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PRODUCT INTEGRALS AND INVERSES IN NORMED RINGS

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This paper concerns product integrals of functions with values in a normed complete ring. The inverses of elements obtained as such integrals are investigated. In particular, the conditions under which $[{}_x\prod^y(1+G)]^{-1}$ exists are shown to be related to the requirement that $\int_x^y |G^2| = 0$. Since the existence of $[{}_x\prod^y(1+G)]^{-1}$ is connected with the existence of the product integrals ${}_y\prod^x(1+G)$ and ${}_x\prod^y(1-G)$, the study of the inverse leads to a study of the conditions under which these integrals exist when ${}_x\prod^y(1+G)$ is known to exist. Commutative and noncommutative rings are considered.

II. Definitions. All integrals and definitions are of the subdivision-refinement type; functions are from $S \times S$ to N, where S denotes a linearly ordered set and N denotes a ring which has a multiplicative identity element represented by 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. The statements that G is bounded, $G \in OP^\circ$, $G \in OP'$, and $G \in OB^\circ$ on $\{a, b\}$ mean there exist a subdivision D of $\{a, b\}$ and a number B such that if $J = \{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|G(u)| < B$ for $u \in J(I)$,

(2) $|\prod_{q=i}^j (1 + G_q)| < B$ for $1 \leqq i \leqq j \leqq n$,

(3) $|\prod_{J(I)} (1 + G)| < B$, and

(4) $\sum_{J(I)} |G| < B$,

respectively, where $G_q = G(x_{q-1}, x_q)$ and $J(I) = \{(x_{q-1}, x_q)\}_{q=1}^n$. Also,

(1) $G \in OA^\circ$ on $\{a, b\}$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$, and

(2) $G \in OM^\circ$ on $\{a, b\}$ only if ${}_x\prod^y(1 + G)$ exists for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$ and $\int_a^b |1 + G - \prod (1 + G)| = 0$.

The statement that $G \in OD^\circ$ on $\{a, b\}$ means if $\varepsilon > 0$ then there exists a subdivision D of $\{a, b\}$ such that if $\{x_q\}_{q=0}^n$ is a refinement of D and $1 \leqq i \leqq j \leqq n$, then

$$\left|1 - \left[\prod_{q=i}^j (1 + G_q)\right]\left[\prod_{q=i}^j (1 - G_{j+i-q})\right]\right| < \varepsilon$$

and

$$\left|1 - \left[\prod_{q=i}^j (1 - G_{j+i-q})\right]\left[\prod_{q=i}^j (1 + G_q)\right]\right| < \varepsilon .$$

If N is commutative, then the preceding inequalities are equivalent to the requirement that

$$\left|1 - \prod_{q=i}^{j} (1 - G_q^2)\right| < \varepsilon \; .$$

In the following treatment it is assumed that $\{a, b\}$ is in the linear ordering of S. Thus, if $\{x_q\}_{q=0}^n$ is a subdivision of $\{a, b\}$, then

(1) $\int_a^b G \sim \sum_{q=1}^n G(x_{q-1}, x_q)$,

(2) $\int_b^a G \sim \sum_{q=1}^n G(x_{n+1-q}, x_{n-q})$,

(3) ${}_a\prod^b (1 + G) \sim \prod_{q=1}^n [1 + G(x_{q-1}, x_q)]$, and

(4) ${}_b\prod^a (1 + G) \sim \prod_{q=1}^n [1 + G(x_{n+1-q}, x_{n-q})]$.

Similar considerations hold for OP°, OP', and OB°. Note that if $G(x, y) = - G(y, x)$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, then

$$\prod_{q=1}^{n} [1 + G(x_{n+1-q}, x_{n-q})] = \prod_{q=1}^{n} [1 - G(x_{n-q}, x_{n+1-q})] \; .$$

We adopt the conventions that $G(x, x) = 0$ and $\prod_{q=r}^s (1 + G_q) = 1$ if $r > s$. See B. W. Helton [2] and J. S. MacNerney [5] for additional details.

III. Results: noncommutative rings.

LEMMA 3.1. *If* $\{a_q\}_{q=1}^n$ *is a sequence of elements of* N, *then*

$$\begin{aligned}&\left|1 - \left[\prod_{q=1}^{n} (1 + a_q)\right]\left[\prod_{q=1}^{n} (1 - a_{n+1-q})\right]\right| \\ &\qquad \leqq \sum_{q=1}^{n} [\,|a_q^2|\,]\left[\left|\prod_{j=1}^{q} (1 + a_j)\right|\right]\left[\left|\prod_{j=2}^{q} (1 - a_{q+1-j})\right|\right] .\end{aligned}$$

Indication of proof. Lemma 3.1 can be established by induction.

