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LOCAL IDEALS IN A TOPOLOGICAL ALGEBRA OF ENTIRE FUNCTIONS CHARACTERIZED BY A NON-RADIAL RATE OF GROWTH

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In this paper a class of locally convex algebras of entire functions is considered: For fixed $\rho > 0$, $\sigma > 0$, and n a positive integer, let $E[\rho, \sigma; n]$ denote the space of all entire functions f in n variables which satisfy $|f(x + iy)| = O\{\exp[A(\|x\|^\rho + \|y\|^\sigma)]\}$ for some $A > 0$. Sufficient conditions are given in order that the local ideal generated by a family in $E[\rho, \sigma; n]$ coincides with the closed ideal generated by the family.

For $z = x+iy = (x_1 + iy_1, x_2 + iy_2, \cdots, x_n + iy_n) \in \boldsymbol{C}^n$, write $\|z\|^2 = \|x\|^2 + \|y\|^2 = \sum_{k=1}^n (x_k^2 + y_k^2)$. For $f: \boldsymbol{C}^n \rightarrow \boldsymbol{C}$ and $A > 0$, let $\|f\|_A = \sup\{|f(z)| \exp[-A(\|x\|^\rho + \|y\|^\sigma)]: z \in \boldsymbol{C}^n\}$. The space $E = E[\rho, \sigma; n]$ is a locally convex algebra over $\boldsymbol{C}$, with the natural inductive limit topology from the Banach spaces $\{f \text{ entire}: \|f\|_A < \infty\}$, $A > 0$.

For $\mathscr{F}$ a family of functions in E, write $\mathscr{I}(\mathscr{F})$, $\mathscr{I}^-(\mathscr{F})$, and $\mathscr{I}_{\text{loc}}(\mathscr{F})$, respectively, for the ideal, closed ideal, and local ideal in E generated by $\mathscr{F}$. The local ideal $\mathscr{I}_{\text{loc}}(\mathscr{F})$ consists of all $H \in E$ such that in a neighborhood of each $z_0 \in \boldsymbol{C}^n$, H has the form $H = \sum_{j=1}^r h_j F_j$ for some $F_1, F_2, \cdots, F_r \in \mathscr{F}$ and $h_1, h_2, \cdots, h_r$ analytic in a neighborhood of z_0. The ideal $\mathscr{I}_{\text{loc}}(\mathscr{F})$ is closed in E and contains $\mathscr{F}$; hence $\mathscr{I}(\mathscr{F}) \subseteqq \mathscr{I}^-(\mathscr{F}) \subseteqq \mathscr{I}_{\text{loc}}(\mathscr{F})$. The problem to be considered is: Under what conditions is $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in the space $E = E[\rho, \sigma; n]$?

Problems of this type have been studied in various algebras E by many authors, among them: L. Ehrenpreis [2, 3], L. Schwartz [14], H. Cartan [1], L. Hörmander [5, 6], B. A. Taylor [15], J. J. Kelleher and B. A. Taylor [7, 8, 9], J. Metzger [11], I. F. Krasičkov [10], P. K. Raševskiĭ [13], and K. V. Rajeswara Rao [12].

