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POINTWISE BOUNDED APPROXIMATION BY FUNCTIONS SATISFYING A SIDE CONDITION

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In this paper necessary and sufficient conditions on a subset S of the unit disc D are given such that every bounded analytic function f on D is a pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$ of uniformly continuous analytic functions on D bounded by the sup norm of f and in addition satisfying $\sup\{|f_n(z)|, z \in S\} \leqq \sup\{|f(z)|, z \in S\}$ for all n.

Let $D = \{z: |z| < 1\}$ denote the open unit disc and $T = \{z: |z| = 1\}$ the unit circle. $H^\infty(D)$ denotes all bounded analytic functions on D and $A(D)$ consists of all uniformly continuous f in $H^\infty(D)$. For any $f \in H^\infty(D)$ and any subset S of D we put $\|f\|_S = \sup\{|f(z)|, z \in S\}$ and we set $\|f\| = \|f\|_D$.

A sequence $\{z_n\}_{n=1}^{\infty}$ in D converging to $z \in T$ converges *nontangentially* to z if for some constant λ we have $|z - z_n| \leqq \lambda(1 - |z_n|)$ for all n. If $f \in H^\infty(D)$ then Fatou's theorem [2, page 34] tells us that f has a nontangential limit at almost every boundary point. Thus at almost every boundary point $\lim f(z_n)$ exists and is independent of the choice of sequence. If $f \in H^\infty(D)$ is known a.e. on T we recapture its values in D by the Cauchy or Poisson integral formula. We will therefore consider functions in $H^\infty(D)$ as defined in D and a.e. on T.

A relatively closed subset S of D is called a *Farrell set* if for each $f \in H^\infty(D)$ there are $f_n \in A$, $n = 1, 2, \cdots$, converging pointwise to f on D with $\|f_n\| \leqq \|f\|$ and such that $\|f_n\|_S \leqq \|f\|_S$. This concept was introduced to us by Professor L. A. Rubel who also raised the question of describing such sets. The object here is to characterize Farrell sets in terms of their cluster points on T. The author is very grateful to Dr. A. M. Davie for valuable conversations on this subject.

First we observe that if $rz \in S$ whenever $0 < r < 1$ and $z \in S$, then S is a Farrell set. Indeed, letting $f_r(z) = f(rz)(0 < r < 1)$, we have: $f_r(z) \to f(z)$ as $r \to 1$.

On the other hand, let $S = \{z_n\}_{n=1}^{\infty}$ where $\sum_n (1 - |z_n|) < \infty$ and assume that the set of cluster points of S on T has positive linear measure. Then there are $f \in H^\infty(D)$ with $f = 0$ on S, but $f \neq 0$, while if $f \in A(D)$ and $f = 0$ on S we must have $f \equiv 0$. The set of cluster points of S on T which are not nontangential limits of sequences from S is here too large. In fact we will prove the following.

THEOREM. *A relatively closed subset S of the open unit disc D is*

a Farrell set if and only if the set S_1 of cluster points of S on T which are nontangential limits of sequences from S has zero linear measure.

Proof. First assume S_1 has positive linear measure. Using ideas from [1] we construct a function $f \in H^\infty(D)$ which shows that S is not a Farrell set. If $t > 0$ and $z \in T$ we define

$$\Delta(z, t) = \{w: 0 < 1 - |w| \leqq t, |z - w| \leqq 2(1 - |w|)\} .$$

For $t > 0$ we also define $E_t = \{z \in \bar{S} \cap T: \Delta(z, t) \cap S = \varnothing\}$.

Since $S_1 \subseteqq \bigcup_t E_t$ has positive measure, we can find for some t_0 a set $E = E_{t_0}$ of positive measure.

Let u be the harmonic extension to D of the function on T which is zero on E and -1 on $T \backslash E$. Define $f = \exp[u + iv]$ where v is a harmonic conjugate to u in D. Then $f \in H^\infty(D)$ and by the theorem of Fatou we have $|f| = 1$ a.e. on E and $|f| = e^{-1}$ a.e. on $T \backslash E$. In particular $\|f\| = 1$.

We claim that $\|f\|_S < 1$. Assume this proved. Let $\{f_n\}_{n=1}^\infty \subset A(D)$, $\|f_n\| \leqq 1$ and assume $f_n(z) \to f(z)$ $z \in D$. We prove that $\|f_n\|_S \to 1$ which in turn will show that S is not a Farrell set.