LEMMA 3.2. *If* G *is a function from* $S \times S$ *to* N *such that* $\int_a^b |G^2| = 0$, $G \in OP^\circ$ *on* $\{a, b\}$ *and* $\{b, a\}$ *and* $G(x, y) = - G(y, x)$ *for each subdivision* $\{a, x, y, b\}$ *of* $\{a, b\}$, *then* $G \in OD^\circ$ *on* $\{a, b\}$.

Indication of proof. Lemma 3.2 follows as a corollary to Lemma 3.1.

LEMMA 3.3. *If* G *is a function from* $S \times S$ *to* N *such that* $\int_a^b |G^2| = 0$, ${}_a\prod^b (1 + G)$ *exists,* $G \in OP^\circ$ *on* $\{a, b\}$ *and* $\{b, a\}$, $G(x, y) = - G(y, x)$ *for each subdivision* $\{a, x, y, b\}$ *of* $\{a, b\}$ *and* $\varepsilon > 0$, *then there exists a subdivision* D *of* $\{a, b\}$ *such that if* $\{x_q\}_{q=0}^n$ *and* $\{y_q\}_{q=0}^m$ *are refinements of* D, *then*

$$\left|1 - \left\{\prod_{q=1}^{n}[1 + G(x_{q-1}, x_q)]\right\}\left\{\prod_{q=1}^{m}[1 - G(y_{m-q}, y_{m+1-q})]\right\}\right| < \varepsilon$$

and

$$\left|1 - \left\{\prod_{q=1}^{m}[1 - G(y_{m-q}, y_{m+1-q})]\right\}\left\{\prod_{q=1}^{n}[1 + G(x_{q-1}, x_q)]\right\}\right| < \varepsilon .$$

Indication of proof. Lemma 3.3 follows by using Lemma 3.2 and the Cauchy criterion for product integrals.

LEMMA 3.4. *If G is a function from $S \times S$ to N such that $G \in OD^\circ$ on $\{a, b\}$, $G \in OP'$ on $\{a, b\}$ and $\{b, a\}$, ${}_a\prod^b(1 + G)$ exists, ${}_b\prod^a(1 + G)$ exists and $G(x, y) = -G(y, x)$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, then $[{}_a\prod^b(1 + G)]^{-1}$ exists and is ${}_b\prod^a(1 + G)$.*

Proof. Let $\varepsilon > 0$. There exists a subdivision D of $\{a, b\}$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|\prod_{q=1}^n(1 + G_q)| < B$,
(2) $|\prod_{q=1}^n(1 - G_{n+1-q})| < B$,
(3) $|1 - [\prod_{q=1}^n(1 + G_q)][\prod_{q=1}^n(1 - G_{n+1-q})]| < \varepsilon/4$,
(4) $|{}_a\prod^b(1 + G) - \prod_{q=1}^n(1 + G_q)| < \varepsilon(4B)^{-1}$
(5) $|{}_b\prod^a(1 + G) - \prod_{q=1}^n(1 - G_{n+1-q})| < \varepsilon(4B)^{-1}$, and
(6) $[\varepsilon(4B)^{-1}]^2 < \varepsilon/4$.

Suppose $\{x_q\}_{q=0}^n$ is a refinement of D. Let P_1 and P_2 denote $\prod_{q=1}^n(1 + G_q)$ and $\prod_{q=1}^n(1 - G_{n+1-q})$, respectively. Thus,

$$\begin{aligned}&|1 - [{}_a\prod^b(1 + G)][{}_b\prod^a(1 + G)]|\\ &\quad \leqq |1 - P_1P_2| + |{}_a\prod^b(1 + G) - P_1||{}_b\prod^a(1 + G) - P_2|\\ &\quad + |{}_a\prod^b(1 + G) - P_1||P_2| + |{}_b\prod^a(1 + G) - P_2||P_1|\\ &\quad < \varepsilon/4 + [\varepsilon(4B)^{-1}]^2 + B[\varepsilon(4B)^{-1}] + B[\varepsilon(4B)^{-1}] < \varepsilon .\end{aligned}$$

Therefore, $[{}_a\prod^b(1 + G)]^{-1}$ exists and is ${}_b\prod^a(1 + G)$.