Let $\mathscr{F} \subseteqq E = E[\rho, \sigma; n]$. It is known (see B. A. Taylor [15]) that for $n = 1$ and $\rho = \sigma \geqq 1$, $\mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in E for any $\mathscr{F}$. If $\rho = \sigma$ and $\mathscr{F} = \{F\}$, but n is arbitrary, then $\mathscr{I}(F) = \mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$ (see L. Ehrenpreis [2]). In [11] this author proved that if $n = 1$, and $\rho \geqq 1$ or $\sigma \geqq 1$, then $\mathscr{I}^-(F) = \mathscr{I}_{\text{loc}}(F)$ for any $F \in E$; if in addition $\rho \neq \sigma$, there exists an $F \in E$ for which $\mathscr{I}(F) \neq \mathscr{I}^-(F)$. Concerning the more general case where n is arbitrary, and ρ and σ do not necessarily agree: Ehrenpreis's Quotient Structure Theorem (see [3]) implies that if $\rho > 1$ and $\sigma > 1$, and if $\mathscr{F} = \{F_1, F_2, \cdots, F_r\}$ consists of polynomials, then $\mathscr{I}(\mathscr{F}) = \mathscr{I}^-(\mathscr{F}) = \mathscr{I}_{\text{loc}}(\mathscr{F})$ in E. Also, a result of Hörmander [5] implies that when $\rho \geqq 1$ and $\sigma \geqq 1$, a family $\mathscr{T} = \{F_1, F_2, \cdots, F_r\}$ in E satisfies $\mathscr{I}(\mathscr{T}) = E$ if and only if there exist $\varepsilon > 0$ and $A > 0$ such that

$$\sum_{j=1}^{r} |F_j(z)| \geqq \varepsilon \exp[-A(||x||^\rho + ||y||^\sigma)]$$

for all $z \in C^n$.

In this papar the following result is proved:

THEOREM 1. *Let n be a positive integer, $\rho > 0$, $\sigma > 0$, and $\tau = \max(\rho, \sigma) \geqq 1$; and let $\mathscr{T} \subseteqq E[\rho, \sigma; n]$. If $\mathscr{I}(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in $E[\tau, \tau; n]$, then $\mathscr{I}^-(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in $E[\rho, \sigma; n]$.*

Since $\mathscr{I}(F) = \mathscr{I}_{\mathrm{loc}}(F)$ in $E[\tau, \tau; n]$, a consequence of Theorem 1 is:

COROLLARY. *Let n be a positive integer, $\rho > 0$, $\sigma > 0$, with $\max(\rho, \sigma) \geqq 1$. Then $\mathscr{I}^-(F) = \mathscr{I}_{\mathrm{loc}}(F)$ in $E = E[\rho, \sigma; n]$ for any $F \in E$.*

This corollary generalizes to several variables the result proved by this author in [11] for the case of one variable.

Theorem 1 follows immediately from an approximation theorem which is proved in the next section. In the third section Theorem 1 is applied to several examples.

2. The main theorem.

The approximation theorem stated below, Theorem 2, yields Theorem 1 as an immediate corollary. The proof of Theorem 2 is based on a technique of L. Hörmander given in [6], which in turn involves the solution of the $\bar{\partial}$ equation (see [4, Chapter IV]).

THEOREM 2. *Let $\sigma \geqq 1$, and $H, F_1, F_2, \cdots, F_r, G_1, G_2, \cdots, G_r$ be entire functions in n variables, with $H = \sum_{j=1}^{r} G_j F_j$ and*

$$|G_j(z)| \leqq C \exp(A ||z||^\sigma) \tag{1}$$

for all $z \in C^n$, $j = 1, 2, \cdots, r$, where A, C denote positive constants. Then there exist positive constants B, K, and M, and entire functions $g_{j,t}$, $0 < t < 1$, $j = 1, 2, \cdots, r$, such that:

$$\begin{aligned} &\left| H(z) - \sum_{j=1}^{r} g_{j,t}(z) F_j(z) \right| \\ &\leqq tK(1 + ||z||^2)^M \left\{ |H(z)| + \left[\sum_{j=1}^{r} |F_j(z)| \exp(B ||y||^\sigma) \right] \right\} \end{aligned} \tag{2}$$

for all $z \in C^n$, $0 < t < 1$, and

$$(3) \qquad |g_{j,t}(z)| \leqq L(t)(1+\|z\|^2)^M \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$, *where* $L(t) > 0$ *may depend on* t *but not on* z.

The proof of Theorem 2 is facilitated by the following:

LEMMA. *Let* n *and* N *be positive integers, with* N *even. There exist* $\alpha > 0$ *and* $\varepsilon > 0$ *such that: If* $z \in \boldsymbol{C}^n$ *with* $\alpha\|x\| \geqq \|y\|$, *then* $\operatorname{Re}(z^N) \geqq \varepsilon\|z\|^N$.

