PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

IDEALS IN CONVOLUTION ALGEBRAS ON ABELIAN GROUPS

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If G is a locally compact Abelian group, any subalgebra A of $M(G)$ that contains a dense ideal of $L^1(G)$ can be mapped homomorphically onto $C(K)$ for any Helson set K in the dual group. Then, by choosing a Helson set homeomorphic to the one-point compactification N_∞ of the natural numbers, the ideal structure of A can be explored from known properties of $C(N_\infty)$. As special cases normed subalgebras are considered and for them — by different techniques — information on their countably generated, closed ideals J can be obtained. Necessarily $Z(J)$ is open-closed; if G is compactly generated and A contains such a nonzero J, G must be $Z^n \times C$[C a compact group] and J must consist of those L^1 functions whose Fourier transforms vanish on $T^n \times E$, where E is a cofinite subset of the dual of C. In particular, a Segal algebra on G (satisfying mild restrictions) can have a countably generated regular maximal ideal if and only if G is finite.

The paper continues the investigation of Banach algebras from the algebraic point of view begun in [5] and extends some of the theorems there. The idea of transferring properties of $C(N_\infty)$ to A has also been explored for uniform algebras A in [6]. All our results confirm the expected algebraic complexity of Banach algebras as well as the more surprising pivotal role which the Silov boundary plays in determining algebraic structure.

1. Prime ideals. Let G be a locally compact Abelian group with dual group Γ, and consider any subalgebra A of $M(G)$ which contains a dense ideal I in $L^1(G)$. I contains the L^1 functions whose Fourier transform have compact support, since I has void hull and $L^1(G)$ is regular. For $\gamma \in \Gamma$, let $I_\gamma[F_\gamma]$ denote the ideal of functions in A whose Fourier transforms vanish at γ [on a neighborhood of γ]. Notice that a prime P contained in the maximal ideal I_γ necessarily contains F_γ. For if $\hat{\mu}$ vanishes on a compact neighborhood U of γ, and $g \in A$ is chosen so that $\hat{g}(\gamma) = 1$ with supp $\hat{g} \subset U$, then $\mu * g = 0 \in P$, but $g \notin P$; whence $\mu \in P$. Thus a prime of A can be contained in at most one maximal ideal of form I_γ, and if G is compact, no prime can be properly contained in any I_γ. As we shall see, the situation is quite different in the noncompact case.

A compact set $K \subset \Gamma$ is called a Helson set if every $f \in C(K)$, the continuous complex-valued functions on K, is the restriction to

K of the Fourier transform of some $g \in L^1(G)$. Notice that for such a K, $\hat{A}|K = C(K)$. For if $f \in C(K)$ with $f = \hat{g}|K$, $g \in L^1(G)$, choose $h \in A$ whose transform is identically 1 on K and observe $f = g \hat{*} h|K$. Thus if K is a Helson set, the map $\varphi: f \to \hat{f}|K$ is an algebra homomorphism of A onto $C(K)$. Notice if $I \subsetneqq J$ are [prime] ideals of $C(K)$, $\varphi^{-1}(I) \subsetneqq \varphi^{-1}(J)$ are [prime] ideals of A. Thus if $\mathscr{C}$ and $\mathscr{D}$ are infinite chains of primes in $C(K)$, $\varphi^{-1}(\mathscr{C})$ and $\varphi^{-1}(\mathscr{D})$ are infinite chains of primes in A and $\varphi^{-1}(\mathscr{C}) \cap \varphi^{-1}(\mathscr{D}) = \varphi^{-1}(\mathscr{C} \cap \mathscr{D})$.

Let $C_r(X)$ denote the real-valued continuous functions on any space X, and for $B \subset C(X)$, set $B_r = B \cap C_r(X)$. We have

LEMMA 1.1. *$P \to P + iP$ is a lattice preserving one-to-one correspondence between the prime ideals of $C_r(X)$ and those of $C(X)$.*

Proof. $P + iP$ is clearly an ideal of $C(X)$; suppose $f, g \in C(X)$ with $fg \in P + iP$. Then $|fg|^2 = fg\overline{fg} \in P + iP$, so actually $|fg|^2 \in P$. But then $|f|^2 \in P$ or $|g|^2 \in P$, hence $|f| \in P$ or $|g| \in P$, and finally $|f|^{1/2} \in P$ or $|g|^{1/2} \in P$. Thus either $f = |f|^{1/2}(|f|^{1/2} \operatorname{sign} f) \in P + iP$ or $g = |g|^{1/2}(|g|^{1/2} \operatorname{sign} g) \in P + iP$. Every prime Q in $C(X)$ is of this form, since Q_r is prime in $C_r(X)$ and $Q = Q_r + iQ_r$ [4, 3.1, p. 71]. Finally if $P + iP = Q + iQ$, $P = (P + iP)_r = (Q + iQ)_r = Q$.

These simple observations lead to the following

THEOREM 1.2. *Let G be a noncompact LCA group, A a subalgebra of $M(G)$ containing a dense ideal of $L^1(G)$. Then for any $\gamma \in \Gamma$, there are 2^c pairwise disjoint infinite chains of prime ideals of A contained in I_γ, in particular, krull* dim $A = \infty$.

Proof. Since Γ is nondiscrete, it contains a Helson set E homeomorphic to the Cantor middle third set on the real line [16, 5.2.2, 5.6.6]. If $\gamma' \in E$, the translate $C = E - \gamma' + \gamma$ is still a Helson set equivalent to the Cantor set, and it contains γ. Since C is metrizable and has no isolated points, we can find a sequence of distinct points $\gamma_n \in C \backslash \{\gamma\}$ so that $\gamma_n \to \gamma$. This sequence together with its limit point forms a Helson set K by Tietze's extension theorem. K is homeomorphic to N_∞, the one point compactification of the natural numbers, and composing the induced isomorphism $C(K) \cong C(N_\infty)$ with $\varphi: A \to C(K)$ above, we obtain an algebra homomorphism Φ of A onto $C(N_\infty)$ such that $\Phi(I_\gamma) = M_\infty = \{f \in C(N_\infty): f(\infty) = 0\}$. According to [8, 14G, p. 213] there are 2^c maximal chains of prime ideals of $C_r(N_\infty)$ contained in $M_\infty \cap C_r(N_\infty)$, any two of which have only $M_\infty \cap C_r(N_\infty)$ in common. For any such chain $\mathscr{C}$, let $\underline{\mathscr{C}} = \{\Phi^{-1}(P + iP): P \in \mathscr{C}, P \neq M_\infty \cap C_r(N_\infty)\}$.