THEOREM 3.1. *If G is a function from $S \times S$ to N such that $\int_a^b|G^2| = 0$, ${}_a\prod^b(1 + G)$ exists, $G \in OP^\circ$ on $\{a, b\}$ and $\{b, a\}$ and $G(x, y) = -G(y, x)$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, then*

(1) *${}_b\prod^a(1 + G)$ exists, and*
(2) *$[{}_a\prod^b(1 + G)]^{-1}$ exists and is ${}_b\prod^a(1 + G)$.*

Proof. We initially use the Cauchy criterion to show that ${}_b\prod^a(1+G)$ exists. Let $\varepsilon > 0$. There exists a subdivision D_1 of $\{a, b\}$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D_1, then

$$\left|\prod_{q=1}^{n}(1 + G_q)\right| < B$$

and

$$|-1|\left|\prod_{q=1}^{n}(1-G_{n+1-q})\right|<B .$$

Lemma 3.3 implies that there exists a subdivision D_2 of $\{a, b\}$ such that if $\{x_q\}_{q=0}^{n}$ and $\{y_q\}_{q=0}^{m}$ are refinements of D_2, then

$$\left|1-\left\{\prod_{q=1}^{n}[1+G(x_{q-1}, x_q)]\right\}\left\{\prod_{q=1}^{m}[1-G(y_{m-q}, y_{m+1-q})]\right\}\right|<\varepsilon(2B)^{-1}$$

and

$$\left|1-\left\{\prod_{q=1}^{m}[1-G(y_{m-q}, y_{m+1-q})]\right\}\left\{\prod_{q=1}^{n}[1+G(x_{q-1}, x_q)]\right\}\right|<\varepsilon(2B)^{-1} .$$

Let $D=D_1\cup D_2$ and suppose $\{x_q\}_{q=0}^{n}$ and $\{y_q\}_{q=0}^{m}$ are refinements of D. Let P_1, P_2, and P_3 denote

$$\prod_{q=1}^{n}[1+G(x_{q-1}, x_q)] ,$$

$$\prod_{q=1}^{n}[1-G(x_{n-q}, x_{n+1-q})] ,$$

and

$$\prod_{q=1}^{m}[1-G(y_{m-q}, y_{m+1-q})] ,$$

respectively. Now,

$$\begin{aligned}\varepsilon/2 &> [|P_3|][\varepsilon(2B)^{-1}]\\ &\geqq |P_3|\,|1-P_1P_2|\\ &\geqq |P_3-P_3P_1P_2|\\ &\geqq |P_3-P_2|-|1-P_3P_1|\,|P_2|\,|-1|\\ &> |P_3-P_2|-B[\varepsilon(2B)^{-1}] .\end{aligned}$$

Therefore, $\varepsilon>|P_3-P_2|$, and hence, ${}_b\prod^a(1+G)$ exists.

Lemma 3.2 implies that $G\in OD^\circ$ on $\{a, b\}$. Hence, it follows from Lemma 3.4 that $[{}_a\prod^b(1+G)]^{-1}$ exists and is ${}_b\prod^a(1+G)$.

LEMMA 3.5. *If G is a bounded function from $S\times S$ to N such that $\int_a^b|G^2|=0$, $G\in OM^\circ$ on $\{a, b\}$, ${}_y\prod^x(1+G)$ exists and is*

$$[{}_x\prod^y(1+G)]^{-1}$$

for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, $G(x, y)=-G(y, x)$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, and $[{}_x\prod^y(1+G)]^{-1}$ is bounded on $\{a, b\}$, then $G\in OM^\circ$ on $\{b, a\}$.

Proof. Since we are given that ${}_y\prod^x (1 + G)$ exists for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, it is only necessary to show that

$$\int_b^a |1 + G - \prod (1 + G)| = 0 .$$

Let $\varepsilon > 0$. There exist a subdivision D of $\{a, b\}$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $|1 - G_q| < B$ for $1 \leqq q \leqq n$,

(2) $|[{}_{x_{q-1}}\prod^{x_q} (1 + G)]^{-1}| < B$ for $1 \leqq q \leqq n$,

(3) $\sum_{q=1}^n |[{}_{x_{q-1}}\prod^{x_q} (1 + G)] - [1 + G_q]| < \varepsilon(2B^2)^{-1}$, and

(4) $\sum_{q=1}^n |-G_q^2| < \varepsilon(2B)^{-1}$.