Here $z^N = \sum_{k=1}^n (x_k + iy_k)^N$.

Proof. Write $q = N/2$; then

$$\operatorname{Re}[(x_k + iy_k)^N] = x_k^{2q} + \sum_{m=1}^{q} a_m x_k^{2(q-m)} y_k^{2m}$$

for all $x_k + iy_k \in \boldsymbol{C}$, where $a_1, a_2, \cdots, a_q$ are integers depending only on N. Hence for $z \in \boldsymbol{C}^n$,

$$\begin{aligned}\operatorname{Re}(z^N) &\geqq \sum_{k=1}^{n} x_k^{2q} - \sum_{m=1}^{n} |a_m| \left(\sum_{k=1}^{n} x_k^{2(q-m)} y_k^{2m}\right) \\ &\geqq 2^{-(n-1)(q-1)}\|x\|^{2q} - \sum_{m=1}^{q} |a_m|\,\|x\|^{2(q-m)}\|y\|^{2m}\,.\end{aligned}$$

The required condition is then satisfied with $\varepsilon = 2^{-[(n-1)(q-1)]-(q-2)}$, and $0 < \alpha < 1$ sufficiently small that $\sum_{m=1}^{q} |a_m|\alpha^{2m} < 2^{-[(n-1)(q-1)]-1}$.

Proof of Theorem 2. Let N be an even integer, $N > \sigma$. By the lemma there exist $\alpha = \alpha(n, N) > 0$ and $\varepsilon = \varepsilon(n, N) > 0$ such that $\alpha\|x\| \geqq \|y\|$ implies $\operatorname{Re}(z^N) \geqq \varepsilon\|z\|^N$. Set $S = \{z \in \boldsymbol{C}^n : \alpha\|x\| \geqq \|y\|$ and $\operatorname{Re}(z^N) \geqq 1\}$. The bounds (1) imply that for some $B > 0$ and $K_1 > 0$,

$$(4) \qquad |G_j(z)| \leqq K_1 \exp(B\|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n \backslash S$, $j = 1, 2, \cdots, r$.

Let $\varphi: \boldsymbol{R} \to \boldsymbol{R}$ be a C^∞ function such that

$$\begin{aligned}\varphi(u) &= 0 \quad \text{if} \quad u \leqq 0\,, \\ &= 1 \quad \text{if} \quad u \geqq 1\,,\end{aligned}$$

and $0 \leqq \varphi(u) \leqq 1$ if $0 \leqq u \leqq 1$. For $0 < t < 1$ and $z \in \boldsymbol{C}^n$, set

$$\omega_t(z) = [\varphi(\operatorname{Re}(z^N))]\exp[-t(z^N)] + [1 - \varphi(\operatorname{Re}(z^N))]\,.$$

Each ω_t is a C^∞ function on $\boldsymbol{C}^n$; and $|\omega_t(z)| \leqq 1$ for all $z \in \boldsymbol{C}^n$, while

$|\omega_t(z)| \leqq \exp(-\varepsilon \|z\|^N)$ for all $z \in S$. Together with (1) and (4), this implies that for some $K_2 > 0$,

$$(5) \qquad |\omega_t(z) G_j(z)| \leqq K_2 \exp(B \|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$. Since $|1 - e^\zeta| \leqq |\zeta|$ if $\operatorname{Re}\zeta \leqq 0$, and since $|z^N| \leqq n \|z\|^N$, it follows that $|1 - \omega_t(z)| \leqq tn \|z\|^N$ for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Consequently,

$$(6) \qquad \left| H(z) - \sum_{j=1}^{r} (\omega_t(z) G_j(z)) F_j(z) \right| \leqq tn \|z\|^N |H(z)|$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Thus the functions $\omega_t G_j$ satisfy conditions of the form (2) and (3).

