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MAXIMAL PURE SUBGROUPS OF TORSION COMPLETE ABELIAN p-GROUPS

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Let N be the set of nonnegative integers, and let $B = \Sigma \oplus [b_i](i \in N)$ be the direct sum of cyclic groups with $0(b_i) = p^{i+1}$. Denote by $\bar{B}$ the torsion-completion of B. This paper is concerned with pure subgroups of the group $\bar{B}$. If G is such a group, let

$$I(G) = \{i \mid i^{\text{th}} \text{ Ulm invariant of } G \text{ is nonzero}\} .$$

Beaumont and Pierce introduced a further invariant for G, namely,

$$U(G) = \{I(A) \mid A \text{ is a pure torsion-complete subgroup of } G\} .$$

$U(G)$ is a (boolean) ideal in $\mathscr{P}(N)$, the power set of N.

If $\mathscr{I}$ is an ideal in $\mathscr{P}(N)$, then the canonical example of a pure subgroup, G, of $\bar{B}$ with $U(G) = \mathscr{I}$ is constructed as follows:

$$G = \mathscr{G}(\mathscr{I}) = \Sigma A_I (I \in \mathscr{I}) \text{ where } A_I \text{ is the torsion-completion of } \Sigma \oplus [b_i](i \in I) .$$

Beaumont and Pierce showed that if $\mathscr{P}(N)/\mathscr{I}$ has no atoms and $\mathscr{I}$ is free, then there exist maximal pure subgroups G of $\bar{B}$ such that $G \supset \mathscr{G}(\mathscr{I})$ and $U(G) = \mathscr{I}$. The purpose of this paper is to give necessary and sufficient conditions for the existence of such a G in the case that $\mathscr{P}(N)/\mathscr{I}$ is finite. In the process, some information is obtained about the number of nonisomorphic extensions of $\mathscr{G}(\mathscr{I})$.

I. **Preliminaries.** For the basic background on p-groups without elements of infinite height see [2] and [3]. The groups G that we consider in this paper will all be pure subgroups of $\bar{B}$, where B is a standard basic subgroup as above. The following definitions and facts may be found in [1].

(i) DEFINITION. $I(G) = \{n \mid n\text{th Ulm invariant of } G \text{ is not zero}\}$.

(ii) DEFINITION. *If* $x \in \bar{B}$ and $x = \Sigma r_i b_i (i \in N)$, then $\delta(x) = \{i \mid r_i b_i \neq 0\}$.

(iii) PROPOSITION. *If $\mathscr{I}$ is an ideal in $\mathscr{P}(N)$, then* $\mathscr{G}(\mathscr{I}) = \{x \in \bar{B} \mid \delta(x) \in \mathscr{I}\}$.

(iv) PROPOSITION. *$\mathscr{G}(\mathscr{I})$ is a pure subgroup of $\bar{B}$ and* $U(\mathscr{G}(\mathscr{I})) = \mathscr{I}$.

(v) PROPOSITION. *If $\mathscr{I}$ contains all finite subsets of N (such an ideal is called free) and is maximal in $\mathscr{P}(N)$, then $\mathscr{G}(\mathscr{I})$ is a maximal pure subgroup of $\bar{B}$.*

In [1] Beaumont and Pierce give an example to show that it is not always possible to extend $\mathscr{G}(\mathscr{I})$ to a maximal pure subgroup G

with $U(G) = \mathscr{I}$, when $\mathscr{I}$ is the intersection of two maximal ideals. It turns out that this is the case which causes most of the difficulties, and the majority of the paper is devoted to showing that their example is typical of the situation where no such G exists.

II. Throughout this section $\mathscr{I}$ will be the intersection of two maximal free ideals. Let $\mathscr{V}$ and $\mathscr{W}$ be distinct maximal free ideals of $\mathscr{P}(N)$ and let $\mathscr{I} = \mathscr{V} \cap \mathscr{W}$. Let $V \in \mathscr{V} - \mathscr{I}$ and let $W = N - V$. Then $W \in \mathscr{W} - \mathscr{I}$ and by the maximality of $\mathscr{V}$ and $\mathscr{W}$ we have $\mathscr{V} = [V, \mathscr{I}]$ and $\mathscr{W} = [W, \mathscr{I}]$. Note that $\mathscr{P}(V) \cap \mathscr{I}$ and $\mathscr{P}(W) \cap \mathscr{I}$ are maximal ideals of $\mathscr{P}(V)$ and $\mathscr{P}(W)$ respectively, and that $\mathscr{G}(\mathscr{I}) = \mathscr{G}(\mathscr{V}) \cap \mathscr{G}(\mathscr{W})$.

Our purpose in this section is to give a necessary and sufficient condition for a group G with $\mathscr{G}(\mathscr{I}) \subset G \subset \bar{B}$ and $G/\mathscr{G}(\mathscr{I}) \cong Z_p(\infty)$ to be of the form $\mathscr{G}(\mathscr{V})$ or $\mathscr{G}(\mathscr{W})$.

II. A. NOTATION.

(i) Let A_1 be the closure in $\bar{B}$ of $\Sigma \bigoplus [b_i](i \in V)$.
Let A_2 be the closure in $\bar{B}$ of $\Sigma \bigoplus [b_i](i \in W)$.