Let $\{x_q\}_{q=0}^n$ be a refinement of D and suppose P_q denotes ${}_{x_{q-1}}\prod^{x_q}(1 + G)$ for $1 \leqq q \leqq n$. Thus,

$$\begin{aligned}
&\sum_{q=1}^n \Big|[1 - G_q] - [{}_{x_q}\prod^{x_{q-1}}(1 + G)]\Big| \\
&= \sum_{q=1}^n |\{[1 - G_q] - [P_q]^{-1}\}\{[P_q][P_q]^{-1}\}| \\
&\leqq \sum_{q=1}^n |[1 - G_q][P_q] - 1|\,|[P_q]^{-1}| \\
&\leqq B \sum_{q=1}^n |[1 - G_q][P_q - (1 + G_q) + (1 + G_q)] - 1| \\
&\leqq B^2 \sum_{q=1}^n |P_q - (1 + G_q)| + B \sum_{q=1}^n |-G_q^2| \\
&< B^2[\varepsilon(2B^2)^{-1}] + B[\varepsilon(2B)^{-1}] = \varepsilon .
\end{aligned}$$

Therefore, $G \in OM^\circ$ on $\{b, a\}$.

The proof of Lemma 3.5 is essentially the same as the proof of a previous result by the author [3, Theorem 4]. However, since the argument is relatively brief and the setting here is somewhat different from that in [3], the proof is included for completeness.

THEOREM 3.2. *If G is a function from $S \times S$ to N such that $\int_a^b |G^2| = 0$, $G \in OP^\circ$ on $\{a, b\}$ and $\{b, a\}$, $G \in OM^\circ$ on $\{a, b\}$ and $G(x, y) = -G(y, x)$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$, then $G \in OM^\circ$ on $\{b, a\}$.*

Proof. It follows from Theorem 3.1 that ${}_y\prod^x (1 + G)$ exists and is $[{}_x\prod^y (1 + G)]^{-1}$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$. Therefore, Theorem 3.2 follows from Lemma 3.5.

We now show that under certain restrictions the Riccati integral equation

$$n(y) = k + \int_a^y n(r)G(r, s)n(r)$$

has a solution in a noncommutative ring. Initially, we state two lemmas on the product integral solution of integral equations. Both of these lemmas are based on a result of B. W. Helton [2, Theorem 5.1, p. 310]. The hypothesis of this result has been modified to produce the lemmas given here. However, with the use of Theorem 3.1 to assure the existence of certain inverses, these lemmas can be established by arguments that are essentially the same as the one given by B. W. Helton.

LEMMA 3.6. *If*

(a) $a \in S$, $k \in N$, k^{-1} *exists and* f *is a function from* S *to* N *such that* $f(a) = k$, *and*

(b) G *is a function from* $S \times S$ *to* N *such that if* $\{a, y\} \in S \times S$, *then* $G \in OP^{\circ}$ *on* $\{a, y\}$ *and* $\{y, a\}$, $G(u, v) = -G(v, u)$ *for each subdivision* $\{a, u, v, y\}$ *of* $\{a, y\}$ *and* $\int_a^y |G^2| = 0$,

then the following statements are equivalent:

(1) *if* $\{a, y\} \in S \times S$, *then* $f(r)G(r, s) \in OA^{\circ}$ *on* $\{a, y\}$ *and*

$$f(y) = k + \int_a^y f(r)G(r, s) ,$$

and

(2) *if* $\{a, y\} \in S \times S$, *then* $G \in OM^{\circ}$ *on* $\{a, y\}$ *and*

$$f(y) = k[{}_a\Pi^y (1 + G)] .$$

LEMMA 3.7. *If*

(a) $a \in S$, $k \in N$, k^{-1} *exists and* f *is a function from* S *to* N *such that* $f(a) = k$, *and*

(b) G *is a function from* $S \times S$ *to* N *such that if* $\{y, a\} \in S \times S$, *then* $G \in OP^{\circ}$ *on* $\{y, a\}$ *and* $\{a, y\}$, $G(u, v) = -G(v, u)$ *for each subdivision* $\{y, u, v, a\}$ *of* $\{y, a\}$ *and* $\int_y^a |G^2| = 0$,

then the following statements are equivalent:

(1) *if* $\{y, a\} \in S \times S$, *then* $G(r, s)f(s) \in OA^{\circ}$ *on* $\{y, a\}$ *and*

$$f(y) = k + \int_y^a G(r, s)f(s) ,$$

and

(2) *if* $\{y, a\} \in S \times S$, *then* $G \in OM^{\circ}$ *on* $\{y, a\}$ *and*

$$f(y) = [{}_y\Pi^a (1 + G)]k .$$

We now establish the integral equation result. The proof presented here was suggested by B. W. Helton's proof for the commutative case [2, Examples 4,5, p. 320].