1.1 and the remarks preceding it guarantee that $\underline{\mathscr{C}}$ is a chain of prime ideals of A contained in $\Phi^{-1}(M_\infty) = I_\gamma$. Plainly if $\underline{\mathscr{D}}$ is any other such chain, $\underline{\mathscr{D}} \cap \underline{\mathscr{C}} = \varnothing$. Each chain $\underline{\mathscr{C}}$ is infinite, since $P \to \Phi^{-1}(P + iP)$ is a bijection $\mathscr{C} \to \underline{\mathscr{C}}$ and $\mathscr{C}$ is infinite. For otherwise, $\mathscr{C}$ would consist of distinct primes $P_1 < P_2 < \cdots < P_n < M_\infty \cap C_r(N_\infty)$. Since $\mathscr{C}$ is maximal there would be no prime ideals of $C_r(N_\infty)$ strictly between P_n and $M_\infty \cap C_r(N_\infty)$, in violation of, say, [4, 3.1, p. 71]. In particular, then, ascending chains of primes in A of arbitrary length can be exhibited, so that krull dim $A = \infty$.

Notice that if Γ is separable, each I_γ will contain *exactly* 2^c primes. For in that case $C(\Gamma)$ has cardinality c, the Fourier transform embeds A in $C(\Gamma)$ and the remark follows from 1.2.

Observe also that none of the primes of A constructed above can be a finitely generated ideal. For if $\Phi^{-1}(P + iP)$ is finitely generated, so is $\Phi(\Phi^{-1}(P + iP) = P + iP$. But if $\{f_1, \cdots, f_n\}$ generates the prime $P + iP$ over $C(N_\infty)$, so does $\{|f_1|^{1/2}, \cdots, |f_n|^{1/2}\}$, and hence this set generates P over $C_r(N_\infty)$. But then [2, §3, Thm. 1] P is generated by an idempotent f, so that $\infty = Z(P) = Z(f)$ is isolated in N_∞, an impossible conclusion.

PROPOSITION 1.3. *If G is noncompact and A is any subalgebra of $M(G)$ containing a dense ideal of $L^1(G)$, then not every finitely generated ideal of A is principal.*

Proof. Otherwise, for any Cantor Helson set $K \subset \Gamma$, $C(K)$, as a homomorphic image of A, would have the same property; K would then be a metric F-space [8, 14.25], hence discrete [8, 14N].

For A as in 1.3 and $E \subset \Gamma$, let $I_E[F_E]$ denote the measures in A whose Fourier-Stieltjes transforms vanish on E [on a neighborhood of E]. For $J \subset$ A set $Z(J) = \bigcap \{\hat{\mu}^{-1}(0)\colon \mu \in J\}$.

PROPOSITION 1.4. *Take G and A as in* 1.3. *If J is an ideal of A strictly between F_γ and I_γ, there are ideals $\underline{J}$, $\bar{J}$ of A so that $F_\gamma \subset \underline{J} \subset J \subset \bar{J} \subset I_\gamma$ with all inclusions proper.*

Proof. As before, there is an algebra homomorphism Φ of A onto $C(N_\infty)$ so that $\Phi(I_\gamma) = M_\infty$. By [4, 4.5, p. 75], there are ideals $\underline{I}$ and $\bar{I}$ of $C(N_\infty)$ so that $F_\infty \subset \underline{I} \subset \Phi(J) \subset \bar{I} \subset M_\infty$ (strictly). The conclusion follows with $\underline{J} = \Phi^{-1}(\underline{I})$, $\bar{J} = \Phi^{-1}(\bar{I})$.

By repeated use of 1.4 we conclude that through any such J we can thread infinite ascending and descending chains of ideals of A lying between F_γ and I_γ. Although in general $\bar{J}$ will not be prime, it can be chosen so if J is countably generated.

PROPOSITION 1.5. *Take G and A as in* 1.3. *If J is a countably generated ideal of A for which $Z(J) = \gamma$, there is a prime ideal P of A strictly between J and I_γ.*

Proof. With $\Phi: A \to C(N_\infty)$ as in 1.4, $\Phi(J)$ is a countably generated ideal of $C(N_\infty)$ and $Z(\Phi(J)) = \infty$. Since ∞ is not isolated, $F_\infty \neq M_\infty$ and [3, Thm. 3, p. 175] yields a prime ideal Q of $C(N_\infty)$ strictly between $\Phi(J)$ and M_∞. Take $P = \Phi^{-1}(Q)$.

In particular a maximal ideal I_γ in A is never countably generated (compare 3.10).

It is not hard to see that in 1.4 and 1.5 γ can be replaced by, say any proper nonvoid closed subset of Cantor Helson set in Γ. Unfortunately this interpolation method cannot be forced much beyond that. Contrast the fluid simplicity above with the halting computations in 3.1 below, where primes are obtained constructively.

2. Normed ideals and subalgebras. Call an ideal [subalgebra] A of $M(G)$ a normed ideal [subalgebra] in $M(G)$ if A is a Banach space under some norm $\|\ \|_A$ which satisfies:

(i) $\|\mu\| \leqq a\|\mu\|_A$ ($\mu \in A$, a constant)

(ii) $K = \{f \in L^1(G)\colon \text{supp } \hat{f} \text{ is compact}\} \subset A$.

