\infty; S) = 0 \, ,$$

so that

(95) $$n(\infty; S) + h(\infty; S) = m \, .$$

We shall show now that (95) holds not only for ∞ but also for every finite $\boldsymbol{a}$; that is, we shall show that

(96) $$n(\boldsymbol{a}; S) + h(\boldsymbol{a}; S) = m \, .$$

Let $\boldsymbol{a}$ be any finite point. By (37):
(i) If $n_a = 0$ and S has a pole at ∞, then

$$m^\circ(r, \boldsymbol{a}; S) = \log (1 + |\boldsymbol{a}|^2) + O(1) \, .$$

(ii) If $n_a = 0$ and S has a $\boldsymbol{b}$-point, $\boldsymbol{b} \neq \boldsymbol{a}$, at ∞, then

$$m^\circ(r, \boldsymbol{a}; S) = \log \frac{(1 + |\boldsymbol{b}|^2)^{1/2}(1 + |\boldsymbol{a}|^2)^{1/2}}{|\boldsymbol{b} - \boldsymbol{a}|} + O(1) \, .$$

(iii) If $n_a > 0$, then

$$m^\circ(r, \boldsymbol{a}; S) = \log r^{n_a} + O(1) \, .$$

In each of the three cases, we have

(97) $$\lim_{r \to \infty} \frac{m^\circ(r, \boldsymbol{a}; S)}{\log r} = n_a \, ,$$

since $n_a = 0$ in the first two cases.

By (52), (55), (86), and (97), we therefore have

(98) $$\lim_{r \to \infty} \frac{N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S)}{\log r} = m - n_a \, .$$

We shall now show that

(99) $$\lim_{r \to \infty} [n(r, \boldsymbol{a}; S) + h(r, \boldsymbol{a}; S)] = m - n_a \, .$$

Since $n(r, a; S)$ and $h(r, a; S)$ are nondecreasing functions of r, if (99) did not hold then either there would be an $\varepsilon > 0$ such that

$$n(r, a; S) + h(r, a; S) \leqq m - n_a - \varepsilon$$

for all r, or there would be an $\varepsilon > 0$ and an r_0 such that

$$n(r, a; S) + h(r, a; S) \geqq m - n_a + \varepsilon$$

for all $r \geqq r_0$. In the former case, by (38) and (45), we would have

$$\varlimsup_{r \to \infty} \frac{N(r, a; S) + H(r, a; S)}{\log r} \leqq m - n_a - \varepsilon \ ,$$

and similarly in the latter case we would have

$$\varliminf_{r \to \infty} \frac{N(r, a; S) + H(r, a; S)}{\log r} \geqq m - n_a + \varepsilon \ .$$

Hence (99) follows from (98).

From (93), (94), and (99), we obtain

$$n(a; S) + h(a; S) = n_a + \lim_{r \to \infty} [n(r, a; S) + h(r, a; S)]$$
$$= n_a + (m - n_a) = m \ .$$

We have thus proved the following result:

THEOREM 3. *If S is a rational or logarithmico-rational minimal surface of degree $m > 0$ in the extended E_n-space, then for each a, including $a = \infty$, in this space,*

(100) $$n(a; S) + h(a; S) = m \ .$$

Since m is a positive integer, since $n(a; S)$ is a nonnegative integer and $h(a; S)$ is nonnegative, and since $h(a; S) = 0$ if and only if either $a = \infty$ or S is a plane surface with a on S, it follows from (100) that:

(i)   For each a, both $n(a; S)$ and $h(a; S)$ are nonnegative integers.

(ii) $n(a; S) \leqq m$, with equality if and only if either $a = \infty$ or S is a plane with a on S.

(iii) $h(a; S) \leqq m$, with equality for almost all points of E_n if $n > 2$.

(iv)   Theorem 3 includes the fundamental theorem of algebra as a special case.

Since many proofs of the fundamental theorem of algebra are known, alternative proofs of Theorem 3 might also be desirable. In particular, $h(a; s)$ will shortly be interpreted, at least for $n = 3$, in terms of areas on a sphere with center at a, and this well might lead to a topological proof of the theorem.

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