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LIPSCHITZ SPACES

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If (S, d) is a metric space and $0 < \alpha < 1$, $\mathrm{Lip}(S, d^\alpha)$ is the Banach space of real or complex-valued functions f on S such that $\|f\| = \max(\|f\|_\infty, \|f\|_{d^\alpha}) < \infty$, where $\|f\|_{d^\alpha} = \sup\{|f(s) - f(t)|d^{-\alpha}(s, t): s \neq t\}$. The closed subspace of functions f such that $\lim_{d(s,t)\to 0} |f(s) - f(t)|\, d^{-\alpha}(s, t) = 0$ is denoted by $\mathrm{lip}(S, d^\alpha)$. The main result is that, when $\inf_{s\neq t} d(s, t) = 0$ $\mathrm{lip}(S, d^\alpha)$ contains a complemented subspace isomorphic with c_0 and $\mathrm{Lip}(S, d)$ contains a subspace isomorphic with l_∞. From the construction, it follows that $\mathrm{lip}(S, d^\alpha)$ is not isomorphic to a dual space nor is it complemented in $\mathrm{Lip}(S, d^\alpha)$.

If E is a normed space, E^* denotes its dual. c_0, l_1, and l_∞ denote the usual sequence spaces; "isomorphism" means "linear homeomorphism"; "projection" means "continuous projection"; and F is a complemented subspace of E if there is a projection of E on F.

In recent years, much work has been done on the Banach space properties (isomorphic and isometric) of Lipschitz functions. Some of the main references are [1], [2], [3], [4], [10], [11], and [12]. There are still many outstanding conjectures concerning these spaces, some of which seem to be fairly difficult; especially certain questions about extreme points. The known results along these lines can be found in [2], [3], [4], [10], and [11].

It is established in [1] that if (S, d) is an infinite compact subset of Euclidean space, then $\mathrm{lip}(S, d^\alpha)(0 < \alpha < 1)$ is isomorphic with c_0 and $\mathrm{Lip}(S, d^\alpha)$ is isomorphic with l_∞. The proof given in [1] is not evidently adaptable to arbitrary compact metric spaces. Thus, many of the natural conjectures one might make concerning properties that Lipschitz spaces may share with these sequence spaces are still unresolved. Of course the main one is whether $\mathrm{lip}(S, d^\alpha)$ with $0 < \alpha < 1$ and S compact and infinite is isomorphic with c_0.

It is shown in [1, Remark, p. 319] that for (S, d) compact and $0 < \alpha < 1$, $\mathrm{lip}(S, d^\alpha)$ and $\mathrm{Lip}(S, d^\alpha)$ are isomorphic to subspaces of c_0 and l_∞ respectively. (Although the sketch of the proof given there is not precisely right, Professor Frampton, in a private communication, has exhibited a correct one for which we thank him.) It is well known that an infinite dimensional subspace A of c_0 contains a subspace B isomorphic to c_0 and that, since A is separable, B is complemented. In § 1 of this paper, we show that $\mathrm{lip}(S, d^\alpha)$, $0 < \alpha < 1$, contains a complemented subspace isomorphic to c_0 when $\inf\{d(s, t): s \neq t\} = 0$ (Theorem 1). It is also shown that $\mathrm{lip}(S, d^\alpha)$ is separable if and only if (S, d) is precompact (Theorem 2). (Let us remark here that if $(\bar{S}, d)$ is the completion of (S, d) then the restriction map is an isometric isomorphism of Lip $(\bar{S}, d^\alpha)$ onto Lip (S, d^α) which sends lip $(\bar{S}, d^\alpha)$ onto lip (S, d^α) $(0 < \alpha \leqq 1)$.)

In addition, we show in Theorem 1 that for any infinite metric space (S, d) and $0 < \alpha \leqq 1$, Lip (S, d^α) contains a subspace isomorphic to l_∞.

In § 2, we discuss some open problems concerning the isomorphic types of the Lipschitz spaces.

In § 3, we consider some questions about the extreme points of the unit ball and dual unit ball of these spaces.