Consider the following conditions on a normed subalgebra A:

(iii) There is an $\alpha > 0$ such that given $\gamma \in \Gamma$ there is a neighborhood V of γ so that $\|k\|_A \leqq \alpha\|k\|_1$ whenever $k \in K$ and supp $\hat{k} \subset V$.

(iv) There is some $r > 0$ so that $\|\gamma k\|_A \leqq r\|k\|_A$ whenever $k \in K$, $\gamma \in \Gamma$.

Notice (iv) implies (iii). For choose any compact neighborhood C of 0 in Γ. $B = \{f \in L^1(G)\colon \text{supp } \hat{f} \subset C)$ is a closed subspace of both A and $L^1(G)$ by (i) and (ii). The open mapping theorem together with (i) yields a constant $\beta > 0$ so that $\|f\|_A \leqq \beta\|f\|_1$ whenever $f \in B$. For $\gamma \in \Gamma$ set $V = \gamma + C$; if $k \in K$ and supp $\hat{k} \subset V$, $\bar{\gamma}k \in B$ so that $\|k\|_A = \|r\bar{\gamma}k\|_A \leqq r\|\bar{\gamma}k\|_A \leqq r\beta\|\bar{\gamma}k\|_1 = r\beta\|k\|_1$. Thus condition (iii) holds with $\alpha = r\beta$. The converse in false (cf. [1, §7, p. 276]).

A finite intersection $A = \bigcap_i A_i$ of normed ideals [subalgebras] is again a normed ideal [subalgebra] when given the norm $\|\mu\|_A = \sum_i \|\mu\|_{A_i}$. If each A_i satisfies (iii) [(iv)] with α_i, then A will satisfy (iii) [(iv)] with $\alpha = \sum_i \alpha_i$.

Tensor products $A \hat{\otimes}_p B$ of normed subalgebras A in $M(G)$ and B in $M(H)$, completed with respect to the projective norm $\|u\|_p = \inf\{\sum_i \|\mu_i\|_A\|\lambda_i\|_B\colon u = \sum_i \mu_i \otimes \lambda_i\}$, can also be viewed as normed subalgebras of $M(G \times H)$. Indeed the map $\sum_i \mu_i \otimes \lambda_i \to \sum_i \mu_i \times \lambda_i$ is an algebra isomorphism of $A \otimes B$ into $M(G \times H)$, since it factors into

$$A \otimes B \xrightarrow{\hat{\otimes}\hat{}} \hat{A} \otimes \hat{B} \xrightarrow{\varphi} C(\Gamma \times \Omega) \xrightarrow{\hat{}\,-1} M(G \times H)$$

where φ is a restriction of the natural algebra embedding $C(\Gamma) \otimes C(\Omega) \to C(\Gamma \times \Omega)$. From the inequality

$$||\sum_i \mu_i \times \lambda_i|| \leqq \sum_i ||\mu_i|| \; ||\lambda_i|| \leqq ab \sum_i ||\mu_i||_A ||\lambda_i||_B$$

we see that $A \otimes B \to M(G \times H)$ is continuous, so that it extends continuously to the completion $A \hat{\otimes}_p B$; in particular (i) holds with constant ab.

If K_1, K_2 and K are the functions on G, H and $G \times H$ whose transforms have compact support, then $K_1 \otimes K_2$ is $|| \; ||_p$-dense in $L^1(G) \otimes L^1(H)$; since the above embedding process gives a Banach space equivalence $L^1(G) \hat{\otimes}_p L^1(H) \cong L^1(G \times H)$, given $f \in K$ with

$$\operatorname{supp} \hat{f} \subset \operatorname{int}(C_1 \times C_2)$$

$[C_1$ and C_2 compact in G and $H]$, we can find a sequence $\{f_n\}$ from $K_1 \otimes K_2$ so that $||f - f_n||_1 \to 0$ and supp $\hat{f}_n \subset C_1 \times C_2$ for each n. Choose $h_i \in K_i$ with $\hat{h}_i | C_i \equiv 1$ and observe that using (i) and the open mapping theorem as in 2.2, we may find $\alpha, \beta > 0$ so that $||g||_A \leqq \alpha ||g||_1$ and $||h||_B \leqq \beta ||h||_1$ whenever g and h are L^1 functions whose transforms are supported on supp $\hat{h}_1$ and supp $\hat{h}_2$ respectively. With

$$M = ||h_1||_1 ||h_2||_1$$

it follows that $||u||_p \leqq \alpha\beta M ||u||_1$ whenever $u \in K_1 \otimes K_2$ and supp $\hat{u} \subset C_1 \times C_2$. For if $\sum_i k_i \otimes k_i'$ is any representation of u, $u = u(h_1 \otimes h_2) = \sum_i k_i * h_1 \otimes k_i' * h_2$, whence

$$||u||_p \leqq \sum_i ||k_i * h_1||_A ||k_i' * h_2||_B \leqq \alpha\beta ||h_1||_1 ||h_2||_1 \sum_i ||k_i||_1 ||k_i'||_1 \,.$$

Since $L^1(G \times H) = L^1(G) \hat{\otimes}_p L^1(H)$ isometrically, we conclude $||u||_p \leqq \alpha\beta M \inf \{\sum_i ||k_i||_1 ||k_i'||_1 \colon u = \sum_i k_i \otimes k_i'\} = \alpha\beta M ||u||_1$. Thus

$$||f_n - f_m||_p \leqq \alpha\beta M ||f_n - f_m||_1 \longrightarrow 0 \,,$$

whence for some $g \in A \hat{\otimes}_p B$, $||f_n - g||_p \to 0$. But $||f - g||_1 \leqq ||f - f_n||_1 + ab||f_n - g||_p \to 0$, so actually $f \in A \hat{\otimes}_p B$; that is, (ii) holds.