Given $\varepsilon > 0$ it follows from Fatou's theorem that there exists $z \in D$ such that

$$|f(z)| > 1 - \varepsilon \tag{1}$$

$$m_z(E) > 1 - \varepsilon . \tag{2}$$

Here m_z denotes harmonic measure on T w.r.t. z.

There exists a number N such that

$$|f_n(z) - f(z)| < \varepsilon \qquad \text{if } n > N . \tag{3}$$

By (1) and (3) we have $|f_n(z)| > 1 - 2\varepsilon$ if $n > N$.

But on the other hand we have

$$|f_n(z)| \leqq \int_E |f_n| \, dm_z + \int_{T \backslash E} |f_n| \, dm_z \leqq \|f_n\|_E + \varepsilon$$

so that $1 \geqq \|f_n\|_S \geqq \|f_n\|_E \geqq 1 - 3\varepsilon$ if $n \geqq N$.

It remains to prove that $\|f\|_S < 1$. Choose $z \in S$ such that $|z| > 1 - t_0$.

Define $J_z = \{w \in T: |w - z| < 2(1 - |z|)\}$. By a well-known estimate of harmonic measure there exists a constant $c \in (0, 1)$ depending only on t_0 such that

$$m_z(J_z) \geqq c . \tag{4}$$

Since $z \in S$ we have $J_z \cap E = \varnothing$.

By (4) we therefore have since $|f| \leqq 1$ a.e. on E and $|f| \leqq e^{-1}$ a.e. on $T\backslash E$:

$$|f(z)| \leqq \int_E |f|\, dm_z + \int_{T\backslash E} |f|\, dm_z \leqq m_z(E) + \frac{1}{e} m_z(T\backslash E)$$
$$\leqq 1 - c + ce^{-1} < 1 - c2^{-1} \quad \text{if} \quad z \in S .$$

Since clearly $\sup\{|f(z)| : |z| \leqq 1 - t_0\} < 1$, the claim is proved.

Suppose now S_1 has zero linear measure.

We first consider the special case where $B = \bar{S} \cap T$ itself has zero measure.

Given $f \in H^\infty(D)$ we first choose functions g_n continuous on $D \cup T\backslash B$ and analytic in D such that $\|g_n\|_D \leqq \|f\|$, $\|g_n\|_S \leqq \|f\|_S$ and $g_n(z) \to f(z)$ if $z \in D$. (See [3] § 1.)

We then choose $\{a_n\}_{n=1}^\infty \subset A(D)$ converging pointwise to 1 on D such that $\|a_n\| = 1$ and $a_n = 0$ on B. See [2], p. 80 for a construction of the a_n's.

Define $f_n = 0$ on B and $f_n = a_n g_n$ in $D\backslash B$. Then $f_n \in A(D)$ and $\{f_n\}_{n=1}^\infty$ approximates f as required.

Finally we consider the case where B has positive linear measure.

Define $H(\theta, z) = (e^{i\theta} + z)/(e^{i\theta} - z)$ if $-\pi < \theta \leqq \pi$ and $z \in D$.

Choose $f \in H^\infty(D)$ with $\|f\| = 1$. Let $\eta = \|f\|_S$. If $\eta = 0$ it follows from Fatou's theorem and that S_1 has zero linear measure, that $|f| = 0$ a.e. on B which is impossible since $f \neq 0$ and B has positive measure. Hence $\eta > 0$.

We can factorize f as $f = I \cdot F$ where I and F denotes the "inner" and "outer" part of f. [[2], Ch. 5.]

Here F is given by

$$F(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta, z)u(\theta)d\theta\right]$$

and it follows from the hypothesis that $u \leqq 0$ a.e. on T and $u \leqq \log \eta$ a.e. on B.

Fix a positive integer n. Let u_1 be a continuously differentiable function on T and $V \supset B$ a set open in T such that $u_1 \leqq 0$, $u_1 \leqq \log \eta$ on V and with the following properties:

$$(5) \qquad \int_{-\pi}^{\pi} |u(\theta) - u_1(\theta)|\, d\theta < \frac{1}{n}$$

$$(6) \qquad \int_{T\backslash V} |H(\theta, z)(u(\theta) - u_1(\theta))|\, d\theta < \frac{1}{n} \text{ for all } z \in S .$$

To obtain u_1 on V one first defines u_1 in a neighborhood of B such that $\int_B |u(\theta) - u_1(\theta)|\, d\theta < 1/2n$. This inequality will still hold if B is

replaced by some open set V containing it if linear measure of $V\backslash B$ is small enough. But then it is easy to extend u_1 to T so that (5) and (6) hold.