1. In this section we prove our main theorems. We begin by stating

THEOREM 1. *Let (S, d) be a metric space with* $\inf \{d(s, t): s \neq t\} = 0$. *Then* Lip (S, d) *contains a subspace isomorphic to l_∞ and* lip (S, d^α), $0 < \alpha < 1$, *contains a complemented subspace isomorphic to c_0.*

REMARK 1. Theorem 1 has been announced in [5] along with Corollaries 1 and 2 below. It is pointed out in [4, Lemma 2.5] that if $\inf \{d(s, t): s \neq t\} > 0$, then both lip (S, d^α) and Lip (S, d^α) are isomorphic with the bounded functions on S.

The next two corollaries are immediate. We assume (S, d) and α are as in Theorem 1.

COROLLARY 1. lip (S, d^α) *is not isomorphic with a dual space.*

Proof. It has been observed in [7, p. 16] by Lindenstrauss (and is not hard to see) that if a Banach space is complemented in some dual space, it is complemented in its second dual. Hence, if lip (S, d^α) is isomorphic with a dual space, c_0 is complemented in $l_\infty \cdots a$ contradiction.

COROLLARY 2. lip (S, d^α) *is not complemented in* Lip (S, d^α).

Proof. Let E and E_0 denote the subspaces isomorphic with l_∞ and c_0 respectively. In the proof of Theorem 1, E and E_0 are constructed so that $E_0 \subset E$. Let P be a projection of lip (S, d^α) on E_0 and suppose Q is a projection of Lip (S, d^α) on lip (S, d^α). It is then easy to see that PQ restricted to E is a projection of E onto E_0. This is a contradiction.

REMARK 2. The theorem also implies the author's result [4, Theorem 2.6].

REMARK 3. l_∞ is complemented in any Banach space containing it. The following notation is used throughout:

$$(S, d) \text{ is a metric space },$$

$$\bar{B}(s, r) = \{t \in S \mid d(s, t) \leqq r\}$$

and

$$\overset{\circ}{B}(s, r) = \{t \in S \mid d(s, t) < r\} .$$

If $A \subset S$ and $B \subset S$, $d(A, B) = \inf \{d(s, t) \mid s \in A, t \in B\}$. $d(A, \{t\})$ is denoted by $d(A, t)$.

In [5], we sketched the proof of Theorem 1 in the case where S has no nonconstant Cauchy sequences. We present here the proof where S is assumed to have a limit point. (Recall that completeness may be assumed without loss of generality.) Since the case $\alpha = 1$ for $\mathrm{Lip}\,(S, d^\alpha)$ is similar to the discrete case appearing in [5], we will omit it.

Proof of the Theorem. We begin by constructing a sequence of closed balls that will serve as supports for certain functions.

Let s_0 be a fixed limit point of S. Choose $s_1 \in S$ with $0 < d(s_1, s_0)$ and define $r_1 = (1/2)d(s_1, s_0)$, $B_1 = \bar{B}(s_1, r_1)$ and $p_1 = d(s_0, B_1)$. Now, assume that s_j ($j \geqq 1$) has been chosen such that $d(s_j, s_0) > 0$. Set $r_j = (1/2)d(s_j, s_0)$, $B_j = \bar{B}(s_j, r_j)$ and $p_j = d(s_0, B_j)$. We first note that

$$(1) \qquad r_j \leqq p_j .$$

Proof. Otherwise, there is a point $t \in B_j$ with $d(s_0, t) < r_j$. Hence, $d(s_0, s_j) \leqq d(s_0, t) + d(t, s_j) < 2r_j$, a contradiction.

Now, $p_j > 0$ implies that there is a point s_{j+1} with $0 < d(s_{j+1}, s_0) < (1/6)p_j$. Set $r_{j+1} = (1/2)d(s_{j+1}, s_0)$, $B_{j+1} = \bar{B}(s_{j+1}, r_{j+1})$, and $p_{j+1} = d(B_{j+1}, s_0)$.