(ii) $$G_1 = \mathscr{G}(\mathscr{P}(V) \cap \mathscr{I})).$$
$$G_2 = \mathscr{G}(\mathscr{P}(W) \cap \mathscr{I})).$$

(iii) $$v_n = \Sigma p^{i-n+1} b_i (i \in V \text{ and } i \geqq n-1).$$
$$w_n = \Sigma p^{i-n+1} b_i (i \in W \text{ and } i \geqq n-1).$$

(iv) $\mathscr{H}_1 = \{G \mid G = \mathscr{G}(\mathscr{I}) + [\{u_n \mid n \in N\}]$, where $u_n - pu_{n+1} \in \mathscr{G}(\mathscr{I})$ and $u_n = v_n + t_n w_n$ with $0 \leqq t_n < p^n\}$.

$\mathscr{H}_2 = \{G \mid G = \mathscr{G}(\mathscr{I}) + [u_n \mid n \in N]$ where $u_n - pu_{n+1} \in \mathscr{G}(\mathscr{I})$ and $u_n = w_n + s_n v_n$ with $0 \leqq s_n < p^n\}$.

The following proposition records the obvious connections between these objects.

II. B. PROPOSITION.

(i) $$A_1 \bigoplus A_2 = \bar{B};\ G_1 \bigoplus G_2 = \mathscr{G}(\mathscr{I}).$$

(ii) $$A_1 = [\{v_n \mid n \in N\}] + G_1.$$
$$A_2 = [\{w_n \mid n \in N\}] + G_2.$$

(iii) $$A_i/G_i \cong Z_p(\infty) \quad \text{for} \quad i = 1, 2.$$

(iv) $$\mathscr{G}(\mathscr{V}) = [\{v_n \mid n \in N\}] + \mathscr{G}(\mathscr{I}).$$
$$\mathscr{G}(\mathscr{W}) = [\{w_n \mid n \in N\}] + \mathscr{G}(\mathscr{I}).$$

(v) *G is a pure subgroup of $\bar{B}$ with $G/\mathscr{G}(\mathscr{I}) = Z_p(\infty)$ iff $G \in \mathscr{H}_1 \cup \mathscr{H}_2$.*

At this point we have explicitly realized $\bar{B}/\mathscr{G}(\mathscr{I})$ as $A_1/G_1 \bigoplus A_2/G_2$ and have definite sets of representatives for $A_1/G_1 \cong Z_p(\infty)$ and $A_2/G_2 \cong Z_p(\infty)$. By I.B.v. the groups G that we are interested in are obtained by taking a rank 1 summand of $A_1/G_1 \bigoplus A_2/G_2$ and adding its representatives to $\mathscr{G}(\mathscr{J})$. Any such summand D will be complementary to either A_1/G_1 or A_2/G_2 (or both). Now if D is complementary to A_2/G_2, for example, let π_1 and π_2 be the projections into A_1/G_1 and A_2/G_2, respectively, with respect to the decomposition $A_1/G_1 \oplus A_2/G_2$. Then, of course, for $d \in D$ we have $d = \pi_1(d) + \pi_2(d)$ and ϕ defined by $\phi(\pi_1(d)) = \pi_2(d)$ is an element of $\mathrm{Hom}\,(A_1/G_1, A_2/G_2)$. In fact, there is a one-to-one correspondence between $H_1 = \mathrm{Hom}\,(A_1/G_1, A_2/G_2)$ and $\mathscr{H}_1$ and $H_2 = \mathrm{Hom}\,(A_2/G_2, A_1/G_1)$ and $\mathscr{H}_2$. The following definition and proposition set forth the precise situation.

II. C. *The Correspondence.*

(i) Let $\phi \in H_1$, then $\phi(v_n + G_1) = t_n w_n + G_2$. Let $u_n = v_n + t_n w_n$ and define $G[\phi] = \mathscr{G}(\mathscr{J}) + [\{u_n \mid n \in N\}]$.

(ii) If $G \in \mathscr{H}_1$ with $G = \mathscr{G}(\mathscr{J}) + [\{v_n + t_n w_n \mid n \in N\}]$, then define $\phi[G] \in H_1$ by $\phi[G](v_n + G_1) = t_n w_n + G_2$.

II. D. PROPOSITION. *For $\phi \in H_1$ and $G \in \mathscr{H}_1$*

(i) *$G[\phi]$ is a uniquely determined element of $\mathscr{H}_1$.*

(ii) *$\phi[G]$ is a uniquely determined element of H_1.*

(iii) $G[\phi[G]] = G$.

(iv) $\phi[G[\phi]] = \phi$.

By interchanging the roles of the w_n and v_n we get a similar one-to-one correspondence between H_2 and $\mathscr{H}_2$. In fact, H_1 and H_2 coordinatize $\mathscr{H}_1$ and $\mathscr{H}_2$ with some overlap as the following proposition makes clear.

II. E. PROPOSITION. *If ϕ and ψ are distinct elements of $H_1 \cup H_2$, then $G[\phi] = G[\psi]$ iff ϕ and ψ are isomorphisms with $\phi = \psi^{-1}$.*

Proof. This is the case where the summand defining $G[\psi]$ is complementary to both A_1/G_1 and A_2/G_2.