As is done by Hörmander, the functions $\omega_t G_j$ will now be altered to obtain the desired analytic functions $g_{j,t}$. First of all, $\bar{\partial}\omega_t = 0$ if $\operatorname{Re}(z^N) \leqq 0$, and $\|\bar{\partial}[\varphi(\operatorname{Re}(z^N))]\| \leqq K_3 \|z\|^{N-1}$ everywhere on $\boldsymbol{C}^n$; therefore, $\|\bar{\partial}\omega_t(z)\| \leqq tnK_3 \|z\|^{2N-1}$ for all $z \in \boldsymbol{C}^n$, $0 < t < 1$. Also, $\bar{\partial}(\omega_t G_j) = (\bar{\partial}\omega_t) G_j$; and $\bar{\partial}\omega_t = 0$ on S. By (4) then, for $0 < t < 1$ and $j = 1, 2, \cdots, r$,

$$(7) \qquad \|\bar{\partial}(\omega_t(z) G_j(z))\| \leqq tK_4 \|z\|^{2N-1} \exp(B \|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, and thus

$$\int_{C^n} \|\bar{\partial}(\omega_t(z) G_j(z))\|^2 e^{-\psi(z)} d\lambda(z) \leqq t^2 K_5$$

where $\psi(z) \equiv 2B \|y\|^\sigma + (2N + n) \log(1 + \|z\|^2)$, and λ denotes Lebesgue measure.

By applying Theorem 4.4.2 of Hörmander [4], functions $\nu_{j,t}$ of class C^∞ on $\boldsymbol{C}^n$ may be chosen such that $\bar{\partial}\nu_{j,t} = \bar{\partial}(\omega_t G_j)$ and

$$\int_{C^n} |\nu_{j,t}(z)|^2 \exp[-\psi(z) - 2\log(1 + \|z\|^2)] d\lambda(z) \leqq t^2 K_5$$

for $0 < t < 1$, $j = 1, 2, \cdots, r$. Together with (7), this implies (see Hörmander [6, p. 314]) that

$$(8) \qquad |\nu_{j,t}(z)| \leqq tK_6 (1 + \|z\|^2)^M \exp(B \|y\|^\sigma)$$

for all $z \in \boldsymbol{C}^n$, $0 < t < 1$, $j = 1, 2, \cdots, r$, where $M = N + 1 + (1/2)n$.

Each of the functions $g_{j,t} = \omega_t G_j - \nu_{j,t}$ is then entire. Further, (3) is satisfied because of (5) and (8). Lastly $H - \sum_{j=1}^r g_{j,t} F_j = [H - \sum_{j=1}^r (\omega_t G_j) F_j] + \sum_{j=1}^r \nu_{j,t} F_j$, and thus (2) follows from (6) and (8).

3. Examples and applications. In this section several examples are given where $\mathscr{I}^-(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in spaces of the form $E[\rho, \sigma; n]$.

EXAMPLE 1. Let $E = E[\rho, \sigma; n]$, with $\tau = \max(\rho, \sigma) \geqq 1$, and let $F \in E$. The corollary to Theorem 1 implies that $\mathscr{I}^-(F) = \mathscr{I}_{\mathrm{loc}}(F)$

in E. Also, $\mathscr{I}(F) = \mathscr{I}^{-}(F) = \mathscr{I}_{\mathrm{loc}}(F)$ in $E[\tau, \tau; n]$. However, it need not be the case that $\mathscr{I}(F) = \mathscr{I}^{-}(F)$ in E; indeed, if $\rho \neq \sigma$ then (see [11]) there exists an $F \in E$ for which $\mathscr{I}(F) \neq \mathscr{I}^{-}(F)$.