We record the following facts concerning our construction which are needed later.

$$(2) \qquad p_j < \frac{1}{6}p_{j-1} \quad \text{for each} \quad j .$$

Proof. $p_j \leqq d(s_0, s_j) < (1/6)p_{j-1}$.

$$(3) \qquad B_j \subset \overset{\circ}{B}\left(s_0, \frac{1}{2}p_{j-1}\right) \quad \text{for each} \quad j .$$

Proof. Let $t \in B_j$. Then $d(t, s_0) \leqq d(t, s_j) + d(s_j, s_0) \leqq r_j + 2r_j \leqq 3p_j < (1/2)p_{j-1}$ by (1) and (2).

$$(4) \qquad \text{If} \quad j > i , \quad d(B_i, B_j) \geqq \frac{1}{2}p_i .$$

Proof. If $s \in B_i$, $d(s, s_0) \geqq p_i$. If $t \in B_j$, $d(t, s_0) < (1/2)p_{j-1} \leqq (1/2)p_i$ by (3) and (2). Hence, $d(s, t) \geqq d(s, s_0) - d(s_0, t) \geqq (1/2)p_i$.

$$\text{(5)} \qquad \begin{array}{l} d(s, \tilde{B}_j) \leqq 3p_j \text{ for each } j \text{ and each } s \in S\,, \\ \text{where } \tilde{B}_j \text{ denotes the complement of } B_j\,. \end{array}$$

Proof. Assume $s \in B_j$ since the assertion is otherwise trivial. Given $\varepsilon > 0$, there is $t \in B_j$ such that $d(t, s_0) < \varepsilon + d(B_j, s_0) = \varepsilon + p_j$. Thus, $d(s, \tilde{B}_j) \leqq d(s, s_0) \leqq d(s, s_j) + d(s_j, t) + d(t, s_0) \leqq r_j + r_j + \varepsilon + p_j \leqq 3p_j + \varepsilon$, and the assertion follows.

If $s \in B_i$ and $t \in B_j$ with $j > i$, then

$$\text{(6)} \qquad \frac{d(s, \tilde{B}_i) + d(t, \tilde{B}_j)}{d(B_i, B_j)} < 7\,.$$

Proof. By (5) and (4),

$$\frac{d(s, \tilde{B}_i) + d(t, \tilde{B}_j)}{d(B_i, B_j)} \leqq \frac{3p_i + 3p_j}{(1/2)p_i} = 6\left[1 + \frac{p_j}{p_i}\right].$$

Since $j > i$, $p_j < (1/6)p_{j-i} \leqq (1/6)p_i$. Hence, the assertion follows.

We next proceed to construct the isomorphism. We will assume in the proof that for each j, $d(s, \tilde{B}_j) \leqq 1$ for all $s \in S$ and $d(s, t) \leqq 1$ for all $s, t \in B_j$. This clearly can be done by taking j large enough (see (5) and (1)).

First choose a sequence $\{\beta_j\}$ converging to α such that $\alpha < \beta_j < 1$, $r_j^{\beta_j - \alpha} \geqq 1/2$, and $d^{\beta_j - \alpha}(s_j, \tilde{B}_j) \geqq 1/2$ for each j. Now, define $f_j(s) = d^{\beta_j}(s, \tilde{B}_j)$ for each j and $s \in S$. Then, given $a = \{a_j\} \in l_\infty$, define

$$f_a = \sum_{j=1}^{\infty} a_j f_j\,.$$

It is easy to see that, since the nonzero functions f_j have disjoint supports B_j, the function f_a is well-defined and the mapping $a \to f_a$ is one-to-one and linear.