We are now ready for the fundamental definition and theorem of this section.

II. F. DEFINITION. Let S_1, S_2, T_1, T_2 be abelian groups and let $\phi \in \mathrm{Hom}\,(S_1/S_2, T_1/T_2)$. We say that ϕ is liftable if there is a $\Phi \in \mathrm{Hom}\,(S_1, T_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} S_1 & \xrightarrow{\ \Phi\ } & T_1 \\ \downarrow & & \downarrow \\ S_1/S_2 & \xrightarrow[\ \phi\ ]{} & T_1/T_2 \, . \end{array}$$

II. G. THEOREM. *Let $G \in \mathscr{H}_1$ with $G = G[\phi]$ for $\phi \in H_1$. Then*

$G = \mathscr{G}(\mathscr{V})$ if and only if ϕ is liftable. A similar theorem holds for $G \in \mathscr{H}_2$.

Proof. Clearly $G = \mathscr{G}(\mathscr{V})$ if and only if there is a pure torsion-complete subgroup A of G with $I(A) = V$.

(i) If ϕ is liftable, then $\Phi \in \operatorname{Hom}(A_1, A_2)$ by definition of liftable. Now Φ can be thought of as an endomorphism of $\bar{B}$ by taking $\Phi(A_2) = 0$. With this understanding, Φ is height increasing on $\bar{B}[p]$, since if $k \in W$, $\Phi(p^k b_k) = 0$, and if $k \in V$, then $\delta(\Phi(p^k b_k)) \subset W$, so $k \notin \delta(\Phi(p^k b_k))$. Let $A = (1 + \Phi)(A_1)$. Then $A \oplus A_2 = \bar{B}$, so A is a pure torsion-complete with $I(A) = V$. If $x \in A_1$, then $x = g + rv_n$, where $g \in G_1$ and $0 \leqq r < p^n$ for some n. Consequently, $(1 + \Phi)(x) \in G$ since Φ liftable implies $\Phi(G_1) \subset G_2$ and $v_n + \Phi(v_n) \in G$. It follows that $A \subset G$.

(ii) Assume that there exists a pure torsion-complete A with $A \subset G$ and $I(A) = V$. Since $U(A) \cap U(G_2) = \mathscr{P}(V) \cap (W \cap \mathscr{I})$ is empty, we have $A + G_2$ is pure and the sum is direct. We claim that, in fact, $A \oplus G_2 = G$. To see this note that $U(A \oplus G_2) = \mathscr{P}(V) + (W \cap \mathscr{I})$ so $U(A \oplus G_2)$ is free. Hence any basic subgroup of $A \oplus G_2$ will be a basic subgroup of $\bar{B}$. One such basic subgroup is $B' = B_1 \oplus B_2$ where B_1 is a basic subgroup of A and $B_2 = B \cap G_2$. One sees easily that with respect to B', $A \oplus G_2$ is of the form $\mathscr{G}(\mathscr{V})$. Hence $\bar{B}/(A \oplus G_2) = Z_p(\infty)$ by 4.2 in [1]. And since $A \oplus G_2 \subseteqq G$, $\bar{B}/G = Z_p(\infty)$, and both G and $A \oplus G_2$ are pure in $\bar{B}$, we have $A \oplus G_2 = G$. Now $\bar{B} = A \oplus A_2$ since $I(A \oplus A_2) = N$. Let Φ be $-(\pi | A_1)$ where π is the projection of $\bar{B}$ on A_2 associated with this decomposition of $\bar{B}$. To show that Φ is a lifting of ϕ we must show the following two things:

(a) $\Phi(G_1) \subseteqq G_2$. This is clear because $G = A \oplus G_2$ implies $(A \oplus G_2) \cap A_1 \supseteqq \mathscr{G}(\mathscr{I}) \cap A_1 = G_1$ and Φ maps into A_2 which is the closure in $\bar{B}$ of G_2.

(b) $\Phi(v_n) \equiv t_n w_n \bmod G_2$ where $G = \mathscr{G}(\mathscr{I}) + [\{v_n + t_n w_n \mid n \in N\}]$. Because $v_n + \Phi(v_n) \in A \subset G$, it follows that $(v_n + \Phi(v_n)) - (v_n + t_n w_n) = \Phi(v_n) - t_n w_n \in G$. Therefore, since $\delta(\Phi(v_n)) \subset W$, $\Phi(v_n) - t_n w_n \in G \cap A_2 = G_2$; that is, $\Phi(v_n) \equiv t_n w_n \bmod G_2$.

II. G. Remark. Note that it is not possible for $G = G[\phi] \in \mathscr{H}_1$ to be isomorphic to $\mathscr{G}(\mathscr{W})$ if $\operatorname{Ker} \phi > 0$. This is the case because $G \cong \mathscr{G}(\mathscr{W})$ implies that there is a pure $A \subset \mathscr{G}(\mathscr{W})$ with $A \cong A_2$ and $\operatorname{Ker} \phi > 0$ implies that $A_1[p] \subset G$ which would give $G[p] \subset A_1[p] + A[p] = \bar{B}[p]$. The purity of G would now give $G = \bar{B}$, a contradiction. If $\operatorname{Ker} \phi = 0$, then we might have $G \cong \mathscr{G}(\mathscr{W})$, this will occur if $\phi^{-1} \in H_2$ is liftable. Therefore, this theorem together with the corresponding theorem for $G \in \mathscr{H}_2$ give a necessary and sufficient condition for G to be isomorphic to $\mathscr{G}(\mathscr{V})$ or $\mathscr{G}(\mathscr{W})$.

