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OSCILLATION AND NONOSCILLATION CRITERIA FOR SOME SELF-ADJOINT EVEN ORDER LINEAR DIFFERENTIAL OPERATORS

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Oscillation and nonoscillation results are presented for the operator

$$L_{2n}y = \sum_{k=0}^{n} (-1)^{n-k}(p_k y^{(n-k)})^{(n-k)}$$

where $p_0(x) > 0$ on $(0, \infty)$ and for $k = 0, 1, \cdots, n$, p_k is a real-valued, $n - k$ times differentiable function on $(0, \infty)$. Also, y is an element of the set of all real-valued, $2n$ — fold continuously differentiable, finite functions on $(0, \infty)$.

In particular, a nonoscillation result is given for L_{2n} without sign restrictions on the coefficients. Oscillation results are given for L_4 without the requirement that p_1 be negative for large x. Finally, the oscillation of

$$L_{2n}y = (-1)^n \ (ry^{(n)})^{(n)} + py$$

is considered for $r(x)$ not necessarily bounded.

The oscillatory behavior of L_4 has been considered by Leighton and Nehari [8], Barrett [1], and Hinton [4]. In general, L_{2n} has been considered by Glazman [2], Hinton [5], Hunt [6], and Hunt and Namboodiri [7].

DEFINITION 0.1. The operator L_{2n} is called *oscillatory on* $[a, b]$ provided there is a function y, $y \not\equiv 0$, and numbers c and d for which $a \leqq c < d \leqq b$ such that $L_{2n}y = 0$ and

$$y^{(k)}(c) = 0 = y^{(k)}(d) \text{ for } k = 0, 1, \cdots, n-1 .$$

Otherwise, L_{2n} is called *nonoscillatory on* $[a, b]$. The operator L_{2n} is called *oscillatory on* $[a, \infty)$ if for any given $c \geqq a$ there is a $d > c$ such that L_{2n} is oscillatory on $[c, d]$.

DEFINITION 0.2. Given a positive integer n and a number a define $\mathfrak{D}_n(b)$ for all $b > a$ to be the set of all real-valued functions y with the following properties:

(a) $y^{(k)}$ is absolutely continuous on $[a, b]$ for $k = 0, 1, \cdots, n-1$,
(b) $y^{(n)}$ is essentially bounded on $[a, b]$, and
(c) $y^{(k)}(a) = 0 = y^{(k)}(b)$ for $k = 0, 1, \cdots, n-1$.

For $y \in \mathfrak{D}_n(b)$ define

$$I_b(y) = \int_a^b \sum_{k=0}^{n} p_k(x)(y^{(n-k)}(x))^2 dx$$

which is called the *quadratic functional* for L_{2n}.

The following theorem has provided the primary motivation for the results which are to follow.

THEOREM 0.1 (Reid [9]). *The following two statements are equivalent.*

(i) *The operator L_{2n} is nonoscillatory on $[a, b]$.*

(ii) *If $y \in \mathfrak{D}_n(b)$ and $y \not\equiv 0$ then $I_b(y) > 0$.*

Consequently, in order to show that L_{2n} is oscillatory on $(0, \infty)$, given any $a > 0$, it will suffice to construct a $y \in \mathfrak{D}_n(b)$ for some $b > a$ for which $I_b(y)$ is not positive and $y \not\equiv 0$. This is the technique of proof for all of the oscillation theorems which follow.

This method of proof is especially conducive to oscillation theorems which require that integral conditions be met by the coefficients of L_{2n}. For example, Glazman [2, p. 104] showed that $(-1)^n y^{(2n)} + py$ is oscillatory on $(0, \infty)$ when $\int^{\infty} p = -\infty$ (see Theorem 3.2).

Initially, the construction of y is suggested by the conditions of the hypothesis on the coefficients of L_{2n} and the corresponding quadratic formula. For example, to establish the above result, Glazman let $y \equiv 1$ over the major portion of the interval $[a, b]$. To show that $y^{iv} - (qy')'$ is oscillatory when $\int^{\infty} q = -\infty$ (see Theorem 2.2) the author let $y(x) = x - a$ over a portion of $[a, b]$. Next, we construct y over the remaining portion of $[a, b]$ to insure that $y \in \mathfrak{D}_n(b)$ and the integral of $p_{n-k} \cdot y^{(n-k)^2}$ is bounded above for $k = 0, 1, \cdots, n$ independent of b.

For other proofs using this method the reader should consult Glazman [2, pp. 95–105] and Hinton [4].

1. The nonoscillation of L_{2n}.

LEMMA 1.1 (Glazman [2, p. 83]).

(i) *If $g(a) = 0$ for some $a > 0$ and g' is continuous on $[a, b]$, then*

$$\int_a^b x^{-2m}(g(x))^2 dx \leqq \left(\frac{2}{2m-1}\right)^2 \int_a^b x^{-2m+2}(g'(x))^2 dx$$

for m a positive integer. Moreover, if $g \not\equiv 0$ on $[a, b]$, the above inequality is strict.