Next, let us note that

$$\|f_a\| \geqq \frac{1}{2^{1+\alpha}}\|a\|$$

for each $a \in l_\infty$. To see this, observe that for each j,

$$\begin{aligned} \|f_a\|_{d^\alpha} &\geqq \frac{|f_a(s_j) - f_a(s_0)|}{d^\alpha(s_j, s_0)} = \frac{|a_j| d^{\beta_j}(s_j, \tilde{B}_j)}{d^\alpha(s_j, s_0)} \\ &= \frac{|a_j| d^{\beta_j}(s_j, \tilde{B}_j)}{(2r_j)^\alpha} \geqq \frac{|a_j| r_j^{\beta_j}}{2^\alpha r_j^\alpha} \geqq \frac{|a_j|}{2^{1+\alpha}} \end{aligned}$$

by our choice of β_j. Since j was arbitrary, the desired inequality follows.

Now, we will show that $a \to f_a$ is bounded and that $f_a \in \operatorname{lip}(S, d^\alpha)$ when $a \in c_0$. The boundedness of $a \to f_a$ will show that f_a is in fact in $\operatorname{Lip}(S, d^\alpha)$.

In what follows, assume that $a \in l_\infty$ is fixed with $|a_j| \leqq 1$ for each j and set $f = f_a$. First note that

$$||f||_\infty = \sup_n ||a_n f_n||_\infty \leqq \sup_n |a_n| \leqq 1 .$$

We next proceed to show that $||f||_{d^\alpha} \leqq 7^\alpha \cdot 2^{1-\alpha} \leqq 7$.

Let $s \in S$, $t \in S$, $s \neq t$. If $s \in B_i$ and

$$t \notin \bigcup_{j=1}^{\infty} B_j ,$$

then

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} = \frac{|a_i| d^{\beta_i}(s, \tilde{B}_i)}{d^\alpha(s, t)} \leqq |a_i| d^{\beta_i - \alpha}(s, \tilde{B}_i) \leqq 1 ,$$

while if $t \in B_i$, then

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} \leqq |a_i| d^{\beta_i - \alpha}(s, t) \leqq 1 .$$

Thus, suppose $s \in B_i$, $t \in B_j$ and $j > i$. Then

$$\begin{aligned} \frac{|f(s) - f(t)|}{d^\alpha(s, t)} &\leqq \frac{|f(s)| + |f(t)|}{d^\alpha(s, t)} \leqq \frac{d^{\beta_i}(s, \tilde{B}_i) + d^{\beta_j}(t, \tilde{B}_j)}{d^\alpha(B_i, B_j)} \\ &\leqq \frac{d^\alpha(s, \tilde{B}_i) + d^\alpha(t, \tilde{B}_j)}{d^\alpha(B_i, B_j)} , \end{aligned}$$

since $\beta_i, \beta_j > \alpha$ and $d(s, \tilde{B}_k) \leqq 1$ for all k. Now, the last quotient does not exceed

$$2^{1-\alpha}\left[\frac{d(s, \tilde{B}_i) + d(t, \tilde{B}_j)}{d(B_i, B_j)}\right]^\alpha$$

since

$$\frac{p^\alpha + q^\alpha}{2} \leqq \left(\frac{p + q}{2}\right)^\alpha$$

for $p \geqq 0$, $q \geqq 0$. Hence, from (6) above,

$$||f||_{d^\alpha} \leqq 1 \vee 7^\alpha \cdot 2^{1-\alpha} \leqq 7^\alpha \cdot 2^{1-\alpha} \leqq 7 .$$

Thus, $\{f_a \mid a \in l_\infty\}$ is isomorphic with l_∞.

Next, let $a \in c_0$, $||a|| \leqq 1$, and let $\varepsilon > 0$ be given. We must find $\delta > 0$ so that $0 < d(s, t) < \delta$ implies that

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} \leqq \varepsilon .$$

Since $a \in c_0$, there is a number N such that $n \geqq N$ implies $|a_n| < (1/14)\varepsilon$. There exists a $\delta_0 > 0$ such that if $i \neq j$ and $d(B_i, B_j) < \delta_0$, then $i \geqq N$ and $j \geqq N$. This follows from (4) above. Having chosen N, take $0 < \delta < \min\{\varepsilon^{1/\beta_j - \alpha} \mid 1 \leqq j < N\}$ and $\delta < \delta_0$. Then $0 < d(B_i, B_j) < \delta$ still implies $i, j \geqq N$.