III. In this section we state the main theorems of the paper. The notation remains the same as that of II.

III. A. DEFINITION. For $\phi \in H_1 \cup H_2$ let $K(\phi) = n$ if $|\mathrm{Ker}\,\phi| = p^n$.

III. B. PROPOSITION. *Let $\phi, \psi \in H_1$. If $K(\phi) = K(\psi)$, then $G[\phi]$ is isomorphic to $G[\psi]$ and similarly for H_2.*

Proof. $G[\phi] = \mathscr{G}(\mathscr{I}) + [\{v_n + t_n w_n \mid n \in N\}]$ and $G[\psi] = \mathscr{G}(\mathscr{I}) + [\{v_n + s_n w_n \mid n \in N\}]$. Let $m = K(\phi) = K(\psi)$. Then $t_n = p^m t'_n \bmod p^n$ and $s_n = p^m s'_n \bmod p^n$ for $(t'_n, p) = 1$ and $(s'_n, p) = 1$ for $n > m$. Let r_n be such that $r_n t'_n = s'_n \bmod p^n$. Define α by:

$$\alpha(b_k) = b_k \quad \text{for} \quad k \in V \cup \{h \in W \mid h < m - 1\}$$
$$\alpha(b_h) = r_{h+1} b_h \quad \text{for} \quad h \in W \quad \text{and} \quad h \geqq m - 1\,.$$

III. C. DEFINITION.

(i) $n_1(\mathscr{V}, \mathscr{W}) = \min\{K(\phi) \mid \phi \in H_1,\ U(G[\phi]) = \mathscr{V}\}$ if this exists, and ∞ otherwise.

(ii) $n_2(\mathscr{V}, \mathscr{W}) = \min\{K(\phi) \mid \phi \in H_2,\ U(G[\phi]) = \mathscr{W}\}$ if this exists, and ∞ otherwise.

III. D. PROPOSITION. *Let $\phi \in H_i$. Then ϕ is liftable if $\infty > K(\phi) \geqq n_i(\mathscr{V}, \mathscr{W})$.*

Proof. Let $a = n_1(\mathscr{V}, \mathscr{W})$. By III. B. we have $G[\phi] \cong \mathscr{G}(\mathscr{V})$ for every $\phi \in H_1$ with $K(\phi) = a$. By II. G. every such ϕ is liftable. If $K(\phi) > a$, then there is a $\psi \in H_1$ with
$p^{K(\phi)-a}\psi = \phi$ and $K(\psi) = a$. Then $p^{K(\phi)-a}\Psi = \Phi$ is a lift of ϕ.

III. E. DEFINITION. If $I \subset N$ and n any integer, then we write $I - n$ for $\{i - n \mid i \in I\} \cap N$. If $\mathscr{V}$ is an ideal of $\mathscr{P}(N)$, then $\mathscr{V}^n = \{I - n \mid I \in \mathscr{V}\}$.

III. F. PROPOSITION. *Let $\mathscr{V}$ be a maximal free ideal of $\mathscr{P}(N)$. Then*

(i) *$\mathscr{V}^n$ is a maximal free ideal of $\mathscr{P}(N)$.*

(ii) *If $n \neq m$, then $\mathscr{V}^n \neq \mathscr{V}^m$.*

(iii) *$n_1(\mathscr{V}, \mathscr{V}^m) = m$; $n_2(\mathscr{V}, \mathscr{V}^m) = 0$.*

Proof. (i) Clearly $\mathscr{V}^n$ is free. If $V \subseteqq N$ and $V \notin \mathscr{V}^n$, then $V + n \notin \mathscr{V}$. Since $N - (V + n \cup (N - V) + n)$ is finite we have $(N - V) + n \in \mathscr{V}$ by maximality of $\mathscr{V}$. Therefore, $N - V \in \mathscr{V}^n$ by definition and $\mathscr{V}^n$ is maximal.

(ii) If $I \notin \mathscr{V}$, then $I - m \notin \mathscr{V}^m$ and $I - n \notin \mathscr{V}^n$. Consequently, if $\mathscr{V}^m = \mathscr{V}^n$, then $I - m \cap I - n \notin \mathscr{V}^m = \mathscr{V}^n$ where $I - m \cap I - n = \{k - m \mid \text{there is a } k' \in I \text{ with } k - m = k' - n\} = \{k - m \mid \text{there exists } k' \in I \text{ with } k - k' = m - n\}$. However, since $\mathscr{V}$ is free, there exist I's such that $I \notin \mathscr{V}$ and having the property that if $k, k' \in I$ with $k \neq k'$, then $|k - k'| > m - n$. For such an I we actually have $I - m$ and $I - n$ disjoint.

(iii) Let $V \subset N$ be such that $V \notin \mathscr{V}$ and $V \cap (V - m)$ is empty. Let $W = (V - m) \cup (N - V)$, $\mathscr{W} = \mathscr{V}^m$ and use the notation of § 2.