THEOREM 3.3. *If*

(a) *$a \in S$, $k \in N$, k^{-1} exists and n is a function from S to N such that $n(a) = k$,*

(b) *G is a function from $S \times S$ to N such that if $\{a, y\} \in S \times S$, then $G(r, s)n(r) \in OP^\circ$ on $\{a, y\}$, $G(r, s)n(s) \in OP^\circ$ on $\{y, a\}$, $G(u, v) = -G(v, u)$ for each subdivision $\{a, u, v, y\}$ of $\{a, y\}$ and*

$$\int_a^y |[G(r, s)n(r)]^2| = 0 ,$$

and

(c) *if $\{a, y\} \in S \times S$, then $n(r)G(r, s)n(r) \in OA^\circ$ on $\{a, y\}$ and*

$$n(y) = k + \int_a^y n(r)G(r, s)n(r) ,$$

then (conclusion) if $\{a, y\} \in S \times S$,

$$n(y) = \left[k^{-1} + \int_y^a G\right]^{-1} .$$

Proof. If $\{a, y\} \in S \times S$ and H denotes the function such that if $\{a, r, s, y\}$ is a subdivision of $\{a, y\}$, then

(1) $H(r, s) = G(r, s)n(r)$, and

(2) $H(s, r) = G(s, r)n(r)$,

then we know the following about H: $H \in OP^\circ$ on $\{a, y\}$ and $\{y, a\}$, $H(u, v) = -H(v, u)$ for each subdivision $\{a, u, v, y\}$ of $\{a, y\}$, $\int_a^y |H^2| = 0$, $n(r)H(r, s) \in OA^\circ$ on $\{a, y\}$ and

$$n(y) = k + \int_a^y n(r)H(r, s) .$$

Therefore, it follows from Lemma 3.6 that if $\{a, y\} \in S \times S$, then $H \in OM^\circ$ on $\{a, y\}$ and

$$n(y) = k[{}_a\Pi^y (1 + H)] .$$

Again, if we suppose $\{a, y\} \in S \times S$ and define H as before, then we know the following about H: $H \in OP^\circ$ on $\{a, y\}$ and $\{y, a\}$, $H(u, v) = -H(v, u)$ for each subdivision $\{a, u, v, y\}$ of $\{a, y\}$, $\int_a^y |H^2| = 0$ and $H \in OM^\circ$ on $\{a, y\}$. Thus, from Theorem 3.1, $[{}_a\Pi^y (1 + H)]^{-1}$ exists and is ${}_y\Pi^a (1 + H)$. Further, from Theorem 3.2, $H \in OM^\circ$ on $\{y, a\}$. Now, since k^{-1} exists,

$$\begin{aligned}[n(y)]^{-1} &= \{k[{}_a\Pi^y (1 + H)]\}^{-1} \\ &= [{}_y\Pi^a (1 + H)]k^{-1} .\end{aligned}$$

Once more, if we suppose $\{a, y\} \in S \times S$ and define H as before, then we know the following about H: $H \in OP^\circ$ on $\{a, y\}$ and $\{y, a\}$,

$H(u, v) = -H(v, u)$ for each subdivision $\{y, u, v, a\}$ of $\{y, a\}$, $\int_y^a |G^2| = 0$, $H \in OM^\circ$ on $\{y, a\}$ and

$$[n(y)]^{-1} = [{}_y\Pi^a (1 + H)]k^{-1} .$$

Thus, from Lemma 3.7,

$$\begin{aligned}[n(y)]^{-1} &= k^{-1} + \int_y^a [H(r, s)][n(s)]^{-1} \\ &= k^{-1} + \int_y^a [G(r, s)n(s)][n(s)]^{-1} \\ &= k^{-1} + \int_y^a G .\end{aligned}$$

We have now established that if $\{a, y\} \in S \times S$, then

$$n(y) = \left[k^{-1} + \int_y^a G\right]^{-1} .$$

IV. Results: commutative rings.

THEOREM 4.1. *Suppose N is commutative. If G is a function from $S \times S$ to N such that $G \in OD^\circ$ on $\{a, b\}$, ${}_a\Pi^b (1 + G)$ exists and $-G \in OP'$ on $\{a, b\}$, then*