If A and B satisfy (iii) with constants α and α', then $A \hat{\otimes}_p B$ satisfies (iii) with $2\alpha\alpha'$. For if $(\gamma_1, \gamma_2) \in \Gamma \times \Omega$ is given and if corresponding neighborhoods V_1 and V_2 are chosen for A and B, find compact, symmetric neighborhoods W_1 and W_2 of 0 in Γ and Ω so that $m(W_i + W_i) \leqq 2m(W_i)$ and $\gamma_i + W_i + W_i \subset V_i$; choose $h_i \in K_i$ so that $\hat{h}_i | \gamma_i + W_i \equiv 1$, supp $\hat{h}_i \subset V_i$ and $||h_i||_1 \leqq \{m(\gamma_i + W_i + W_i)/(m(W_i))\}^{1/2} \leqq \sqrt{2}$ [16, 2.6.1]. If $k \in K_1 \otimes K_2$ with supp $\hat{k} \subset \gamma_1 + W_1 \times \gamma_2 + W_2$ and if $\sum_i k_i \otimes k_i'$ is a representation of k, $\sum_i k_i * h_i \otimes k_i' * h_i$ is also and we have $||k||_p \leqq \sum_i ||k_i * h_1||_A ||k_i' * h_2||_B \leqq \alpha\alpha' ||h_1||_1 ||h_2||_1 \sum_i ||k_i||_1 ||k_i'||_1$. Thus $||k||_p \leqq 2\alpha\alpha' \inf \{\sum_i ||k_i||_1 ||k_i'||_1 \colon k = \sum_i k_i \otimes k_i'\} = 2\alpha\alpha' ||k||_1$. Since $K_1 \otimes$

K_2 is $\| \|_p$-dense in K this inequality holds for any $k \in K$ whose transform is supported on $\gamma_1 + W_1 \times \gamma_2 + W_2$. Also if (iv) holds for A and B with constants r and r', then $\|(\gamma, \eta)u\|_p = \inf\{\sum_i \|\gamma f_i\|_A \|\eta g_i\|_B$: $u = \sum_i f_i \otimes g_i\} \leqq rr'\|u\|_p$, so that (iv) holds for $A \hat{\otimes}_p B$ with constant rr'.

Unfortunately $A \otimes B$ completed in some other cross norm cannot always be realized as a normed subalgebra of $M(G \times H)$ in this way. For example consider $L^1(G) \otimes L^1(H)$ completed with respect to the biequicontinuous norm

$$\|u\|_e = \sup\left\{\left|\sum_i \int_G f_i(x)f'(x)dx \int_H g_i(y)g'(y)dy\right| : f' \in L^\infty(G),\ g' \in L^\infty(H)\ ,\right.$$
$$\left. u = \sum_i f_i \otimes g_i,\ \|f'\|_\infty \leqq 1,\ \|g'\|_\infty \leqq 1\right\}\ .$$

If $L^1(G) \hat{\otimes}_e L^1(H)$ is a normed subalgebra of $M(G \times H)$, then it embeds in $L^1(G \times H)$ and $\|u\|_1 \leqq a\|u\|_e$; since $L^1(G \times H) = L^1(G) \hat{\otimes}_p L^1(H)$ isometrically, $\|u\|_p = \|u\|_1 \leqq a\|u\|_e \leqq a\|u\|_p$, so that actually

$$L^1(G) \hat{\otimes}_p L^1(H) = L^1(G) \hat{\otimes}_e L^1(H)\ .$$

Applying a theorem of Grothendieck [10, Chp. II, §2.1, p. 42] when $G = Z$ we conclude that $L^1(H)$ is a nuclear space. But since no infinite dimensional Banach space can be nuclear, this conclusion is absurd whenever H is infinite.

EXAMPLES. The essential $L^1(G)$-modules A in $L^1(G)$ defined in [12, 39.32] are examples of normed ideals. For $A = L^1(G) * A$ [12, 39.32.b.ii], so that $M(G) * A = [M(G) * L^1(G)] * A = L^1(G) * A = A$; that is, A is an ideal of $M(G)$. (i) is [12, 39.32.j]; (ii) follows since A is a dense ideal of $L^1(G)$ which, by regularity of $L^1(G)$, must contain $\{f \in K: \operatorname{supp} \hat{f} \cap Z(A) = \emptyset\} = K[Z(A) = \emptyset]$. Algebras of this type include $A^p(G)$ [13], the Wiener algebra $\mathscr{M}_1(R)$ [12, 39.33] and its generalizations [15, p. 127]; and for compact G, both $L^p(G)$ and $C(G)$. Each of these satisfy (iv) with $r = 1$.

The Segal algebras S defined in [15, 6.2.0, p. 126] are also normed ideals. For it follows from [15, 6.2.3, p. 128] that $L^1(G) * S$ is dense in S; the Hewitt factorization theorem [12, 32,22] then implies

$$L^1(G) * S = S\ ,$$

and as above, S is an ideal of $M(G)$. (i)-(ii) are [15, 6.2.2]. Actually, we have

LEMMA 2.1. *Let A be a normed ideal in $M(G)$. The following are equivalent*: (a) *A is an essential $L^1(G)$-module in $L^1(G)$*; (b) *A is a Banach algebra with an approximate identity on K*; (c) *A is a Segal* *algebra.*

Proof. (a) implies (b) is [12, 39.32.g]; if (b) holds, $K \subset A \subset L^1(G)$. For if $\{f_\alpha\} \subset K$ is an approximate identity for A, then for each $\mu \in A$, $\{f_\alpha * \mu\}$ contains a sequence $\{f_{\alpha_n} * \mu\}$ converging to μ. This sequence is Cauchy in $L^1(G)$ by (i), and converges to some $f \in L^1(G)$. Again by (i) we see that for fixed $\gamma \in \Gamma$, $\hat{f}(\gamma) = \lim_{n\to\infty} \hat{f}_{\alpha_n}(\gamma)\hat{\mu}(\gamma) = \hat{\mu}(\gamma)$, so that $\mu = f \in L^1(G)$. Since $L^1(G) * A$ is obviously dense in A, (a) holds.

If (c) holds, then so does (b) [15, 6.2.3, p. 128]. If (b) holds, we see as above that K is A-dense in $A \subset L^1(G)$. It follows that A is a Segal algebra [1, p. 275].

