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THE NON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

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The paper is devoted partly to the study of non-absolute Nörlund summability of Fourier series of $\varphi(t)$ under the condition $\varphi(t)\chi(t) \in AC[0, \pi]$ for suitable $\chi(t)$. The other aspect is to determine the order of variation of the Harmonic mean of the Fourier series whenever $\varphi(t) \log k/t \in BV[0, \pi]$.

1. Let L denote the class of all real functions f with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$ and let the Fourier series of $f \in L$ be given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) ,$$

assuming, as we may, the constant term to be zero.

We write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$$

$$g(n, t) = \int_0^t \frac{\cos nu}{\chi(u)} du$$

$$h(n, t) = \int_t^{\pi} \frac{\cos nu}{\chi(u)} du .$$

Let $\{p_n\}$ be a sequence of constants such that $P_n = \sum_{v=0}^{n} p_v \neq 0$ $(n \geqq 0)$ and $P_{-1} = p_{-1} = 0$. For the definition of absolute Nörlund or (N, p) method, see, for example, Pati [9]. When $\sum_{n=0}^{\infty} a_n$ is absolutely (N, p) summable, we shall write, for brevity, $\sum_{n=0}^{\infty} a_n \in |N, p|$.

We define the sequence of constants $\{c_n\}$ formally by $(\sum_{n=0}^{\infty} p_n x^n)^{-1} = \sum_{n=0}^{\infty} c_n x^n$, $c_{-1} = 0$.

2. One of the objects of this paper is to study the non-absolute (N, p) summability factors of Fourier series and generalize the following outstanding result of Pati in Theorems 1–2. Besides, the proof of Theorems 1–2 are short and simple and avoids the direct technique of Pati which is somewhat long and complicated.

If we write

$$G = \left\{f : f \in L,\ \varphi(t) \log k/t \in AC[0, \pi] \text{ and } \sum_{n=1}^{\infty} A_n(x) \notin \left|N, \frac{1}{n+1}\right|\right\}$$

then Pati's theorem is in the following form:

THEOREM P [9]. *G is nonempty.*

Mohanty and Ray [8] subsequently constructed an example of $f \in G$.

We now establish

THEOREM 1. *Let χ be a real differentiable function and $\{\varepsilon_n\}$ be a sequence satisfying the following conditions:*

(1) $$\phi(t)\chi(t) \in AC[0, \pi] ,$$

(2) $$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |g(n, \pi)| < \infty ,$$

(3) $$\frac{|\chi^1(t)|}{\chi^2(t)} \nearrow \text{ as } t \searrow 0 ,$$

(4) $$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n^2 |P_n|} \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} < \infty ,$$

(5) $$\sum_{n=1}^{\infty} \left| \Delta\left(\frac{\varepsilon_n}{nP_n}\right)\right| < \infty ,$$

(6) $$\varepsilon_n = 0(nP_n) ,$$

(7) *$\exists$ a set E: $mE > 0$ and $\exists$ a constant $\eta > 0$ such that $\chi(t)^{-1} > \eta$ $\forall t \in E$.*

Then

(8) $$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(t)| = \infty \qquad (\forall t \in E) ,$$

if and only if

(9) $$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty .$$

Now, if we denote, $G^* = \{f: f \in L$, conditions (1) through (7) and (9) hold and $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|\}$ then we establish

THEOREM 2. *Let*

(10) $$\sum_{v=0}^{n} |p_v| = O(|P_n|) , \quad \sum_{n=0}^{\infty} |c_n| < \infty .$$

Then G^* is nonempty.

In §3, we discuss some special cases of interest of Theorem 2.

Since Theorem 2 implies that the total variation of the (N, p) mean of the series $\sum_{n=1}^{\infty} \varepsilon_n A_n(x)$ is unbounded, the natural question now is to determine the order of the variation. And this is achieved in Theorem 3 in §4.