Let $\phi \in H_2$ with $\phi(w_n + G_2) = v_n + G_1$. A computation shows that the following Φ is a lift of ϕ:

$\Phi(b_j) = p^m b_{j+m}$ for $j \in V - m$ and $\Phi(b_j) = 0$ for $j \in W - (V - m)$. For this Φ we have $K(\phi) = 0$, so $n_2(\mathscr{V}, \mathscr{V}^m) = 0$.

Let $\phi \in H_1$ with $\phi(v_n + G_1) = p^m w_n + G_2$. Define a lifting Φ of ϕ as follows.

$\Phi(b_i) = b_{i-m}$ for $i \in V$ and $i \geqq m$

$\Phi(b_i) = 0$ for $i \in V$ and $i < m$.

Once again, a straightforward check shows that Φ is a lift of ϕ. Hence, $n_1(\mathscr{V}, \mathscr{V}^m) \leqq m$. By III. G. (ii) below, if $n = n_1(\mathscr{V}, \mathscr{V}^m) < m$, we have $W = \mathscr{V}^n = \mathscr{V}^m$ in contradiction to III. F. (ii).

We now show that if $\mathscr{G}(\mathscr{I})$ possesses at least one extension of the form $\mathscr{G}(\mathscr{V})$ and at least one of the form $\mathscr{G}(\mathscr{W})$ (other than the ones corresponding to the zero homomorphisms in H_1 and H_2), then $\mathscr{W} = \mathscr{V}^n$ for some n, or vice-versa.

III. G. THEOREM. *Let $\mathscr{V}$ and $\mathscr{W}$ be maximal free ideals. If $n_1(\mathscr{V}, \mathscr{W}) < \infty$ and $n_2(\mathscr{V}, \mathscr{W}) < \infty$, then*

(i) *$n_i(\mathscr{V}, \mathscr{W}) = 0$ for one of $i = 1$ or $i = 2$.*

(ii) *If $n_2(\mathscr{V}, \mathscr{W}) = 0$, then $\mathscr{W} = \mathscr{V}^n$, where $n = n_1(\mathscr{V}, \mathscr{W})$.*

Proof. Deferred to § IV.

III. H. COROLLARY. *If $\mathscr{V}$ and $\mathscr{W}$ are maximal free ideals with $\mathscr{I} = \mathscr{V} \cap \mathscr{W}$, then there does not exist a G with $\mathscr{G}(\mathscr{I}) \subset G$, $G/\mathscr{G}(\mathscr{I}) \cong Z_p(\infty)$, and $U(G) = \mathscr{I}$ if and only if $\mathscr{W} = \mathscr{V}^1$ or $\mathscr{V} = \mathscr{W}^1$.*

Proof. (i) If $\mathscr{W} = \mathscr{V}^1$, then $n_1(\mathscr{V}, \mathscr{W}) = 0$ and $n_2(\mathscr{V}, \mathscr{W}) = 1$ by III. F., so $G \in \mathscr{H}_1$ implies $G \cong \mathscr{G}(\mathscr{V})$ and $G \in \mathscr{H}_2 - \mathscr{H}_1$ implies $G \cong \mathscr{G}(\mathscr{W})$.

(ii) If the G mentioned does not exist, then for every $G \in \mathscr{H}_1 \cap \mathscr{H}_2$ we have either $G \cong \mathscr{G}(\mathscr{V})$ or $G \cong \mathscr{G}(\mathscr{W})$. Hence, one of the $n_i(\mathscr{V}, \mathscr{W})$ is zero and the other is 1. By III. G. we have either $\mathscr{V} = \mathscr{W}^1$ or $\mathscr{W} = \mathscr{V}^1$.

III. I. THEOREM. *If $\mathscr{V}_1, \cdots, \mathscr{V}_n$ are distinct maximal free ideals* *and* $\mathscr{J} = \bigcap \mathscr{V}_i (1 \leqq i \leqq n)$, *then there is a* G, *pure in* $\bar{B}$, *with* $\mathscr{G}(\mathscr{J}) \subset G$, $U(G) = \mathscr{J}$ *and* $\bar{B}/G \cong Z_p(\infty)$ *if and only if there is no pair* i, j *with* $\mathscr{V}_j = \mathscr{V}_i^1$.

Proof. As in the case of $n = 2$ we may choose I_i such that $I_i \in \mathscr{V}_i$, $N = \bigcup I_i (1 \leqq i \leqq n)$ with I_i and I_j disjoint for $i \neq j$. Define A_i as the torsion-completion of $\Sigma \bigoplus [b_j] (j \in I_i)$ and G_i as $\mathscr{G}(\mathscr{P}(I_i) \cap \mathscr{J})$. For each $k \in N$ and each $1 \leqq i \leqq n$ let $v_{i,k} = \Sigma p^{j-k+1} b_j (j \in I_i)$. Then $\{v_{i,k} \mid k \in N\}$ is a canonical set of generators for $A_i/G_i \cong Z_p(\infty)$. Let $H_{ij} = \operatorname{Hom}(A_i/G_i, A_j/G_j)$ and $\mathscr{H}_{ij} = \{G \mid G$ is pure in $A_i \bigoplus A_j$ with $(A_i \bigoplus A_j)/G \cong Z_p(\infty)$ and $G \supset G_i \bigoplus G_j\}$.