(ii) *If $g \not\equiv 0$, $g^{(m)}$ is continuous on $[a, b]$, and $g(a) = \cdots = g^{(m-1)}(a) = 0$, then*

$$\int_a^b x^{-2m}(g(x))^2 dx < \left(\frac{2^m}{1\cdot 3\cdot \cdots \cdot (2m-1)}\right)^2 \int_a^b (g^{(m)})^2 dx .$$

A well known result in oscillation theory is a sufficient condition for the nonoscillation of L_2 due to Hille [3]. A generalization of this result for L_{2n} is given in the next theorem.

THEOREM 1.1. *For L_{2n} defined above with $p_0(x) \equiv 1$ let $P_k^0(x) = p_k(x)$ and*

$$P_k^m(x) = \int_x^\infty P_k^{m-1}(t)dt$$

for m an integer greater than or equal to one when

$$-\infty < \int^\infty P_k^{m-1}(t)dt < \infty .$$

If for $k = 1, \cdots, n$ and $x \geqq a$ we have $-\infty < \int_a^\infty P_k^m < \infty$ for $m = 0, 1, \cdots, k-1$, $x^k|P_k^k| \leqq a_k$, and $\sum_{k=1}^n a_k M_k \leqq 1$ where $M_k = k!\, 2^{4k-1}/(2k)!$, then L_{2n} is nonoscillatory on $[a, b]$ for all $b > a$.

Proof. The proof is given only for $n > 1$. Suppose L_{2n} is oscillatory on $[a, b]$. Then, there are numbers c and d and a function y which is not identically zero such that $L_{2n}y = 0$ and $y^{(k)}(c) = 0 = y^{(k)}(d)$ for $k = 0, 1, \cdots, n-1$. Since $(L_{2n}y)y = 0$ then

$$-\sum_{k=1}^n (-1)^{n-k} \int_c^d (p_k y^{(n-k)})^{(n-k)} y = (-1)^n \int_c^d y^{(2n)}\, y = \int_c^d [y^{(n)}]^2 ,$$

by integrating by parts n times. By integrating by parts $n-k$ times we find that

$$-\int_c^d (-1)^{n-k} (p_k y^{(n-k)})^{(n-k)} y = -\int_c^d p_k (y^{(n-k)})^2 .$$

However, by integrating by parts k times and using Leibniz's rule we obtain

$$\begin{aligned}
-\int_c^d p_k (y^{(n-k)})^2 &= -\int_c^d P_k^k [(y^{(n-k)})^2]^{(k)} = -\int_c^d P_k^k \cdot \sum_{i=0}^k \binom{k}{i} (y^{(n-k)})^{(k-i)} (y^{(n-k)})^{(i)} \\
&\leqq \sum_{i=0}^k \binom{k}{i} \int_c^d \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \cdot t^k |P_k^k| \\
&\leqq a_k \sum_{i=0}^k \binom{k}{i} \cdot \int_c^d \frac{|y^{(n-i)}|}{t^k} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \\
&= a_k \left[2\int_c^d |y^{(n)}| \cdot \frac{|y^{(n-k)}|}{t^k} + \sum_{i=1}^{k-1} \binom{k}{i} \int_c^d \frac{|y^{(n-i)}|}{t^i} \cdot \frac{|y^{(n-k+i)}|}{t^{k-i}} \right] \\
&\leqq a_k \left[2\|y^{(n)}\|_2 \left\| \frac{y^{(n-k)}}{t^k} \right\|_2 + \sum_{i=1}^{k-1} \binom{k}{i} \left\| \frac{y^{(n-i)}}{t^i} \right\|_2 \left\| \frac{y^{(n-k+i)}}{t^{k-i}} \right\|_2 \right]
\end{aligned}$$

$$< a_k\Big[\|y^{(n)}\|_2^2\Big(\frac{2^{k+1}}{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}\Big)$$
$$+\sum_{i=1}^{k-1}\binom{k}{i}\|y^{(n)}\|_2^2\Big(\frac{2^i}{1\cdot 3\cdot\ \cdots\ \cdot(2i-1)}\Big)\Big(\frac{2^{k-i}}{1\cdot 3\cdot\ \cdots\ \cdot(2(k-i)-1)}\Big)\Big]$$
$$= a_k C_k\int_c^d [y^{(n)}]^2$$

by Lemma 1.1 and the Cauchy inequality where

$$C_k = \frac{2^{k+1}}{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}$$
$$+\sum_{i=1}^{k-1}\binom{k}{i}\cdot\Big(\frac{2^k}{[1\cdot 3\cdot\ \cdots\ \cdot(2i-1)][1\cdot 3\cdots(2(k-i)-1)]}\Big).$$