(1) *${}_a\Pi^b (1 - G)$ exists, and*

(2) *$[{}_a\Pi^b (1 + G)]^{-1}$ exists and is ${}_a\Pi^b (1 - G)$.*

Proof. We use the Cauchy criterion for product integrals to show that ${}_a\Pi^b (1 - G)$ exists. Let $\varepsilon > 0$. There exist a subdivision D of $\{a, b\}$ and a number B such that if K and L are refinements of D, then

(1) $|\Pi_{K(I)} (1 + G)| < B$ and $|\Pi_{K(I)} (1 - G)| < B$,

(2) $|\Pi_{K(I)} (1 + G) - \Pi_{L(I)} (1 + G)| < \varepsilon(3B^2)^{-1}$, and

(3) $|1 - \Pi_{K(I)} (1 - G^2)| < \varepsilon(6B)^{-1}$.

Suppose K and L are refinements of D. Let P_1, P_2, P_3, and P_4 denote $\Pi_{K(I)} (1 - G)$, $\Pi_{L(I)} (1 - G)$, $\Pi_{K(I)} (1 - G^2)$, and $\Pi_{L(I)} (1 - G^2)$, respectively. Thus,

$$\begin{aligned}|P_1 - P_2| &= |\{P_1 - P_2\}\{P_3 + [1 - P_3]\}| \\ &< |P_1P_3 - P_2P_3| + [2B][\varepsilon(6B)^{-1}] \\ &\leqq |P_1P_3 - P_1P_4| + |P_1P_4 - P_2P_3| + \varepsilon/3 \\ &\leqq |P_1|[|P_3 - 1| + |1 - P_4|] \\ &\quad + |P_1||P_2||\Pi_{L(I)} (1 + G) - \Pi_{K(I)} (1 + G)| + \varepsilon/3 \\ &< B[\varepsilon(6B)^{-1}] + \varepsilon(6B)^{-1} + B^2[\varepsilon(3B^2)^{-1}] + \varepsilon/3 \\ &= \varepsilon .\end{aligned}$$

Therefore, ${}_a\prod^b (1 - G)$ exists.

Since $G \in OD^\circ$ on $\{a, b\}$, it follows readily that $[{}_a\prod^b (1 + G)]^{-1}$ exists and is ${}_a\prod^b (1 - G)$.

LEMMA 4.1. *Suppose N is commutative. If G is a function from $S \times S$ to N, then the following statements are equivalent*:

(1) $G \in OD^\circ$ *on* $\{a, b\}$ *and* $-G^2 \in OM^\circ$ *on* $\{a, b\}$, *and*

(2) $\int_a^b |G^2| = 0$.

Proof $(1 \to 2)$. Since $G \in OD^\circ$ on $\{a, b\}$, ${}_x\prod^y (1 - G^2)$ exists and is 1 for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$. Thus, since $-G^2 \in OM^\circ$ on $\{a, b\}$,

$$0 = \int_a^b | \prod (1 - G^2) - (1 - G^2)|$$
$$= \int_a^b |1 - (1 - G^2)| = \int_a^b |G^2| \, .$$

Proof $(2 \to 1)$. Note that $|-G^2| \in OP^\circ$ on $\{a, b\}$. There exist a subdivision D of $\{a, b\}$ and a number B such that if $\{x_q\}_{q=0}^n$ is a refinement of D, then

(1) $\prod_{q=i}^j (1 + |-G_q^2|) < B$ for $1 \leqq i \leqq j \leqq n$, and

(2) $\sum_{q=1}^n |G_q^2| < \varepsilon/B$.

Suppose $\{x_q\}_{q=0}^n$ is a refinement of D. Thus, if $1 \leqq i \leqq j \leqq n$, then

$$\begin{aligned}
&\left|1 - \prod_{q=i}^j (1 - G_q^2)\right| \\
&\quad = \left|1 - \left\{1 + \sum_{q=i}^j [-G_q^2]\left[\prod_{k=q+1}^j (1 - G_k^2)\right]\right\}\right| \\
&\quad \leqq \sum_{q=i}^j [|G_q^2|]\left[\prod_{k=q+1}^j (1 + |-G_k^2|)\right] \\
&\quad < B\varepsilon/B = \varepsilon \, .
\end{aligned}$$

Therefore, $G \in OD^\circ$ on $\{a, b\}$.

Since $\int_a^b |G^2| = 0$, it follows that $-G^2 \in OA^\circ$ and OB° on $\{a, b\}$. Therefore, $-G^2 \in OM^\circ$ on $\{a, b\}$ by a result of B. W. Helton [2, Theorem 3.4, p. 301].