Since our earlier theorems could have been stated and proved for a standard subbasic we know that H_{ij} coordinatizes $\mathscr{H}_{ij}$, that if $G \in \mathscr{H}_{ij}$ then $G \cong A_i \bigoplus G_j$ if and only if the associated $\phi \in H_{ij}$ is liftable, and that Theorem III. G. holds.

We can realize $\bar{B}/\mathscr{G}(\mathscr{J}) = D$ as $\Sigma A_i/G_i (1 \leqq i \leqq n)$. Observe that if G is a pure subgroup of $\bar{B}$ with $\bar{B}/G \cong Z_p(\infty)$ such that $G \supset \mathscr{G}(\mathscr{J})$, then G is obtained by taking a rank $n-1$ summand D_G of D and adding a set of representatives for D_G to $\mathscr{G}(\mathscr{J})$. Since D_G is of rank $n-1$ there must be at least one summand of the form A_i/G_i, with $D = (A_i/G_i) \bigoplus D_G$. For $j \neq i$ let $\phi_{j,i} = -\pi_{j,i}$ where $\pi_{j,i}$ is the projection of A_j/G_j into A_i/G_i associated with this decomposition. Then $D_G = \Sigma \bigoplus Z_j (j = 1, \cdots, n$ and $j \neq i)$ where a complete set of generators for Z_j is $\{v_{j,k} + \phi_{j,i}(v_{j,k}) \mid k \in N\}$. Alternatively if $S_{j,i} \in \mathscr{H}_{j,i}$ is the group associated with $\phi_{j,i} \in H_{j,i}$, then G is the group generated by $\{S_{j,i}\}$.

In fact, G will contain other groups of the form $S_{j,h} \in \mathscr{H}_{j,h}$. If $i \neq j \neq h \neq i$ and $K(\phi_{j,i}) \geqq K(\phi_{h,i})$, then let $\phi_{j,h} \in H_{j,h}$ be the map defined by $\phi_{j,h}(v_{j,k}) = r_k v_{h,k}$ if $\phi_{j,i}(v_{j,k}) = r_k \phi_{h,i}(v_{h,k})$. If $S_{j,h} \in \mathscr{H}_{j,h}$ is the group associated with $\phi_{j,h}$, then we have $S_{j,h} \subset G$. It follows that for every pair j, h with $1 \leqq j, h \leqq n$, G contains an element of either $\mathscr{H}_{j,h}$ or $\mathscr{H}_{h,j}$. Consequently, in view of III. H., if $U(G) = \mathscr{J}$, then we cannot have $\mathscr{V}_i = \mathscr{V}_j^1$ for any pair i, j.

Suppose now that we do not have $\mathscr{V}_i = \mathscr{V}_j^1$ for any pair i, j. Then for every pair i, j either $n_1(\mathscr{V}_i, \mathscr{V}_j) \geqq 2$ or $n_2(\mathscr{V}_i, \mathscr{V}_j) \geqq 2$. A simple combinational argument shows we may assume that $n_1(\mathscr{V}_i, \mathscr{V}_j) \geqq 2|j-i|$. For $i > 1$ choose $\phi_{i,1} \in H_{i,1}$ with $K(\phi_{i,1}) = i - 1$ and let $S_{i,1} \in \mathscr{H}_{i,1}$ be the group associated with $\phi_{i,1}$.

If G is generated by $\{S_{i,1} \mid i = 2, \cdots, n\}$, then G is pure, $\bar{B}/G \cong Z_p(\infty)$ and $G \supset \mathscr{G}(\mathscr{J})$. We claim that $U(G) = \mathscr{J}$. If not, then $I_i \in U(G)$ for some i. Note that $I_i \neq I_1$ since this would mean that there existed a pure torsion-complete A with $A \subset G$ and $I(A) = I_1$. However, $A \bigoplus A_2 \bigoplus \cdots \bigoplus A_n = \bar{B}$ and $(A_2 \bigoplus \cdots \bigoplus A_n)[p] \subset G$ by construction, so $A \subset G$ would imply $G[p] = \bar{B}[p]$. Since G is pure this would give

$G = \bar{B}$, a contradiction.

As noted above, for every pair k, j, G contains a subgroup of $\mathscr{H}_{k,j}$ or $\mathscr{H}_{j,k}$. Because of our assumed ordering of $\mathscr{V}_1, \cdots, \mathscr{V}_n$ we know that for $j < k$, $G \supset S_{k,j} \in \mathscr{H}_{k,j}$. Let T be the group generated by $\{S_{k,j} | k \neq i \neq j\}$. Then clearly $G = T \oplus S_{i,1}$. On the other hand, $U(T)$, $U(A)$, and $U(G_1)$ are pairwise disjoint, so $T \oplus A \oplus G_1$ is pure and since $(A_1 \oplus \cdots \oplus A_{i-1} \oplus A_{i+1} \oplus \cdots \oplus A_n)/(T \oplus G_1) \cong Z_p(\infty)$ we have $G = T \oplus A \oplus G_1$. Hence $S_{i,1} \cong A \oplus G_1$. But this contradicts our choice of $S_{i,1}$. Hence $U(G) = \mathscr{I}$.