Now, let $0 < d(s, t) < \delta$. Suppose $s \in B_i$ and $t \notin \bigcup_{j \neq i} B_j$. If $i < N$, then

$$\frac{|f(s) - f(t)|}{d^\alpha(s, t)} \leqq |a_i| d^{\beta_i - \alpha}(s, t) \leqq \delta^{\beta_i - \alpha} \leqq \varepsilon .$$

If $i \geqq N$, $|a_i| d^{\beta_i - \alpha}(s, t) \leqq |a_i| \delta^{\beta_i - \alpha} \leqq |a_i| < \varepsilon/14 < \varepsilon$. Next, suppose $s \in B_i$ and $t \in B_j$ $(j > i)$. Then

$$\begin{aligned} \frac{|f(s) - f(t)|}{d^\alpha(s, t)} &\leqq \frac{|a_i| f_i(s) + |a_j| f_j(t)}{d^\alpha(s, t)} \\ &\leqq (|a_i| + |a_j|)\frac{d^{\beta_i}(s, \widetilde{B}_i) + d^{\beta_j}(t, \widetilde{B}_j)}{d^\alpha(s, t)} \\ &\leqq \left(\frac{\varepsilon}{14} + \frac{\varepsilon}{14}\right)\cdot 7 = \varepsilon ; \end{aligned}$$

this is because $0 < d(s, t) < \delta$, $s \in B_i$ and $t \in B_j$ imply $d(B_i, B_j) < \delta$ and hence $i, j \geqq N$.

The only part now remaining is to show that "c_0" is complemented in $\operatorname{lip}(S, d^\alpha)$. Our proof will entail the construction of a projection of $\operatorname{Lip}(S, d^\alpha)$ onto the image of l_∞ which sends $\operatorname{lip}(S, d^\alpha)$ onto the image of c_0. As mentioned before, since l_∞ is injective, it is already known that it must be complemented in $\operatorname{Lip}(S, d^\alpha)$.

Given $f \in \operatorname{Lip}(S, d^\alpha)$, define

$$Pf = \sum_n a_n f_n$$

where

$$a_n = \frac{f(s_n) - f(s_0)}{f_n(s_n)} .$$

It is easy to see that P is linear and that $P^2 = P$. Let $||f|| \leqq 1$. We must first find a constant M such than $|a_n| \leqq M$ for each n. $d(s_n, s_0) = 2r_n$, for each n, by definition; thus,

$$d^\alpha(s_n, s_0) = 2^\alpha r_n^\alpha \leqq 2^\alpha d^\alpha(s_n, \widetilde{B}_n) \leqq 2^{1+\alpha} d^{\beta_n}(s_n, \widetilde{B}_n)$$

for each n, since $\beta_n - \alpha$ was chosen small enough so that

$$d^{\beta_n - \alpha}(s_n, \widetilde{B}_n) \geqq \frac{1}{2} .$$

Hence,

$$|f(s_n) - f(s_0)| \leqq d^\alpha(s_n, s_0) \leqq 2^{1+\alpha} d^{\beta_n}(s_n, \tilde{B}_n) = 2^{1+\alpha} f_n(s_n)$$

for each n. Therefore, $|a_n| \leqq 2^{1+\alpha}$ when $\|f\| \leqq 1$.

Now, let $f \in \operatorname{lip}(S, d^\alpha)$. We must show that

$$\lim_{n\to\infty} \frac{|f(s_n) - f(s_0)|}{f_n(s_n)} = 0 .$$

As above, we have $d^\alpha(s_n, s_0) \leqq 2^{1+\alpha} f_n(s_n)$, so

$$\lim_{n\to\infty} \frac{|f(s_n) - f(s_0)|}{f_n(s_n)} \leqq 2^{1+\alpha} \lim_{n\to\infty} \frac{|f(s_n) - f(s_0)|}{d^\alpha(s_n, s_0)} = 0 .$$

This completes the proof of the theorem.