A simplification shows that

$$C_k = [2^{2k}k!/(2k)!]\sum_{i=0}^{k}\binom{2k}{2i}.$$

Since

$$0 = (1-1)^{2k} = \sum_{i=0}^{2k}(-1)^i\binom{2k}{i} = \sum_{i=0}^{k}\binom{2k}{2i} - \sum_{i=1}^{k}\binom{2k}{2i-1},$$

then

$$2^{2k-1} = \frac{1}{2}(1+1)^{2k} = \frac{1}{2}\Big[\sum_{i=0}^{k}\binom{2k}{2i} + \sum_{i=1}^{k}\binom{2k}{2i-1}\Big] = \sum_{i=0}^{k}\binom{2k}{2i}.$$

Therefore,

$$C_k = [2^{4k-1}k!]/(2k)! = M_k.$$

Consequently,

$$\int_c^d [y^{(n)}]^2 = -\sum_{k=1}^{n}(-1)^{n-k}\int_c^d (p_k y^{(n-k)})^{(n-k)} y$$
$$= -\sum_{k=1}^{n}\int_c^d p_k(y^{(n-k)})^2 < \sum_{k=1}^{n} a_k M_k\int_c^d [y^{(n)}]^2 \leqq \int_c^d [y^{(n)}]^2$$

which is a contradiction. Therefore, L_{2n} is nonoscillatory on $[a, b]$.

It will be useful in applying Theorem 1.1 to note that $M_{k+1} = 8M_k/(2k+1)$.

For the remainder of this paper we will assume that $p_k(x)$ is identically zero for $k = 1$ to $n-2$ and will denote $p_0(x)$, $p_{n-1}(x)$, and $p_n(x)$ by $r(x)$, $q(x)$, and $p(x)$ respectively. Similarly, $P_n^k(x)$ and $P_{n-1}^k(x)$ will be denoted by $P_k(x)$ and $Q_k(x)$ respectively.

If $p(x) = kx^{-4}$, $r \equiv 1$, and $q \equiv 0$, then $L_4 y = 0$ is the familiar Euler equation. In this case, L_4 is oscillatory if and only if $k < -9/16$.

Also, $k < -9/16$ and $p(x) = kx^{-4}$ implies $x^2 P_2(x) < -3/32$. Theorem 1.1 shows that $L_4 y = y^{iv} + py$ is nonoscillatory when $x^2 | P_2(x) | \leqq 3/32$.

2. The oscillation of L_4.

Using Theorem 0.1, Hinton [4] showed that L_4 is oscillatory when $\int^\infty 1/r = \infty$, $q \leqq 0$, and $\int^\infty p = -\infty$. The same technique yields the following results.

THEOREM 2.1. *Suppose that* $r(x) \leqq N$, $q(x) \leqq M$ *and* $\int^\infty p = -\infty$ *for* $x > 0$, *then* $L_4 y = (ry'')'' - (qy')' + py$ *is oscillatory on* $(0, \infty)$.

THEOREM 2.2. *If* $0 < r(x) \leqq M$, $\int^\infty q = -\infty$, *and* $\int^\infty x^2 | p(x) | < \infty$ *then* $L_4 y = (ry'')'' - (qy')' + py$ *is oscillatory on* $(0, \infty)$.

Proof. Let $\xi(x) = x^2/2$. Define $y(x)$ as follows:

$$y(x) = \begin{cases} 0 & x < a \\ \xi(x - a) & a \leqq x < a + 1 \\ x - a - 1/2 & a + 1 \leqq x < b_1 \\ -\xi(x - b_2) + b_1 - a & b_1 \leqq x < b_2 = b_1 + 1 \\ b_1 - a & b_2 \leqq x < b_3 \\ -\xi(x - b_3) + b_1 - a & b_3 \leqq x < b_4 \\ -\xi'(b_4 - b_3)(x - b_4) + b_1 - a - \xi(b_4 - b_3) & b_4 \leqq x < b_5 \\ \xi(x - b) & b_5 \leqq x < b \\ 0 & b \leqq x\,. \end{cases}$$

It is easy to show that

$$\int_a^d r(y'')^2 + py^2 < 16M + \int_a^\infty x^2 | p |$$

if we require that $b_4 - b_3 = b - b_5 \leqq 1$. There is a number c such that

$$1 + 16M + \int_a^\infty x^2 | p | + \int_a^{a+1} q(y')^2 + \int_{a+1}^x q \leqq 0$$

for all $x \geqq c$.

Let $Y_1(x) = \int_c^x q(t)dt$. Since $Y_1(x)$ tends to $-\infty$ as x tends to ∞ there is a number b_1 which is the last zero of $Y_1(x)$. Hence,

$$\int_{b_1}^{b_2} q(y')^2 = Y_1(x)(y')^2 \,|_{b_1}^{b_2} - 2\int_{b_1}^{b_2} y'y'' Y_1 < 0$$

since $Y_1(b_1) = 0 = y'(b_2)$, $y'' = -1$, $y' \geqq 0$, and $Y_1 < 0$ on $(b_1, b_2]$.