Since $u_1 \leqq \log \eta$ on V and $u_1 \leqq 0$ there is a compact subset K of S such that

$$(7) \qquad \int u_1 dm_z \leqq \log\left(\eta + \frac{1}{n}\right) \quad \text{if} \quad z \in S\backslash K .$$

Let $A = \{z \in V: u(z) \leqq u_1(z)\}$ and M be a compact subset of A. We choose a continuously differentiable function $a \leqq 0$ such that supp a (the support of a) is contained in V and we have

$$(8) \qquad \left|\int_V H(\theta, z)(u_1 + a - u)(\theta)d\theta\right| < \frac{1}{n} \quad \text{if} \quad z \in K .$$

If a approximates $u - u_1$ sufficiently well on M and if the linear measures of $A\backslash M$ and supp $a\backslash M$ are sufficiently small we can obtain that (5) and (6) still hold if we replace u_1 by $(u_1 + a)$ there.

We define $g_n(z) = \exp\left[1/2\pi\int_{-\pi}^{\pi}(H(\theta, z)(u_1 + a)(\theta)d\theta\right]$, $z \in D$.

Since $u_1 + a$ is smooth $g_n \in A(D)$ and $\| g_n \| \leqq 1$ since $u_1 + a \leqq 0$.

By (5), $g_n(z) \to F(z)$ if $z \in D$. When $z \in S\backslash K$, (7) implies that $|g_n(z)_1| \leqq \eta + 1/n$ and if $z \in K$ we get from (6) and (8) that $|g_n(z)| \leqq \eta e^{1/n}$.

Define now $h_n = (\eta/(\eta + 3/n))g_n$ and $B_n(z) = I(z(1 - n^{-1}))$ if $z \in D$ and $n = 1, 2, \cdots$.

The sequence $f_n = B_n h_n n = 1, 2, \cdots$ approximates f as required.

We have completed the proof that S is a Farrell set if S_1 has zero linear measure and the proof also shows how to construct the functions $\{f_n\}$ given f such that they satisfy the requirements given in the definition of a Farrell set.

If S_1 has zero linear measure and B has positive linear measure there is a proof based on functional analysis showing that S is a Farrell set. This proof is due to Dr. A. M. Davie. Since the proof is short and rather different from the one given above we would like to include it here.

So we assume $f \in H^\infty(D)$, $|f| = 1$ and $\| f \|_S = \eta > 0$.

Let $N = \{f \in C(\bar{D}) \| f \|_S \leqq \eta$ and $\| f \| \leqq 1\}$. We have to show that f is in the closure of $N \cap A(D)$ in the topology of uniform convergence on compact subsets of D.

By the separation theorem and Riez representation theorem it is sufficient to prove that $|\mu(f)| \leqq 1$ whenever μ is a regular complex Borel measure with compact support in D such that $|\mu(h)| \leqq 1$ for all $h \in N \cap A(D)$. ($N \cap A(D)$ is a convex set in the space of all continuous functions in D, with the topology of uniform convergence on compact subsets of D and the dual space is the space of regular

complex Borel measures with compact support in D.)

N is the unit ball w.r.t. some norm on $C(\bar{D})$ which is equivalent to sup norm on $\bar{D}$ since $\eta > 0$.

Hence we can extend the functional $g \to \mu(g)$ from $A(D)$ to $C(\bar{D})$ and represent it by a measure ν on $\bar{D}$ such that $|\nu(g)| \leqq 1$ for all $g \in N$.

But the last fact implies

$$(I): |\nu|(\bar{D}\backslash\bar{S}) + \eta|\nu|\bar{S} \leqq 1$$

where $|\nu|$ denotes the total variation of ν.

We claim that $\nu(E) = 0$ if $E \subset T$ has zero linear measure. To see this let $K \subset E$ be compact. Choose $\{a_n\}_{n=1}^{\infty} \subset A(D)$ such that $\|a_n\| = 1$, $a_n = 0$ on K, and $a_n \to 1$ on $\bar{D}\backslash K$. (The sequence $\{a_n\}$ mentioned in the previous proof will do.) Then $0 = (\mu - \nu)(1 - a_n) \to (\mu - \nu)(K) = \nu(K)$ by dominated convergence.

This means that f is defined a.e. ν and if $\{f_n\} \subset A(D)$ converges pointwise to f on D and a.e. to f on T such that $\|f_n\| \leqq \|f\|$ we again have by dominated convergence that $0 = \mu(f) - \nu(f)$. But $|f| \leqq 1$ a.e. ν and $|f| \leqq \eta$ a.e. ν on S so $|\mu(f)| = |\nu(f)| \leqq 1$ by (I).

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