Other examples of normed ideals include $M(G)$ itself; in fact any closed ideal of $M(G)$ containing $L^1(G)$. Thus $M_0(G)$, the measures whose Fourier-Stieltjes transforms vanish at ∞, and $L^1(G)^{1/2}$, the intersection of all maximal ideals of $M(G)$ containing $L^1(G)$ are normed ideals in $M(G)$. All such examples satisfy (iv).

Suppose σ is a positive, regular Borel measure on Γ (not necessarily finite). For $p \in [1, \infty]$, set $A^p(\sigma) = \{\mu \in M(G): \hat{\mu} \in L^p(\sigma)\}$. With the norm $||\mu||^p = ||\mu|| + ||\hat{\mu}||_p$, $A^p(\sigma)$ is a normed ideal of $M(G)$. Indeed, $K \subset A^p(\sigma)$ since σ is finite on compact sets; completeness is standard. By judicious choice of σ, one obtains normed ideals exclusive of those above. If σ is continuous, $A^p(\sigma)$ satisfies (iii) but typically not (iv).

All the examples above satisfy $||\lambda * \mu||_A \leqq b||\lambda|| \, ||\mu||_A (\lambda \in M(G)$, $\mu \in A$, b constant), and finite intersections of such will also. However, since this module property plays no role in what follows, we have excluded it from our definition of normed ideal.

Plainly any closed subalgebra of a normed ideal which contains K is a normed subalgebra. Examples include the closed subalgebras of $M(G)$ containing $L^1(G)$— in particular the L-subalgebras of Taylor [15, 16] whose maximal ideal spaces coincide with Γ, as well as $M_u(G)$, the measures whose transforms are uniformly continuous on Γ. The "normed ideals" of $L^1(G)$ introduced by Cigler [1] are also normed subalgebras of $M(G)$.

It is not hard to see that the completed tensor product of Segal algebras A, B is again a Segal algebra. Indeed it follows from

$$L^1(G) \hat{\otimes}_p L^1(H) \cong L^1(G \times H)$$

that $A \hat{\otimes}_p B$ is a (dense) ideal of $L^1(G \times H)$; if $\{\mu_\alpha\}$ and $\{\nu_\beta\}$ are approximate identity for A and B, $\{\mu_\alpha \otimes \nu_\beta\}$ is an approximate identity for $A \hat{\otimes}_p B$. Since $A \hat{\otimes}_p B$ is a Banach algebra, the remark follows from 2.1. We obtain new examples of Segal algebras in this way: $A^r(R) \hat{\otimes}_p A^s(R)$, for instance, is not embedded in $L^1(R^2)$ as $A^q(R^2)$ for *any* q. A corresponding observation can be made for Cigler's normed ideals in $L^1(G)$ [1].

3. Countably generated ideals. Since K is a dense ideal of $L^1(G)$, 1.2–1.5 apply to any normed subalgebra in $M(G)$. We also have

THEOREM 3.1. *Let A be a normed subalgebra in $M(G)$ and $\{\mu_n\}$ a sequence in $M(G)$ such that $\mu_n * A \subset A$ for all n. If $E = \bigcap \hat{\mu}_n^{-1}(0)$ has a nonempty boundary ∂E in Γ and J is a closed ideal of A with $Z(J) \subset E$, then there is a prime ideal P of A so that for some $\gamma \in \partial E$, $\sum \mu_n * A \subset P \subset I_\gamma$ but $J \not\subset P$.*

Proof. $I = \sum \mu_n * A$ denotes the vector sum of the subspaces $\{\mu_n * A\}$. Clearly we may assume $\|\mu_n\|_A \leqq 2^{-n}$. Fix $\gamma_0 \in \partial E$ and take some compact neighborhood C of γ_0. $\phi = \sum_{n=1}^{\infty} |\hat{\mu}_n|$ is a continuous function on Γ whose zero set is E, so we may select a sequence $\{\gamma_n\}$ of distinct points of $\mathring{C}$ so that $0 < \phi(\gamma_k) < 1/k!$. The set $\{\gamma_n\}$ has some accumulation point $\gamma \in C \cap \partial E$. Standard arguments (exhibited in [9, 2.1]) yield a sequence $\{f_n\} \subset K$ so that $\hat{f}_n(\gamma_n) = 1 = \|f_n\|_1$, $\operatorname{supp} \hat{f}_n \subset C$ and $\operatorname{supp} \hat{f}_n \cap \operatorname{supp} \hat{f}_m = \emptyset$ if $n \neq m$. Since $\gamma_n \notin Z(J)$, there is some $\lambda_n \in J$ with $\hat{\lambda}_n(\gamma_n) = 1$. Modification of [16, 2.6.3] yields for fixed $\varepsilon > 0$ and each n, some $k_n \in L^1(G)$ so that $\hat{k}_n(\gamma_n) = 1 = \|k_n\|_1$ and $\|k_n * \lambda_n\|_1 < 1 + \varepsilon$.

For assuming $\gamma_n = 0$, set $\lambda = \lambda_n$ and $\delta = \varepsilon/4(1 + \|\lambda\|)$, pick a compact $K \subset G$ so that $|\lambda|(G \backslash K) < \delta$, let $W = \{\gamma \in \Gamma: |1 - (x, \eta)| < \delta, x \in K\}$ and find a compact neighborhood of 0 in Γ so that $V - V \subset W$. Choose $s, t \in L^2(G)$ whose Plancherel transforms are χ_V and χ_{-V}, and define $k(x) = s(x)t(x)/m(V)$ where m is Haar measure on Γ. $\hat{k}(0) = 1 = \|k\|_1$ and since

$$k * \lambda(x) = \int_G \{k(x - y) - k(x)\} d\lambda(y) + \int_G k(x) d\lambda(y) ,$$

$$\|k * \lambda\|_1 \leqq \int_G \|k_y - k\|_1 d|\lambda|(y) + |\hat{\lambda}(0)| \, \|k\|_1 .$$