LEMMA 4.2. *Suppose N is commutative. If G is a function from $S \times S$ to N such that $G \in OD^\circ$ and OP° on $\{a, b\}$ and $-G \in OP'$ on $\{a, b\}$, then $-G \in OP^\circ$ on $\{a, b\}$.*

Proof. There exist a subdivision $H = \{y_q\}_{q=0}^m$ of $\{a, b\}$ and a number $B > 1$ such that if $J = \{x_q\}_{q=0}^n$ is a refinement of H, then

(1) $|\prod_{J(I)}(1-G)| < B$,

(2) $\left|\prod_{q=i}^{j}(1+G_q)\right| < B$ for $1 \leqq i \leqq j \leqq n$, and

(3) $\left|\prod_{q=i}^{j}(1-G_q^2)\right| < B$ for $1 \leqq i \leqq j \leqq n$.

Suppose $-G \notin OP^\circ$ on $\{a, b\}$. Hence, there exist sequences $\{p_i\}_{i=1}^\infty$, $\{q_i\}_{i=1}^\infty$ and $\{H_i\}_{i=1}^\infty$ and positive integers r and s such that

(1) $1 \leqq r \leqq s \leqq m$, $\{y_{r-1}, p_i, y_r\}$ is a subdivision of $\{y_{r-1}, y_r\}$ and $\{y_{s-1}, q_i, y_s\}$ is a subdivision of $\{y_{s-1}, y_s\}$, and

(2) H_i is a subdivision of $\{p_i, q_i\}$ such that $i < |\prod_{H_i(I)}(1-G)|$ and if $r < s$ then $\{y_q\}_{q=r}^{s-1} \subseteqq H_i$.

Let i be a positive integer such that $i > 3 + B^3$. Further, let $P = |\prod_{H_i(I)}(1-G)|$ and let $J = H \cup H_i$. Since $G \in OD^\circ$ on $\{a, b\}$, there exist subdivisions K and L of $\{a, p_i\}$ and $\{q_i, b\}$, respectively, such that

(1) $\{y_q\}_{q=0}^{r-1} \subseteqq K$ and $\{y_q\}_{q=s}^{m} \subseteqq L$,

(2) $|U| < (PB)^{-1}$, where $U = 1 - \prod_{K(I)}(1-G^2)$, and

(3) $|V| < (PB)^{-1}$, where $V = 1 - \prod_{L(I)}(1-G^2)$.

Thus,

$$\begin{aligned} i &< |\textstyle\prod_{H_i(I)}(1-G)| \\ &= |[U + \textstyle\prod_{K(I)}(1-G^2)][\prod_{H_i(I)}(1-G)][\prod_{L(I)}(1-G^2)+V]| \\ &\leqq |U||\textstyle\prod_{H_i(I)}(1-G)||\prod_{L(I)}(1-G^2)| \\ &\quad + |U||\textstyle\prod_{H_i(I)}(1-G)||V| \\ &\quad + |\textstyle\prod_{K(I)}(1+G)||\prod_{J(I)}(1-G)||\prod_{L(I)}(1+G)| \\ &\quad + |\textstyle\prod_{K(I)}(1-G^2)||\prod_{H_i(I)}(1-G)||V| \\ &< (PB)(PB)^{-1} + P(PB)^{-2} + B^3 + (PB)(PB)^{-1} \\ &< 3 + B^3 < i. \end{aligned}$$

This is a contradiction, and therefore, $-G \in OP^\circ$ on $\{a, b\}$.

THEOREM 4.2. *Suppose N is commutative. If G is a function from $S \times S$ to N such that $G \in OP^\circ$ and OM° on $\{a, b\}$, $-G \in OP'$ on $\{a, b\}$ and $\int_a^b |G^2| = 0$, then $-G \in OM^\circ$ on $\{a, b\}$.*

Proof. It follows from Theorem 4.1 that ${}_x\prod^y(1-G)$ exists and is $[{}_x\prod^y(1+G)]^{-1}$ for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$. Further, since $G \in OD^\circ$ on $\{a, b\}$ by Lemma 4.1, Lemma 4.2 implies that $-G \in OP^\circ$ on $\{a, b\}$. Therefore, it follows from Lemma 3.5 that $-G \in OM^\circ$ on $\{a, b\}$.