Let $Y_2(x) = \int_{b_2}^x q(t)dt$ and let b_3 be the last zero of Y_2. Pick b_4 so that $-1/2 \leqq Y_2(x) \leqq 0$ on $[b_3, b_4]$ and $b_4 - b_3 \leqq 1$. Since $y' = 0$ on $[b_2, b_3]$, $-1 \leqq y' \leqq 0$ on $[b_3, b]$, $Y_2 \leqq 0$ on $[b_3, \infty)$, $\int_{b_1}^{b_2} q(y')^2 < 0$ we have that

$$\begin{aligned}\int_{b_1}^{b} q(y')^2 < \int_{b_3}^{b} q(y')^2 &= Y_2(x)(y')^2 \,|_{b_3}^{b} - 2\int_{b_3}^{b} y'y''Y_2 \\ &= -2\int_{b_3}^{b_4} y'(-1)Y_2 - 2\int_{b_5}^{b} y'(1)Y_2 \\ &< 2\int_{b_3}^{b_4} y'Y_2 < 2\int_{b_3}^{b_4} |\, y'Y_2 \,| < 1 \,.\end{aligned}$$

Consequently,

$$\begin{aligned}I_b(y) &= \int_a^b r(y'')^2 + q(y')^2 + py^2 \\ &< 16M + \int_a^\infty x^2 |\, p \,| + \int_a^{a+1} q(y')^2 + \int_{a+1}^{b_1} q + 1 \leqq 0\end{aligned}$$

which completes the proof.

We now know that L_4 is oscillatory on $(0, \infty)$ is for r bounded either $\int^\infty p = -\infty$ and $q \leqq 0$ or $\int^\infty q = -\infty$ and $p \leqq 0$. These facts suggest the results of the following theorem.

THEOREM 2.3. *If $\int^\infty p = -\infty$, $\int^\infty q = -\infty$, and $0 < r(x) \leqq M$ then L_4 is oscillatory on $(0, \infty)$.*

Proof. Except for some changes in the parameters we may define $y(x)$ as in the proof of Theorem 2.2. As before, if $b_4 - b_3 = b_6 - b_5 \leqq 1$ then $\int_a^b r(y'')^2 \leqq 16M$.

There is a number c such that

$$1 + 16M + \int_a^{a+1} q(y')^2 + \int_{a+1}^{x} q < 0$$

for all $x \geqq c$. Let $Y(x) = \int_c^x q(t)dt$ and let b_1 be the last zero of $Y(x)$. Integrating by parts we obtain the fact that

$$\int_{b_1}^{b} q(y')^2 = -2\int_{b_1}^{b} y'y''Y$$

since $Y(b_1) = 0 = y'(b)$. Since $y' \geqq 0$, $y'' = -1$, and $Y \leqq 0$ on $[b_1, b_2]$ then

$$-2\int_{b_1}^{b} y'y''Y < -2\int_{b_2}^{b} y'y''Y \,.$$

Since $y'' = 0$ on $[b_2, b_3]$ and $[b_4, b_5]$, $y' \leqq 0$ on $[b_5, b]$, $y'' = 1$ on $[b_5, b]$, and $Y < 0$ on $[b_5, b]$, then

$$-2\int_{b_2}^{b} y'y''Y < -2\int_{b_3}^{b_4} y'y''Y = 2\int_{b_3}^{b_4} y'Y .$$

But, on $[b_3, b_4]$, $y' \leqq 1$. Consequently,

$$\int_{b_1}^{b} q(y')^2 < 2\int_{b_3}^{b_4} y'Y \leqq 2\int_{b_3}^{b_4} | Y | .$$

Since $\int^{\infty} p = -\infty$, there is a number $d > b_2$ such that for $x \geqq d$

$$\int_{a}^{b_2} py^2 + (b_1 - a)^2 \int_{b_2}^{x} p < 0 .$$

Let $W(x) = \int_{d}^{x} p$ and $b_3 \geqq d$ be the last zero of W. Hence,

$$\int_{b_3}^{b} py^2 = -2\int_{b_3}^{b} yy'W(t)dt < 0 .$$

Let $N = \max\{| Y(x) | : x \in [b_3, b_3 + 1]\}$ which we may assume is greater than or equal to one. Pick b_4 so that $b_4 - b_3 = 1/(2N)$. Consequently,

$$\int_{b_1}^{b} q(y')^2 < 1 .$$

Pick b_5 so that $\lim_{x \to b_5^-} y(x) = (b_4 - b_3)^2/2$ and pick b so that $b - b_5 = b_4 - b_3$. We now have that

$$I_b(y) < 16M + \int_{a}^{a+1} q(y')^2 + \int_{a+1}^{x} q + 1 + \int_{a}^{b_2} py^2 + \int_{b_2}^{b_3} (b_1 - a)^2 p < 0 .$$

THEOREM 2.4. *If* $0 < r(x) \leqq M$, $-\infty < \int^{\infty} p < \infty$, $\int^{\infty} P_1 = -\infty$, $\int^{\infty} | q | x^{-1} < \infty$, *and* $q(x) \to 0$ *as* $x \to \infty$ *then* L_4 *is oscillatory on* $(0, \infty)$.