IV. In this section we provide the proof of Theorem III. G.

For simplicity, the following remarks and proposition will be stated for $\phi \in H_1$ The case of $\phi \in H_2$ is exactly the same. The notation is that of II.

Let $\phi \in H_1$ and assume that Φ is a lift of ϕ, with $\Phi(b_k) = \Sigma\, m_{k,h} b_h (h \in W)$ for each $k \in V$. If r is any integer, then we can write $\Phi = \Phi_{1,r} + \Phi_{2,r}$, where $\Phi_{1,r}(b_k) = \Sigma m_{k,h} b_h (h \in W$ and $h < k + r)$ and $\Phi_{2,r}(b_k) = \Sigma m_{k,h} b_h (h \in W$ and $h \geqq k + r)$ with the understanding that either of these sums is zero if its index set is empty. Clearly $\Phi_{1,r}$ and $\Phi_{2,r}$ are elements of $\mathrm{Hom}\,(A_1, A_2)$.

IV. A. LEMMA. *$\Phi_{1,r}(G_1) \subseteqq G_2$ and $\Phi_{2,r}(G_1) \subseteqq G_2$.*

Proof. Since $\Phi(G_1) \subseteqq G_2$ we need only show that $\Phi_{1,r}(G_1) \subseteqq G_2$. Let $x \in G_1$ and assume $\Phi_{1,r}(x) \notin G_2$. Then for some $J \in \mathscr{V}$, $x = \Sigma m_k b_k (k \in J)$ with $m_k b_k \neq 0$. If $J_s = \{k \in J | 0(m_k b_k) = p^{s+1}\}$ and $x_s = \Sigma m_k b_k (k \in J_s)$, then $x = \Sigma x_s (0 \leqq s \leqq n-1)$ where $0(x) = p^n$. Since $\Phi_{1,r}(x) \notin G_2$, there exists an s such that $\Phi_{1,r}(x_s) \notin G_2$. Hence we may assume that $x = x_s$ and $J = J_s$.

Since $0(m_k b_k) = p^{s+1}$ for $k \in J$, we may write $m_k b_k = p^{k-s} l_k b_k$ where $(l_k, p) = 1$. It follows that

$$\Phi_{1,r}(m_k b_k) = \Sigma p^{k-s} l_k m_{k,h} b_h (k - s - 1 < h < k + r) \tag{1}$$

because $h \leqq k - s - 1$ implies $p^{k-s} b_h = 0$.

For $0 \leqq t < s + r + 1$ let $K_t = \{t + n(s + r + 1) | n \in N\} \cap J$. Since the K_t are pairwise disjoint and $J = \bigcup K_t (0 \leqq t < s + r + 1)$ we have $x = \Sigma x_t (0 \leqq t < s + r + 1)$ where $x_t = \Sigma m_k b_k (k \in K_t)$. It follows that there exists a t with $\Phi_{1,r}(x_t) \notin G_2$. Hence, we may assume that $x = x_t$ and $J = K_t$. As a consequence, if $j \in J$ and $k \in J$ with $j \neq k$, then j and k differ by at least $s + r + 1$ in absolute value and by (1) above $\delta(\Phi_{1,r}(m_j b_j))$ and $\delta(\Phi_{1,r}(m_k b_k))$ are disjoint.

We now know that the $\delta(\Phi_{1,r}(m_k b_k))$ are pairwise disjoint for $k \in J$ and that their cardinality is less than $s + r + 1$. Hence, $\delta(\Phi_{1,r}(x)) =$

$\bigcup \delta(\Phi_{1,r}(m_k b_k))(k \in J)$ and, since $\Phi_{1,r}(x) \notin G_2$, $\delta(\Phi_{1,r}(x)) \in \mathscr{P}(W) \cap \mathscr{F}$. Therefore, it is possible to choose one integer t_k from each $\delta(\Phi_{1,r}(m_k b_k))$ such that $\{t_k | k \in J\} \notin \mathscr{P}(W) \cap \mathscr{F}$. For $k \in J$ we define $n_k b_k$ inductively as follows:

(i) $n_k b_k = m_k b_k$ if k is the least element of $\mathscr{F}$.

(ii) $n_k b_k = 0$ if $w_k \in \delta(\Phi(\Sigma n_j b_j))$ for $j < k$ and $j \in J$

$n_k b_k = m_k b_k$ if $w_k \notin \delta(\Phi(\Sigma n_j b_j))$ for $j < k$ and $j \in J$.

Let

$$y = \Sigma n_k b_k (k \in J) .$$

Then $\delta(y) \subset J$ so $y \in G_1$. On the other hand, $\delta(\Phi(y)) \supset \{w_k | k \in J\}$ since $i \in \delta(\Phi(m_k b_k))$ implies $i > k - s - 1$ and $w_j \leqq k - s - 1$ for $j < k$ and $j \in J$. Hence $\Phi(y) \notin G_2$ and this is a contradiction to the assumption that Φ is a lift of ϕ.