We close this section with Theorem 2 which answers a question raised in [5].

THEOREM 2. *The following are equivalent for* $0 < \alpha < 1$.

(a) (S, d) *is precompact.*

(b) $\operatorname{lip}(S, d^\alpha)^*$ *is separable.*

(c) $\operatorname{lip}(S, d^\alpha)$ *is separable.*

Proof. In [3] Jenkins showed that if (S, d) is compact, the span of the point evaluations is dense in $\operatorname{lip}(S, d^\alpha)^*$. It is clear that $\|\varepsilon_s - \varepsilon_t\| \leqq d^\alpha(s, t)$, where $\varepsilon_s(f) = f(s)$. Thus, $\{\varepsilon_s \mid s \in S\}$, and hence $\operatorname{lip}(S, d^\alpha)^*$, is separable. (See [4] for further discussion.) Thus (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) is true for any Banach space, so assume (a) fails. Then there exists a sequence $\{s_n\} \subset S$ and a number $p > 0$ such that $d(s_n, s_m) \geqq p$ for each $n \neq m$. Let $\{s_{n_k}\}$ be any subsequence of $\{s_n\}$ and let $A = \{s_{n_{2k+1}}\}$. Now, the function f defined by $f(s) = \min\{d(s, A), 1\}$ is an element of $\operatorname{lip}(S, d^\alpha)$, where $d(s, A) = \inf\{d(s, t) \mid t \in A\}$. However, $f(s_{n_{2k}}) \geqq \min(p, 1) > 0$ for each k, while $f(s_{n_{2k+1}}) = 0$ for each k. Thus, $\lim_k f(s_{n_k}) = \lim_k \varepsilon_{s_{n_k}}(f)$ does not exist; i.e., $\{\varepsilon_{s_n}\}$ has no weak*-convergent subsequence. This implies that the dual unit ball of $\operatorname{lip}(S, d^\alpha)$ is not w^*-metrizable, which completes the proof of the proposition by contradicting (c).

2. An investigation of the Banach space properties of the Lipschitz spaces is far from complete, and questions concerning those properties of (S, d^α) that give rise to corresponding properties of the Lipschitz spaces are abundant. The ultimate problem of classifying these spaces as to isomorphic type does not appear easy. Even the following two problems are still open:

If (S, d) is compact and infinite, and $0 < \alpha < 1$,

(1) is $\operatorname{lip}(S, d^\alpha)$ isomorphic with c_0 and

(2) is Lip (S, d^α) isomorphic with l_∞?

By [4, Theorem 4.7] it is known that Lip (S, d^α) is isometrically isomorphic with the bidual of lip (S, d^α). Hence, a positive answer to (1) yields a positive answer to (2). As we mentioned in the introduction, the best result in this direction appears in [1].

One possible avenue of attack on question (2) may be furnished by the following proposition, since it seems that the problem of showing (b) or (c) may be more tractable than showing (e) directly. (For the definitions of $\mathscr{L}_1$ and $\mathscr{L}_\infty$ spaces see [9]. A Banach space is injective if it is complemented in every Banach space containing it.)

PROPOSITION 1. *Let (S, d) be compact and infinite. If $0 < \alpha < 1$, the following assertions are equivalent.*

(a) Lip (S, d^α) *is injective.*

(b) Lip (S, d^α) *is an $\mathscr{L}_\infty$ space.*

(c) lip (S, d^α) *is an $\mathscr{L}_\infty$ space.*

(d) lip $(S, d^\alpha)^*$ *is an $\mathscr{L}_1$ space.*

(e) Lip (S, d^α) *is isomorphic with l_∞.*

(a) *and* (b) *are equivalent even if $\alpha = 1$ and (S, d) is arbitrary.*

Proof. [9, Remark 2, p. 337] yields (a) $\Leftrightarrow$ (b) immediately since Lip (S, d) is a dual space for *any* metric space [4]. (a) $\Leftrightarrow$ (c) is [9, Corollary, p. 335]. (d) $\Leftrightarrow$ (b) is [9, Theorem I (iii), p. 327]. (d) $\Leftrightarrow$ (e) is from the observation in [9, Problem 2a, p. 344] and the fact that lip $(S, d^\alpha)^*$ is separable (see [3]).