Proof. Let $\xi(x) = -(3x^3 - 5x^2)/2$, $\alpha(x) = \sqrt{x}$, and $\beta(x) = x^2$. Let

$$y(x) = \begin{cases} 0 & x < a \\ \xi(x - a) & a \leqq x < a + 1 \\ \alpha(x - a) & a + 1 \leqq x < b_1 \\ -\beta(x - b_2) + \alpha(b_1 - a) + \beta(b_1 - b_2) & b_1 \leqq x < b_2 \\ \alpha(b_1 - a) + \beta(b_1 - b_2) & b_2 \leqq x < b_3 \\ -\beta(x - b_3) + y(b_2) & b_3 \leqq x < b_4 \\ \alpha(b - x) & b_4 \leqq x < b_5 \\ \xi(b - x) & b_5 \leqq x < b \\ 0 & b \leqq x . \end{cases}$$

Given b_1 and b_3 we choose b_4, b_5, and b so that $b_4 - b_3 = b_2 - b_1$, $b_5 - b_4 = b_1 - (a + 1)$, and $b = b_5 + 1$. Actually, only b_1 and b_3 will be chosen for reasons other than symmetry and the continuity of y and y'.

First, note that since we are going to pick $b_1, \cdots, b_5$ so that $y \in \mathfrak{D}_2(b)$ then

$$\int_a^b py^2 = -P_1 y^2 |_a^b + \int_a^b P_1(y^2)' = \int_a^b P_1(y^2)' .$$

Hence,

$$I_b(y) \leqq \int_a^b M(y'')^2 + q(y')^2 + P_1(y^2)' .$$

Calculations show that

$$\int_a^b M(y'')^2 \leqq M\left[2\int_0^1 (5 - 9x)^2 + 2 + \frac{1}{2}\int_1^\infty x^{-3}dx\right] \equiv M_1$$

since y' being continuous requires that $0 < b_2 - b_1 = b_4 - b_3 \leqq 1/4$.

Since $\lim_{x\to\infty} q(x) = 0$ and q is continuous then q is bounded by some number, B, on $[a, \infty)$. Let

$$A = 4B\int_0^1 u^2 + B\int_0^1\left(-\frac{9}{2}u^2 + 5u\right)^2 du + 1 .$$

There is a number c so that

$$M_1 + 2 + \int_a^{a+1} [q(y')^2 + P_1(y^2)'] \\ + A + B + (a + 1)\int_{a+1}^\infty x^{-1}|\, q(x)\,| \leqq -\int_{a+1}^x P_1(t)dt$$

and $|\, P_1(x)\,| \leqq 1$ for all $x \geqq c$ since $P_1 \to 0$ as $x \to \infty$.

Let $R(x) = \int_c^x P_1(t)dt$ and b_1 be the last zero of $R(x)$. Pick b_2 so that $1/(2\sqrt{b_1 - a}) = -2(b_1 - b_2)$ which insures that y' is continuous at b_1. We now have that

$$\int_{a+1}^{b_1} q(y')^2 \leqq \int_{a+1}^{b_1} (x - a)^{-1}|\, q\,| = \int_{a+1}^{b_1} x(x - a)^{-1}x^{-1}|\, q\,| \\ < (a + 1)\int_{a+1}^{b_1} x^{-1}|\, q\,| < (a + 1)\int_{a+1}^{\infty} x^{-1}|\, q\,|$$

and

$$\int_{b_1}^{b_2} q(y')^2 \leqq B\int_{b_1}^{b_2} 4(x - b_2)^2 < B$$

since $b_2 - b_1 \leqq 1/4$. Also,

$$\begin{aligned}\int_{b_1}^{b_2} P_1(y^2)' &= 2yy'R\,|_{b_1}^{b_2} - 2\int_{b_1}^{b_2}[(y')^2 + yy'']R \leqq -2\int_{b_1}^{b_2}(y')^2R\\ &= -2\Big[R(t)\int_{b_2}^{t}(y')^2\,|_{b_1}^{b_2} - \int_{b_1}^{b_2}\Big(P_1(t)\int_{b_2}^{t}(y')^2\Big)\Big]\\ &\leqq 2\int_{b_1}^{b_2}|P_1(t)|\int_{b_1}^{b_2}(y')^2 < 1\end{aligned}$$

since $y'(b_2) = 0 = R(b_1)$, $y'' \leqq 0$, $y \geqq 0$, and $R \leqq 0$ on $[b_1, b_2]$. Pick b_3 so that $|P_1(x)| \leqq [6y(b_2)(b_2 - a)]^{-1}$ and $|q(x)| \leqq 4(b_1 - a - 1)^{-1}$ for $x \geqq b_3$. Consequently,

$$\int_{b_2}^{b} P_1(y^2)' = \int_{b_3}^{b} P_1(y^2)' \leqq 2\int_{b_3}^{b}|P_1||y||y'| \leqq 1$$

since $|y| \leqq y(b_2)$, $|y'| \leqq 3$, and $b - b_3 = b_2 - a$. Also,

$$\begin{aligned}\int_{b_2}^{b} q(y')^2 &\leqq B\int_{b_3}^{b_4}4(x - b_3)^2 + (1/4)\int_{b_4}^{b_5}(b_6 - x)^{-1}|q(x)|\\ &\quad + B\int_{b_5}^{b}[5(b_6 - x) - 9(b_6 - x)^2/2]^2\\ &< (A - 1) + (1/4)\int_{b_4}^{b_5}|q| \leqq A\,.\end{aligned}$$