IV. B. *Proof of III. G.* Let $n = n_1(\mathscr{V}, \mathscr{W})$ and $m = n_2(\mathscr{V}, \mathscr{W})$. Choose $\phi \in H_1$ with $\phi(v_k + G_1) = p^n w_k + G_2$ and $\psi \in H_2$ with $\psi(w_k + G_2) = p^m v_k + G_1$. Let Φ and Ψ be the lifts of ϕ and ψ. In the notation of Lemma IV. A., let $\Phi = \Phi_{1,1} + \Phi_{2,1}$ and $\Psi = \Psi_{1,1} + \Psi_{2,1}$. By IV. A., $\Phi_{2,1}$ and $\Psi_{2,1}$ induce maps $\phi_2 \in H_1$ and $\psi_2 \in H_2$. If $(\psi_2\phi_2)(v_k + G_1) = r_k v_k + G_1$, then $\theta(b_k) = r_{k+1} b_k$ for $k \in V$ is a lift of $(\psi_2\phi_2)$ since $b_k = v_{k+1} - p v_{k+2}$ for $k \in V$ by definition.

We claim that $\psi_2 = 0$ or $\phi_2 = 0$ or, equivalently, that $\Psi_{2,1}(A_2) \subset G_1$ or $\Phi_{2,1}(A_1) \subset G_2$. To show this we need only prove that $\psi_2\phi_2 = 0$ since the images of ψ_2 and ϕ_2 must be either $Z_p(\infty)$ or zero. If $\psi_2\phi_2 \neq 0$, then $\{k | r_{k+1} b_k \neq 0\} \notin \mathscr{P}(V) \cap \mathscr{F}$ since θ is a lift of $\psi_2\phi_2$. If $j \in \delta(\Psi_{2,1}\Phi_{2,1}(b_k))$, then $j > k$ since $h \in \delta(\Phi_{2,1}(b_k))$ implies $h > k$ and $j \in \delta(\Psi_{2,1}(b_h))$ implies $j > h$ by definition of $\Phi_{2,1}$ and $\Psi_{2,1}$. Hence $(\theta - \Psi_{2,1}\Phi_{2,1})(b_k) = r_{k+1} b_k + y_k$ where $j \in \delta(y_k)$ implies $j > k$. Since $\{k | r_{k+1} b_k \neq 0\} \notin \mathscr{P}(V) \cap \mathscr{F}$, a construction exactly like the one at the end of Lemma IV. A. produces an $x \in A_1 - G_1$ with $(\theta - \Psi_{2,1}\Phi_{2,1})(x) \in A_1 - G_1$. However, θ and $\Psi_{2,1}\Phi_{2,1}$ are both clearly lifts of $\psi_2\phi_2$, so $(\theta - \Psi_{2,1}\Phi_{2,1})(A_1) \subset G_1$. This contradiction shows that $\psi_2\phi_2 = 0$.

Assume that $\phi_2 = 0$. It follows that $\Phi_{1,1}$ is a lift of ϕ. Let $\theta_t = \Phi_{1,-t+1} - \Phi_{1,-t}$ for $0 \leqq t \leqq n - 1$ and $\theta_n = \Phi_{1,-n+1}$. Then $\theta_t(G_1) \subseteqq G_2$ since $\Phi_{1,r}$ has this property for every r. Therefore, θ_t induces $\theta_t \in H_1$ and, since $\Phi_{1,1} = \Sigma\theta_t(0 \leqq t \leqq n)$ we have $\phi = \Sigma\theta_t(0 \leqq t \leqq n)$.

If $\theta_t \neq 0$, then there is a k with $\theta_t(v_k) \equiv s_k w_k \not\equiv 0 \bmod G_2$. Since $\theta_t(v_k) = \Sigma\theta_t(p^{j-k+1} b_j) = - \Sigma p^{j-k+1} m_{j-t} b_{j-t} (j \in V)$, we have

$$V' = \{j \in V | p^{j-k+1} m_{j-t} b_{j-t} \neq 0\} \notin \mathscr{P}(V) \cap \mathscr{F}$$

and $W' = V' - t \notin \mathscr{P}(W) \cap \mathscr{F}$. However, since $\theta_t(G_1) \subseteqq G_2$, it follows that for any subset K of V', $\Sigma p^{j-k+1} b_j (j \in K)$ is an element of G_1 if and only if $- \Sigma p^{j-k+1} m_{j-t} b_{j-t} (j \in K)$ is an element of G_2. Equivalently,

$K \in \mathscr{P}(V') \cap \mathscr{I}$ if and only if $K - t \in \mathscr{P}(W') \cap \mathscr{I}$. Since $V' \subset V$, $W' \subset W$, $V' \notin \mathscr{I}$ and $W' \notin \mathscr{I}$ we clearly have $\mathscr{W} = \mathscr{V}^t$. By III. F. (iii) we have $n = n_1(\mathscr{V}, \mathscr{W}) \leqq t \leqq n$. It follows that $\theta_t = 0$ for $t < n$ and, since $\phi \neq 0$, $\theta_n = \phi$ which implies $\mathscr{W} = \mathscr{V}^n$.

References

1. R. A. Beaumont and R. S. Pierce, *Some invariants of p-groups*, Michigan Math. J, **11** (1964), 137-149.
2. L. Fuchs, *Abelian Groups*, Pergamon Press, New York, 1960.
3. J. Irwin and T. Koyama, *On Tpological Methods in Abelian Groups*, Studies on Abelian Groups, Springer-Verlag, New York. 1968.

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