Although questions (1) and (2) are the most important, the following questions are also open in general.

(3) Does Lip (S, d) have the approximation property?

(4) If (S, d) is compact and $0 < \alpha < 1$, do lip $(S, d^\alpha)^*$ and lip (S, d^α) have Schauder bases?

Let us remark that in [3] it was shown that lip$(S, d^\alpha)^*$ is separable.

Added in proof: By Enflo's example, there is a (non-compact) metric space for which Lip(S, d) fails the approximation property. Using an idea due to Lindenstrauss it can be shown that Lip(S, d) is not injective if (S, d) is the Hilbert cube.

3. In addition to the questions in § 2 concerning the isomorphism types of the Lipschitz spaces, there are some interesting problems dealing with the extreme points of their unit balls and dual unit balls.

We begin by stating a theorem due to Lindenstrauss and Phelps

[8, Theorem 3.1]:

(I) If E is a normed space whose dual unit ball has countably many extreme points, then E^* is separable and E contains no infinite dimensional reflexive subspaces.

Quite recently William Johnson and Haskell Rosenthal [6] proved:

(II) If E is an infinite dimensional Banach space with E^{**} separable, then E and E^* have infinite dimensional reflexive subspaces.

(The author would like to thank Professors Johnson and Rosenthal for access to a preprint of [6].)

In view of (I), (II) now has as an immediate corollary the following:

(III) The unit ball of E^{**}, for any infinite dimensional Banach space, has uncountably many extreme points.

The aforementioned results have some immediate applications to Lipschitz spaces. We proceed to mention a few.

Since it is known that $\mathrm{Lip}\,(S, d^\alpha)$ is a second dual space for $0 < \alpha < 1$ (see [3] and [4]), it follows from (III) that its unit ball has uncountably many extreme points. In view of (I), Theorem 1 and the fact [4, Theorem 4.1] that $\mathrm{Lip}\,(S, d)$ is a dual space for *any* metric space, we can state the following:

PROPOSITION 2. *If* (S, d) *is any metric space with* S *infinite, then the unit ball of* $\mathrm{Lip}\,(S, d)$ *has uncountable many extreme points.*

Of course, since $\mathrm{Lip}\,(S, d)$ is a dual space, its unit ball is the w^*-closed convex hull of its extreme points. As shown in [4, Corollary 4.4], convergence of bounded nets in the w^*-topology coincides with pointwise convergence in general, and with uniform convergence when (S, d) is compact. Thus, in both senses, the unit ball of $\mathrm{Lip}\,(S, d)$ has many extreme points. The problem of characterizing the extreme points of the unit ball of $\mathrm{Lip}\,(S, d)$ appears to be quite difficult. The only results we know of this kind are in [10] and [11], and these are for $S = [0, 1]$. In both papers a proof is given that the unit ball of $\mathrm{Lip}\,[0, 1]$ is the *norm*-closed convex hull of its extreme points. This problem is also open for more general metric spaces.

Assuming S is compact and countable, (II) yields another previously unknown result. Again in [3] the extreme points of the unit ball of $\mathrm{Lip}\,(S, d)^*$ are shown to be of two types: one corresponding to a subset of $S \cup [(S \times S) \sim \Delta]$ and the other a set Q "arising from" the Stone-Čech compactification of $(S \times S) \sim \Delta$ (see [3] or [4]). It was shown in [4] that, in general, Q must be nonempty. It now follows from (III) that if S is compact and countably infinite, Q must be uncountable. The work of Sherbert [12] shows that the functionals in Q must be point derivations. However, a complete description of Q still appears difficult. We sum up this result in

PROPOSITION 3. *If S is compact and countably infinite, the set Q of extreme points of the unit ball of* Lip $(S, d)^*$ *that are point derivations is uncountable.*

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