In conclusion,

$$\begin{aligned}I_b(y) &\leqq M_1 + \int_{a}^{a+1} q(y')^2 + (a + 1)\int_{a+1}^{\infty} x^{-1}|q| + B + A\\ &\quad + \int_{a}^{a+1} P_1(y^2)' + \int_{a+1}^{b_1} P_1 + 1 + 1 \leqq 0\,.\end{aligned}$$

The conditions of Theorem 2.4, $\int^{\infty}|q|/x < \infty$ and $\lim_{x\to\infty} q(x) = 0$, could be replaced by the conditions, $\int^{\infty}|q| < \infty$ and q bounded, to obtain the same result with a similar proof.

THEOREM 2.5. *Suppose* $0 < r(x) \leqq M$, $\int^{\infty} p < \infty$, $\int^{\infty} P_1 < \infty$, *and* $P_1(x) \leqq Cx^{-4}$ *for* $x > 0$. *If* $\lim_{x\to\infty}\inf x^2P_2(x) < -7\frac{1}{32}M$ *then* $L_4y = (ry'')'' + py$ *is oscillatory on* $(0, \infty)$.

Proof. We will use the fact that for $a > 0$

$$I_b(y) = \int_{a}^{b}[r(y'')^2 + 2yy'P_1]$$

for y given below. Let $\xi(x) = -(3x^3 - 5x^2)/2$, $\alpha(x) = \sqrt{x}$, and $\beta(x) = x^2$. For $0 < \mu < 1$, $0 < \sigma \leqq 1$, $\rho > 0$, and $0 < \gamma \leqq 1$ define y as follows:

$$y(x)=\begin{cases} 0 & x<\mu\rho \\ \xi\left(\dfrac{x-\mu\rho}{\rho[1-\mu]}\right) & \mu\rho\leqq x<\rho \\ \alpha\left(\dfrac{x-\mu\rho}{\rho[1-\mu]}\right) & \rho\leqq x<R \\ -\beta(x-(R+\sigma))+\beta(\sigma)+\alpha\left(\dfrac{R-\mu\rho}{\rho[1-\mu]}\right) & R\leqq x<R+\sigma \\ \beta(\sigma)+\alpha\left(\dfrac{R-\mu\rho}{\rho[1-\mu]}\right) & R+\sigma\leqq x<N \\ -\beta(x-N)+y(R+\sigma) & N\leqq x<N+\gamma \\ -2\gamma(x-N-\gamma)+y(R+\sigma)-\beta(\gamma) & N+\gamma\leqq x<b-\gamma \\ \beta(x-b) & b-\gamma\leqq x<b \\ 0 & b\leqq x\,. \end{cases}$$

Calculations show that

$$\int_{\mu\rho}^{b} r(y'')^2$$
$$< (1/[\rho^3(1-\mu)])\left[7\frac{1}{32}M(1-\mu)^{-2}+4\sigma M\rho^3(1-\mu)+8\gamma M\rho^3(1-\mu)\right].$$

Since

$$\liminf x^2\int_x^\infty P_1 < -7\frac{1}{32}M\,,$$

there is a $\delta>0$ and a sequence $\langle\rho_k\rangle\to\infty$ for which

$$\lim_{k\to\infty}\rho_k^2\int_{\rho_k}^\infty P_1 \leqq -\left(7\frac{1}{32}M+2\delta\right).$$

Pick μ so close to zero that $7(1/32)M(1-\mu)^{-2}=7(1/32)M+7\delta/8$. There is a positive integer N so large that $\mu\rho_k>a$, $C(\mu^{-3}-1)/\rho_k<\delta/8$, and

$$\rho_k^2\int_{\rho_k}^\infty P_1 \leqq -\left(7\frac{1}{32}M+7\delta/4\right)$$

for all $k\geqq N$. Let $\rho=\rho_N$.