Computation [16, p. 50] reveals that the integral on the right hand side is less than ε; thus $\|k * \lambda\|_1 < 1 + \varepsilon$. If γ_n is arbitrary, apply the above argument to $\bar{\gamma}_n \lambda_n$, and for the k so obtained, set $k_n = \gamma_n k$. Certainly $\hat{k}_n(\gamma_n) = 1 = \|k_n\|_1$ and

$$\|k_n * \lambda_n\|_1 = \|\bar{\gamma}_n(k_n * \lambda_n)\|_1 = \|k * \bar{\gamma}_n \lambda_n\|_1 < 1 + \varepsilon .$$

$B = \{g \in L^1(G): \operatorname{supp} \hat{g} \subset C\}$ is a closed subspace of both A and $L^1(G)$ because of (i) and (ii). The open mapping theorem together with (i) yield a constant $\beta > 0$ so that $\|g\|_A \leqq \beta \|g\|_1$ for all $g \in B$. Thus the series $\sum_{n=1}^{\infty} 2^{-n} f_n * k_n * \lambda_n$ is normally convergent in A and defines an element f in the closed ideal J. $S = \{f^m * \sigma: \sigma \in A \backslash I_\gamma$ and $m \geqq 0$ or

$\sigma = \delta_0$ and $m > 0\}$ is a multiplicatively closed set in A which we claim is disjoint from $\sum(\mu_n * A + \boldsymbol{C}\mu_n)$. For suppose on the other hand, $f^m * \sigma = \sum_{i=1}^n \mu_i * (\sigma_i + \alpha_i)$ for some nonnegative integer m, $\sigma \in \{\delta_0\} \cup A \backslash I_\gamma$, $\sigma_i \in A$, $\alpha_i \in \boldsymbol{C}$. Find a $\delta > 0$ and a neighborhood V of γ so that $|\hat{\sigma}(\beta)| \geqq \delta$ for $\beta \in V$. Infinitely often γ_k lies in V and for such k,

$$\frac{\delta k!}{(2^m)^k} = \frac{\delta\left(\frac{1}{2^k}\right)^m}{1/k!} \leqq \frac{|\hat{f}(\gamma_k)^m \hat{\sigma}(\gamma_k)|}{\phi(\gamma_k)} \leqq \sum_{i=1}^n \frac{|\hat{\mu}_i(\gamma_k)|}{\phi(\gamma_k)} |\hat{\sigma}_i(\gamma_k) + \alpha_i|$$
$$\leqq \sum_{i=1}^n ||\sigma_i|| + |\alpha_i| \, .$$

But this is quite impossible, since, as the kth term in the MacLaurin series for e^a, $a^k/k! \to 0$ for all $a \in \boldsymbol{C}$.

In particular, $I \cap S = \phi$ and applying Zorn's lemma, we find a prime ideal P of A containing I and disjoint from S. Clearly $I \subset P \subset I_\gamma$ and $J \not\subset P$.

The following easy corollary significantly generalizes [5, 3.1].

COROLLARY 3.2. *Let A be a normed subalgebra in $M(G)$. If J is a closed, countably generated ideal in A, then $Z(J)$ is open-closed in Γ.*

Proof. If $\{\mu_n\}$ generates J, $J = \sum \mu_n * A + \boldsymbol{C}\mu_n$ and $Z(J) = \bigcap \hat{\mu}_n^{-1}(0)$. But if $Z(J)$ is not open, the proof of 2.2 yields some $f \in J \backslash \sum \mu_n * A + \boldsymbol{C}\mu_n$.

For a normed subalgebra A of $M(G)$ and a closed ideal I of $L^1(G)$, set $I' = \{\mu \in A\colon K * \mu \subset I\}$. Notice I' is a closed ideal of A. For (i) makes it closed; if $\mu \in I'$, $\lambda \in A$ and $f \in K$, $f * (\mu * \lambda) \in I * \lambda \subset I$. This last inclusion follows from a corollary to Cohen's factorization theorem [5, 1.5]: $I = I * I_p[p \in Z(I)]$, so that $I * \lambda = I * (I_p * \lambda) \subset I * L^1(G) \subset I$. Further, the map $I \to I'$ is injective. For if $g \in I' \cap K$ with supp $\hat{g} = C$, we may find some $k \in K$ whose transform is identically 1 on C, so that $g = k * g \in K * g \subset I$. Thus if $I' = J'$, $I \cap K = I' \cap K = J' \cap K = J \cap K$, and taking closures in $L^1(G)$ and using an L^1 approximate identity from K, we have $I = J$.

Since plainly $Z(I) = Z(I')$, Helson's theorem [12, 39.42] indirectly yields uncountably many closed ideals of A with common zero set E whenever E is a set of nonsynthesis for $L^1(G)$. Since such a set cannot be open in Γ, 3.2 implies that none of them are countably generated.

Using an L^1 approximate identity, we see that if A is contained in $L^1(G)$, $I' = I \cap A$; since every closed ideal of a Segal algebra A is of this form [12, 39.32.k], $I \to I'$ is a bijection, and we conclude

COROLLARY 3.3. *If E is a set of nonsynthesis for $L^1(G)$, no closed ideal in a Segal algebra A whose zero set is E can be countably generated.*

Notice also in passing that Malliavin's theorem [16, 7.6.1], together with the above injection, implies that spectral synthesis fails for every normed subalgebra of $M(G)$ whose maximal ideal space is Γ whenever Γ is not discrete. Examples include the L-subalgebras of Taylor such as $L^1(G)^{1/2}$, as well as tensor products of such, since the spectrum of a product is the product of the factors' spectrums [14, 4.2].

Since all the specific examples mentioned in §2 satisfy (iii), their intersection with $M_0(G)$ will also, and we have a sufficient plenty to which the following generalization of [5, 3.6] applies.