EXAMPLE 2. Let $n = 1$, and $E = E[\rho, \sigma; 1]$, with $\tau = \max(\rho, \sigma) \geqq 1$. Let $\mathscr{T} \subseteqq E$ and suppose some $F_0 \in E$ has only finitely many zeros. Then $\mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in E. To prove this, write $F_0 = PH$ where P is a polynomial and $H \in E$ has no zeros. There exists a polynomial Q such that $\mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in E is $\{G \in E: G/Q \text{ is analytic}\}$. Set $P = P_0Q$, so that $F_0 = P_0QH \in \mathscr{T} \subseteqq \mathscr{I}(\mathscr{T})$. The factors of P_0 can bo divided out (see Taylor [15]) to yield $QH \in \mathscr{I}(\mathscr{T})$ in E. Since $1/H \in E[\tau, \tau; 1]$, it follows that $Q \in \mathscr{I}(\mathscr{T})$ in $E(\tau, \tau; 1)$, which implies that $\mathscr{I}(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in $E[\tau, \tau; 1]$, Then by Theorem 1, $\mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ in $E = E[\rho, \sigma; 1]$.

Note that if $1/H \in E$—e.g., if $\rho \geqq 1$, $\sigma \geqq 1$, and F_0 is an exponential polynomial $F_0(z) \equiv P(z)e^{az}$—then $Q \in \mathscr{I}(\mathscr{T})$ in E and thus $\mathscr{I}(\mathscr{T}) = \mathscr{I}^{-}(\mathscr{T}) = \mathscr{I}_{\mathrm{loc}}(\mathscr{T})$ trivially. On the other hand, if $1/H \notin E$ then $\mathscr{I}(\mathscr{T})$ need not coincide with $\mathscr{I}^{-}(\mathscr{T})$ in E—for instance, if $\rho = 1$, $\sigma = 2$, and $\mathscr{T} = \{e^{-(z^2)}, e^{iz} - 1\}$.

EXAMPLE 3. Let $1 \leqq \rho < \sigma$ and $E = E[\rho, \sigma; 1]$. Choose γ, $\rho < \gamma < \sigma$; let $\varepsilon_m = \exp(-(2^{m\gamma}))$, $a_m = 2^m$, $b_m = 2^m + \varepsilon_m$, $m = 1, 2, \cdots$; and let

$$F_1(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{a_m}\right)$$

$$F_2(z) = \prod_{m=1}^{\infty}\left(1 - \frac{z}{b_m}\right)$$

for all $z \in C$. Each F_j is an entire function of order 0 and thus is in E. Clearly $\mathscr{I}_{\mathrm{loc}}(F_1, F_2) = E$. It is easily argued that for $\rho < \rho' < \gamma$, $|F_2(2^m)| = O[\exp(-(2^{m\rho'}))]$ as $m \to \infty$. Consequently $1 \notin \mathscr{I}(F_1, F_2)$ in E. On the other hand, letting $\gamma < \sigma' < \sigma$ and using standard estimates on infinite products yields:

$$|F_1(z)| \geqq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \notin \bigcup_m \left\{z: |z - a_m| < \frac{1}{2}\varepsilon_m\right\},$$

$$|F_2(z)| \geqq \delta \exp(-|z|^{\sigma'}) \quad \text{if} \quad z \notin \bigcup_m \left\{z: |z - b_m| < \frac{1}{2}\varepsilon_m\right\},$$

where $\delta > 0$. Thus $|F_1(z)| + |F_2(z)| \geqq \delta \exp(-|z|^{\sigma'})$ for all $z \in C$. It then follows (Hörmander [5]) that $1 \in \mathscr{I}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$. Hence $\mathscr{I}(F_1, F_2) = \mathscr{I}^{-}(F_1, F_2) = \mathscr{I}_{\mathrm{loc}}(F_1, F_2)$ in $E[\sigma, \sigma; 1]$, while $\mathscr{I}(F_1, F_2) \subsetneqq \mathscr{I}^{-}(F_1, F_2) = \mathscr{I}_{\mathrm{loc}}(F_1, F_2)$ in $E = F[\rho, \sigma; 1]$.

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