Given R, we will pick σ so that $y'(x)$ is continuous at $x=R$. Therefore,

$$\sigma=1/(4\sqrt{\rho[1-\mu](R-\mu\rho)})\,.$$

Since $\sigma\to0$ as $R\to\infty$ and P_1 is bounded on $[a,\infty)$ pick R so large that

$$\rho^2\int_\rho^R P_1 \leqq -\left(7\frac{1}{32}M+13\delta/8\right),$$

$$\sigma < (\delta/8)/[4M\rho^3(1-\mu)] ,$$

and $\sigma|P_1| < \delta/(8\rho^2)$ for all $x \geqq R$. On $[R, R+\sigma]$,

$$0 < y(x) \leqq \alpha((x-\mu\rho)/(\rho[1-\mu]))$$

and

$$0 \leqq y'(x) \leqq \alpha'((x-\mu\rho)/(\rho[1-\mu]))\cdot 1/(\rho[1-\mu])$$

which implies that $0 \leqq 2yy' \leqq 1/(\rho[1-\mu])$ on $[R, R+\sigma]$.

On $[\mu\rho, \rho]$ $0 \leqq 2yy' < 3/(\rho[1-\mu])$. Hence,

$$\begin{aligned} 2\int_a^b yy'P_1 &< 3(\rho[1-\mu])^{-1}\int_{\mu\rho}^{\rho} P_1^+ + (\rho[1-\mu])^{-1}\int_{\rho}^{R} P_1 \\ &\quad + (\rho[1-\mu])^{-1}\int_R^{R+\sigma} |P_1| + 2\int_N^b |yy'||P_1| \\ &< 3C(\rho[1-\mu])^{-1}\int_{\mu\rho}^{\rho} x^{-4}dx + (\rho[1-\mu])^{-1}\int_{\rho}^{R} P_1 \\ &\quad + (\delta/8)/(\rho^3[1-\mu])^{-1} + 2\int_N^b |yy'||P_1| \end{aligned}$$

where $P_1^+(x) = P_1(x)$ when $P_1(x) \geqq 0$ and zero otherwise.

On $[N, b]$ $0 \leqq y(x) \leqq y(R+\sigma)$ and $|y'| \leqq 2\gamma$. Since y is linear on $[N+\gamma, b-\gamma]$ we have that

$$[y(R+\sigma) - 2\gamma^2]/[b-N-2\gamma] = 2\gamma$$

or

$$b - N = [y(R+\sigma)]/(2\gamma) + \gamma .$$

Since $P_1(x) \to 0$ as $x \to \infty$ we can pick N so large that

$$|P_1| \leqq (\delta/8)/(2[y(R+\sigma)]^2\rho^3[1-\mu])$$

for all $x \geqq N$. Pick γ so small that $2\gamma^2[y(R+\sigma)]^{-1} < 1$ and

$$8M\gamma\rho^3[1-\mu] < \delta/8 .$$

Pick b so that

$$\lim_{x\to(b-\gamma)^-} y(x) = \gamma^2 .$$

We now have that

$$\begin{aligned} 2\int_N^b |yy'||P_1| &\leqq 4\gamma y(R+\sigma)\int_N^b |P_1| \\ &\leqq 2\gamma(b-N)\cdot(\delta/8)/(y(R+\sigma)\rho^3[1-\mu]) \\ &= 2\gamma([y(R+\sigma)/(2\gamma)] + \gamma)\cdot(\delta/8)/(y(R+\sigma)\rho^3[1-\mu]) \\ &= (\delta/8)/(\rho^3[1-\mu]) + 2\gamma^2(\delta/8)/(y(R+\sigma)\rho^3[1-\mu]) \\ &< (\delta/4)/(\rho^3[1-\mu]) . \end{aligned}$$

Consequently,

$$2\int_a^b yy'P_1 < (\rho^3[1-\mu])^{-1}\Big(C(\mu^{-3}-1)\rho^{-1} + \rho^2\int_\rho^R P_1 + 3\delta/8\Big)$$
$$< (\rho^3[1-\mu])^{-1}\Big(\delta/2 + \rho^2\int_\rho^R P_1\Big)\,.$$

Hence,

$$I_b(y) = \int_a^b r(y'')^2 + 2yy'P_1$$
$$< (\rho^3[1-\mu])^{-1}\Big(7\frac{1}{32}M + 7\delta/8 + \delta/8 + \delta/8 + \delta/2 + \rho^2\int_\rho^R P_1\Big) \leqq 0$$

which completes the proof.

3. The oscillation of $L_{2n}y = (-1)^n(ry^{(n)})^{(n)} + py$.

THEOREM 3.1. *If $p(x) \leqq 0$, $0 < r(x) \leqq Mx^\alpha$ for $\alpha < 2n-1$, and*

$$\limsup_{x\to\infty} x^{2n-1-\alpha}\int_x^\infty |\,p(t)\,|dt > MA_n^2$$

where

$$A_n^{-1} = \sqrt{2n-1}\,/[(n-1)!]\sum_{k=1}^n (-1)^{k-1}\binom{n-1}{k-1}(2n-k)^{-1}$$

then $L_{2n}y = (-1)^n(ry^{(n)})^{(n)} + p(x)y$ is oscillatory on $(0, \infty)$.