THEOREM 3.4. *Let G and H be LCA group with duals Γ and Ω respectively; suppose H is compact and Γ is connected. Let A be a normed subalgebra of $M_0(G \times H)$ satisfying* (iii). *Then*

(a) *if G is nondiscrete, 0 is the only closed, countably generated ideal of A whereas*

(b) *if G is discrete, a closed ideal J in A is countably generated if and only if $J = I_{\Gamma \times E} \subset K$ where E is a cofinite subset of Ω.*

Proof. Let J be a closed, countably generated ideal in A. $Z(J)$ is open-closed (3.2), and hence of the form $\Gamma \times E$, $E \subset \Omega$.

$$J = \sum \mu_n * A + \sum C\mu_n$$

for some sequence $\{\mu_n\} \subset J \subset M_0(G \times H)$ with $\|\mu_n\| \leqq 1/2^n$. Let $\sum A \times C$ denote the direct sum of countably many copies of $A \times C$, and A_n the subspace of sequences whose entries past the nth one are all zero. A_n is a Banach space with norm $\|\{\lambda_i, \alpha_i\}\| = \sum_{i=1}^n \|\lambda_i\|_A + |\alpha_i|$, and with the final topology induced by the inclusions $A_n \subsetneq \sum A \times C$, $\sum A \times C$ is the strict inductive limit of the A_n. The mapping $T: \sum A \times C \to J$ given by $T(\{\lambda_i, \alpha_i\}) = \sum \mu_i * (\lambda_i + \alpha_i)$ is linear, continuous and surjective. A theorem of Dieudonné and Schwartz [7, Thm. 1, p. 72] implies T is open. Hence if $U = \{\{\lambda_i, \alpha_i\} \in \sum A \times C: \|\lambda_i\|_A \leqq 1$ and $|\alpha_i| \leqq 1$ for all $i\}$, there is a $\delta > 0$ so that $B(0, \delta) \subset T(U)$. It follows that there is a constant $M > 0$ so that given $\mu \in J$, there are $\lambda_1, \cdots, \lambda_r \in A$ and $\alpha_1, \cdots, \alpha_r \in C$ with

$$(*) \quad \mu = \sum_{i=1}^{r} \mu_i * (\lambda_i + \alpha_i), \quad \|\lambda_i\|_A \leqq M\|\mu\|_A$$

and

$$|\alpha_i| \leqq M\|\mu\|_A, \; i = 1, 2, \cdots, r\,.$$

Since $||\mu_i|| \leqq 1/2^i$, we may choose n so large that $\sum_{i=n+1}^{\infty}(1 + a)M\alpha||\mu_i|| < 1/2$, where a and α are the universal constants in (i) and (iii). Let $\varepsilon = 1/2(1 + a)Mn\alpha$ and choose $C \subset \Gamma \times \Omega$ compact so that $|\hat{\mu}_i(\gamma, \eta)| < \varepsilon/2$ off C for $i = 1, \cdots, n$.

(a) If $E = \Omega$ we are done. If $\eta_0 \in E^c$, then since Γ is not compact, $(\gamma_0, \eta_0) \notin C$ for some $\gamma_0 \in \Gamma$. Define $\sigma_i \in M(G)$ by

$$\sigma_i(F) = \int_{F \times H} \overline{\eta_0(y)} d\mu_i(x, y) .$$

Since $\hat{\sigma}_i(\gamma) = \hat{\mu}_i(\gamma, \eta_0)$, $|\hat{\sigma}_i(\gamma_0)| < \varepsilon/2$ for $i = 1, \cdots, n$. Choose a neighborhood V of (γ_0, η_0) in $\Gamma \times \Omega$ satisfying condition (iii), and then a compact neighborhood W of γ_0 in Γ so that $W \times \eta_0 \subset V$. Exactly as in 2.2, we can find some $k \in L^1(G)$ so that supp $\hat{k} \subset W$, $\hat{k}(\gamma_0) = 1 = ||k||_1$ and $||\sigma_i * k||_1 < \varepsilon$ for $i = 1, \cdots, n$. Since $\eta_0 k\colon (x, y) \to \eta_0(y)k(x)$ is in K, $\alpha_i\mu_i * \eta_0 k \in J$, where $\alpha_i = |\hat{\mu}_i(\gamma_0, \eta_0)|/\hat{\mu}_i(\gamma_0, \eta_0)$ whenever $\hat{\mu}_i(\gamma_0, \eta_0) \neq 0, 1$ otherwise. Computations with transforms show that $\alpha_i\mu_i * \eta_0 k(x, y) = \alpha_i\eta_0(y)\sigma_i * k(x)$. Applying (iii) we see

$$\begin{aligned} ||\alpha_i\mu_i * \eta_0 k||_A &\leqq \alpha||\alpha_i\mu_i * \eta_0 k||_1 \leqq \alpha|\alpha_i|\,||\eta_0||_1||\sigma_i * k||_1 \\ &\leqq \alpha||k||_1||\mu_i|| \leqq \alpha/2^i . \end{aligned}$$

The series $\sum_{i=1}^{\infty} \alpha_i\mu_i * \eta_0 k$ is therefore normally convergent in A and defines an element μ in the closed ideal J.

Select $\nu_i \in A$ and $\zeta_i \in C$ satisfying (*) for μ. For $\lambda_i = \nu_i/\alpha_i$ and $\beta_i = \zeta_i/\alpha_i$, we see that $\mu = \sum_{i=1}^{r} \alpha_i\mu_i * (\lambda_i + \beta_i)$ and $||\lambda_i||_A \leqq M||\mu||_A$, $|\beta_i| \leqq M||\mu||_A$. In particular,

$$\begin{aligned} \sum_{i=1}^{\infty} |\hat{\mu}_i(\gamma_0, \eta_0)| &= \sum_{i=1}^{\infty} \alpha_i\hat{\mu}_i(\gamma_0, \eta_0)\widehat{\eta_0 k}(\gamma_0, \eta_0) = \hat{\mu}(\gamma_0, \eta_0) \\ &\leqq \sum_{i=1}^{r} |\hat{\mu}_i(\gamma_0, \eta_0)|\,|\hat{\lambda}_i(\gamma_0, \eta_0) + \beta_i| . \end{aligned}$$