THEOREM 4.3. *Suppose N is commutative. If G is a function from $S \times S$ to N such that ${}_x\prod^y(1+G)$ and ${}_x\prod^y(1-G)$ exist and*

$[_x\prod^y (1 + G)]^{-1}$ *exists and is* $_x\prod^y (1 - G)$ *for each subdivision* $\{a, x, y, b\}$ *of* $\{a, b\}$, *then* $G \in OD^\circ$ *on* $\{a, b\}$.

Proof. Suppose $G \notin OD^\circ$ on $\{a, b\}$. Hence, there exists a positive number ε such that if H is a subdivision of $\{a, b\}$ then there exist a refinement $\{x_q\}_{q=0}^n$ of H and integers i and j such that $1 \leqq i \leqq j \leqq n$ and

$$\left|1 - \prod_{q=i}^{j} (1 - G_q^2)\right| \geqq \varepsilon .$$

For convenience, suppose $1 > \varepsilon$. Note that $_x\prod^y (1 - G^2)$ exists and is 1 for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$. Thus, there exists a subdivision H of $\{a, b\}$ such that if J is a refinement of H, then

$$\varepsilon/4 > |1 - \prod_{J(I)} (1 - G^2)| .$$

Hence, there exist a refinement $\{x_q\}_{q=0}^n$ of H and integers i and j such that $1 \leqq i \leqq j \leqq n$ and

$$\left|1 - \prod_{q=i}^{j} (1 - G_q^2)\right| \geqq \varepsilon .$$

Let K_1 and K_2 represent $\{x_q\}_{q=0}^{i-1}$ and $\{x_q\}_{q=j}^n$, respectively. Further, let

$$P = \left|\prod_{q=i}^{j} (1 - G_q^2)\right| + 1 .$$

There exist refinements L_1 and L_2 of K_1 and K_2, respectively, such that
(1) $|-1 + \prod_{L_1(I)} (1 - G^2)| < \varepsilon(4P)^{-1}$, and
(2) $|-1 + \prod_{L_2(I)} (1 - G^2)| < \varepsilon(4P)^{-1}$.
Let $J = L_1 \cup H \cup L_2$. Thus,

$$\begin{aligned}
\varepsilon/4 &> |1 - \prod_{J(I)} (1 - G^2)| \\
&= \left|1 - [1 - 1 + \prod_{L_1(I)} (1 - G^2)]\left[\prod_{q=i}^{j} (1 - G_q^2)\right]\right. \\
&\qquad \left.\times [1 - 1 + \prod_{L_2(I)} (1 - G^2)]\right| \\
&\geqq \left|1 - \prod_{q=i}^{j} (1 - G_q^2)\right| \\
&\qquad -P|-1 + \prod_{L_2(I)} (1 - G^2)| - P|1 - \prod_{L_1(I)} (1 - G^2)| \\
&\qquad -P|-1 + \prod_{L_1(I)} (1 - G^2)|\,|1 - \prod_{L_2(I)} (1 - G^2)| \\
&> \varepsilon - P[\varepsilon(4P)^{-1}] - P[\varepsilon(4P)^{-1}] - P[\varepsilon(4P)^{-1}]^2 > \varepsilon/4 .
\end{aligned}$$

This is a contradiction, and therefore, $G \in OD^\circ$ on $\{a, b\}$.

REMARK 1. B. W. Helton [2, §6] gives product integral techniques for solving certain types of integral equations in commutative rings.

An important condition in this development is that

$$[{}_x\prod^y (1 + G)]^{-1} = {}_x\prod^y (1 - G) .$$

In particular, see Examples 4 and 5 [2, p. 320]. Thus, the results in this section can be applied to the solution of certain Riccati type integral equations.

REMARK 2. Related results are obtained by J. S. MacNerney [5, §7]. However, in that development the functions under consideration are required to have bounded variation. We do not require bounded variation in this development. However, the functions here are often required to belong to the set OP°. As noted by B. W. Helton [2, p. 299], the set of functions of bounded variation is a proper subset of the set OP°. Also, the requirement that $\int_a^b |G^2| = 0$ does not imply that G has bounded variation. W. P. Davis and J. A. Chatfield [1, p. 747] give a function G such that $\int_a^b |G^2| = 0$, ${}_a\prod^b (1 + G)$ exists and is not zero, and G does not have bounded variation. In addition, J. V. Herod [4] has also investigated the existence of inverses in a setting similar to the one studied by MacNerney.

REMARK 3. Related results are also obtained in a previous paper by the author [3, Theorems 2, 3, 4, 5]. However, conditions relating to commutativity or the existence of inverses are required there that are not required here.

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