Proof. Let $\xi(x)$ be the polynomial of degree $2n-1$ such that $\xi(0) = \xi^{(k)}(0) = \xi^{(k)}(1) = 0$ for $k = 1, 2, \cdots, n-1$ and $\xi(1) = 1$. Given $a > 0$, define $y(x)$ as follows:

$$y(x) = \begin{cases} 0 & x < \mu\rho \\ \xi([x-\mu\rho]/[\rho(1-\mu)]) & \mu\rho \leqq x < \rho \\ 1 & \rho \leqq x < R \\ \xi([\nu R - x]/[R(\nu-1)]) & R \leqq x < \nu R \\ 0 & \nu R \leqq x\,. \end{cases}$$

It can be shown that $\int_0^1 (\xi^{(n)}(x))^2dx = A_n^2$.

A result due to Glazman [2, p. 100] considers the case when $\alpha \leqq 0$. Consequently, we will consider here only the case in which $\alpha > 0$. Since

$$\int_{\mu\rho}^{\nu R} r(y^{(n)})^2 \leqq M\rho^\alpha\int_{\mu\rho}^\rho (y^{(n)})^2 + M(\nu R)^\alpha\int_R^{\nu R}(y^{(n)})^2$$
$$= MA_n^2/[\rho^{2n-1-\alpha}(1-\mu)^{2n-1}]$$
$$+ MA_n^2/[R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}]$$

and $p(x) \leqq 0$, then

$$I_{\nu R}(y) = \int_{\mu\rho}^{R} [r(y^{(n)})^2 + py^2]$$
$$\leqq \frac{1}{\rho^{2n-1-\alpha}}\left(\frac{MA_n^2}{(1-\mu)^{2n-1}} + \frac{MA_n^2\rho^{2n-1-\alpha}}{R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}} - \rho^{2n-1-\alpha}\int_{\rho}^{R}|p|\right).$$

There is a sequence $\langle \rho_k \rangle \to \infty$ and a number $\delta > 0$ such that

$$\lim_{k\to\infty} \rho_k^{2n-1-\alpha}\int_{\rho_k}^{\infty} |p| \geqq MA_n^2 + \delta .$$

Choose $\mu > 0$ so small that

$$MA_n^2/[(1-\mu)^{2n-1}] < MA_n^2 + \delta/4 .$$

There is a number K so that $\mu\rho_k > a$ and

$$\rho_k^{2n-1-\alpha}\int_{\rho_k}^{\infty} |p| > MA_n^2 + 3\delta/4$$

for all $k \geqq K$. Set $\rho = \rho_K$. Choose R so large that

$$\rho^{2n-1-\alpha}\int_{\rho}^{R} |p| > MA_n^2 + \delta/2 .$$

Choose $\nu > 1$ so large that

$$MA_n^2\rho^{2n-1-\alpha}/[R^{2n-1-\alpha}(\nu^{1-\alpha/(2n-1)} - \nu^{-\alpha/(2n-1)})^{2n-1}] < \delta/4 .$$

We now have that $I_{\nu R}(y) < 0$ which implies that L_{2n} is oscillatory on $(0, \infty)$.

THEOREM 3.2. *If there are numbers M and α such that $0 < r(x) \leqq Mx^{\alpha}$ and if for some $\nu > 1$ and A_n as in Theorem 3.1*

$$\lim_{x\to\infty}\left(Kx^{\alpha-2n+1} + \int_a^x p\right) = -\infty$$

where $K = MA_n^2\nu^{\alpha}/(\nu-1)^{2n-1}$ then $L_{2n}y = (-1)^n(ry^{(n)}) + py$ is oscillatory on $(0, \infty)$.

Proof. For μ, ρ, R, and ν below, let $y(x)$ be as in the proof of Theorem 3.1. Pick μ and ν so that $0 < \mu < 1$ and $\nu > 1$. Pick ρ so large that $\mu\rho \geqq a$. As in the proof of Theorem 3.1

$$\int_{\mu\rho}^{\nu R} r(y^{(n)})^2 \leqq MA_n^2\left(\frac{\rho^{\alpha-2n+1}}{(1-\mu)^{2n-1}} + \frac{R^{\alpha-2n+1}\nu^{\alpha}}{(\nu-1)^{2n-1}}\right).$$

There is a number c such that

$$MA_n^2\left(\frac{\rho^{\alpha-2n+1}}{(1-\mu)^{2n-1}}+\frac{x^{\alpha-2n+1}\nu^{\alpha}}{(\nu-1)^{2n-1}}\right)+\int_{\mu\rho}^{\rho}py^2+\int_{\rho}^{x}p<0$$

for all $x \geqq c$. Let $T(x)=\int_c^x p$. Since $T(x)\to-\infty$ as $x\to\infty$, there is a last zero of $T(x)$. Let R be the last zero of $T(x)$. This implies that

$$\int_R^{\nu R}py^2=-2\int_R^{\nu R}yy'T(x)<0\ .$$

Since

$$\int_{\mu\rho}^{\nu R}py^2<\int_{\mu\rho}^{\rho}py^2+\int_{\rho}^{R}p$$

then $I_R(y)<0$.

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