But then if $|\hat{\lambda}_i(\gamma_0, \eta_0) + \beta_i| < 1$ for $i = 1, \cdots, r$, the inequality forces $\hat{\mu}_i(\gamma_0, \eta_0) = 0$ for all i. But this implies $(\gamma_0, \eta_0) \in Z(J)$, a contradiction. We conclude $|\hat{\lambda}_j(\gamma_0, \eta_0) + \beta_j| \geqq 1$ for some j, and since $||\alpha_i\mu_i * \eta_0 k||_A \leqq \alpha||\sigma_i * k||_1$ for all i, we reach the following absurdity:

$$\begin{aligned} 1 \leqq a||\lambda_j||_A + |\beta_j| \leqq (1 + a)M||\mu||_A \leqq \sum_{i=1}^{n}(1 + a)M\alpha||\sigma_i * k||_1 \\ + \sum_{i=n+1}^{\infty}(1 + a)M\alpha||k||_1||\mu_i|| < (1 + a)Mn\alpha\varepsilon + \frac{1}{2} = 1 . \end{aligned}$$

(b) Since Ω is discrete, the compact set C is contained in a product $\Gamma \times F$, $F \subset \Omega$ finite. But then if the complement of E is infinite, there is some $\eta_0 \notin F \cup E$. Taking any $\gamma_0 \in \Gamma$, we have $(\gamma_0, \eta_0) \notin C \cup Z(J)$, and the argument above yields the required contradiction; that is, E^c

must be empty or a finite set $\{\eta_1, \cdots, \eta_r\}$. Since Γ is compact, $J \subset I_{\Gamma \times E} = \{f \in L^1(G \times H): \operatorname{supp} \hat{f} \subset \Gamma \times E^c\} \subset K$; in fact the A-topology and the L^1-topology agree on $I_{\Gamma \times E}$ by the open mapping theorem and (i). J is therefore a closed ideal of $L^1(G \times H)$ with zero set $\Gamma \times E$, and the regularity of $L^1(G \times H)$ yields $I_{\Gamma \times E} = J$. The converse is obvious, since $g(x, y) = \sum_{i=1}^{r} \delta_0(x)\eta_i(y)$ generates $I_{\Gamma \times E}$.

COROLLARY 3.5. *Suppose A is a normed subalgebra of $M_0(G)$ satisfying* (iii). *Then A is a countably generated ideal of itself if and only if G is discrete.*

Proof. Certainly if G is discrete, $M(G) = L^1(G) = K \subset A$ and A is generated by its identity. Suppose on the other hand, G is nondiscrete and $A = \sum \mu_n * A + \sum C\mu_n$ for some sequence $\{\mu_n\} \subset A$. Examine the proof of 3.4 with $H = 0, J = A$: since $Z(J) = \varnothing$ and Γ is not compact, we can still pick $\gamma_0 \notin C \cup Z(J)$ and obtain the required contradiction.

COROLLARY 3.6. *If G is a normed subalgebra of $M_0(G)$ satisfying* (iii) *and Γ is connected, then no nonzero closed ideal of A can be countably generated.*

Proof. Suppose J is a countably generated, closed ideal in A. Then $Z(J) = \Gamma$ or ϕ (3.2). Plainly if $Z(J) = \Gamma, J = 0$; if $Z(J) = \phi$, 3.4.a with $H = 0$ implies G is discrete.

COROLLARY 3.7. (Compare [9, §4, p. 424].) *If G is nondiscrete, A is a normed ideal in $M_0(G)$ satisfying* (iii) *and Γ is connected, then no nonzero subspace of the form $\sum \mu_n * A$, $\mu_n \in M_0(G)$, is closed in A.*

Proof. Otherwise, 3.1 and a simplification of the argument in 3.4 shows, as in 3.6, that G is discrete.

Condition (iii) can be deleted if we assume A is Tauberian on Γ. For example we have

PROPOSITION 3.8. *If A is a Segal algebra on G and Γ is connected, then no nonzero proper subspace of the form $\sum \mu_n * A$, $\mu_n \in M(G)$, is closed in A.*

Proof. Otherwise, the zero set of the closed ideal $J = \sum \mu_n * A$ is ϕ or Γ (3.1); since A is Tauberian (2.1, [12, 39.32.g], [12, 39.27]) and semisimple, $J = A$ or 0.

THEOREM 3.9. *Let G be a compactly generated LCA group and A a normed subalgebra of $M_0(G)$ satisfying* (iii). *Then a nonzero closed ideal J of A is countably generated if and only if G is $Z^n \times C$, C a compact group, and J consists of those $f \in K$ whose Fourier transforms vanish on $T^n \times E$ where E is a cofinite subset of the dual of C.*

Proof. G is a group of the form $R^m \times Z^n \times C$, C a compact group [11, p. 90]. If A has a nonzero closed, countably generated ideal J, then 3.4.a applied to $H = C$ shows that $m = 0$. Thus $G = Z^n \times C$ and 3.4.b applied to $H = C$ gives J the required form.

COROLLARY 3.10. *Let G be an LCA group, A a normed subalgebra of $M_0(G)$ satisfying* (iii). *Then A has a countably generated ideal of the form I_γ, $\gamma \in \Gamma$, if and only if G is finite.*

Proof. If I_γ is countably generated, 3.2 implies γ is isolated in Γ, so that G is compact. But 3.9 then implies the singleton $\{\gamma\}$ is cofinite in Γ, so that actually Γ is finite; since finite Abelian groups are selfdual, we see G is also. Of course if G is finite, the proof of 3.4 shows that every ideal is principal.

Since Γ is the maximal ideal space of a Segal algebra on G, we have in particular

COROLLARY 3.11. *A Segal algebra on G which satisfies* (iii) *has a countably generated regular maximal ideal if and only if G is a finite.*

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Received October 4, 1972 and in revised form February 28, 1973.

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