3. We need the following lemmas for the proof of Theorem 1.

LEMMA 1. (2) *Suppose that $\{f_n(x)\}$ is measurable in (a, b) where $b - a \leqq \infty$, for $n = 1, 2, \cdots$. Then a necessary and sufficient condition that, for every function $\psi(x)$ integrable in the sense of Lebesgue over (a, b), the functions $f_n(x)\psi(x)$ should be integrable L over (a, b) and*

$$\sum_{n=1}^{\infty} \left| \int_a^b \psi(x) f_n(x) dx \right| \leqq K$$

is that

$$\sum_{n=1}^{\infty} |f_n(x)| \leqq K ,$$

where K is an absolute constant for almost every x in (a, b).

LEMMA 2. *Let condition* (3) *hold. Then*

$$h(n, t) = \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} .$$

Proof. We have, by integration by parts, and second mean value theorem,

$$\begin{aligned} h(n, t) &= \left(\int_{\pi/n}^{\pi} - \int_{\pi/n}^{t}\right) \frac{\cos nu}{\chi(u)}\, du \\ &= \frac{\sin nt}{n\chi(t)} + \frac{1}{n}\left(\int_{\pi/n}^{\pi} - \int_{\pi/n}^{t}\right) \frac{\chi^1(u)}{\chi^2(u)} \sin nu du \\ &= \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} \left(\int_{\pi/n}^{\zeta^1} - \int_{\pi/n}^{\zeta}\right) \sin nu du \\ &= \frac{\sin nt}{n\chi(t)} + O\left(\frac{1}{n^2}\right) \frac{|\chi^1(\pi/n)|}{\chi^2(\pi/n)} , \end{aligned}$$

where $\pi/n \leqq \zeta \leqq \pi$, $\pi/n \leqq \zeta^1 \leqq t$.

This completes the proof.

Proof of Theorem 1. We have, by integration by parts,

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos ntdt = F(0)\, g(n, \pi) + \int_0^{\pi} F'(t) h(n, t) dt ,$$

where $F(t) \equiv \phi(t)\chi(t)$. Hence by condition (2) the statement (8) is equivalent to proving that:

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} \left| \int_0^{\pi} F'(t) h(n, t) dt \right| = \infty \qquad (\forall t \in E) . \tag{11}$$

Since, by hypothesis (1)

$$\int_0^{\pi} |F'(t)| dt < \infty ,$$

by Lemma 1, the statement (11) is equivalent to proving that $\exists$ a set E: $mE > 0$ and

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |h(n, t)| = \infty \qquad (\forall t \in E) . \tag{12}$$

Whenever conditions (3) and (4) hold, by virtue of Lemma 2, the statement (12) is easily seen to be equivalent to proving that

$$M(t) = \frac{1}{|\chi(t)|} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} |\sin nt| = \infty \qquad (\forall t \in E) . \tag{13}$$

Now, since

$$|\sin nt| \geqq \sin^2 nt = \frac{1}{2}(1 - \cos 2nt) ,$$

we have

$$M(t) \geqq \frac{1}{2\chi(t)} \left(\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} - \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt \right) .$$

Using conditions (5) and (6) and using Dedekind's theorem we observe that the series

$$\sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} \cos 2nt$$

is convergent for $0 < t < \pi$. Hence (13) is equivalent to showing that

$$\frac{1}{\chi(t)} \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{n|P_n|} = \infty \qquad (\forall t \in E) . \tag{14}$$

Now the result follows from (14) by using the conditions (7) and (8).

Proof of Theorem 2. Das [4], in particular, proved that whenever condition (10) holds, then

$$\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, p| \Longrightarrow \sum_{n=1}^{\infty} \frac{|\varepsilon_n|}{|P_n|} |A_n(x)| < \infty .$$

Now the result follows from Theorem 1.

4. In this section we apply Theorem 2 to some special cases. If we take $\chi(t) = \log k/t$, $E = \{t: k/e \leqq t < \pi\}$ we get

COROLLARY 1. *Let* $\{\varepsilon_n\}$ *satisfy the conditions*:
(i) $\varepsilon_n = O(\log n)$,
(ii) $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log^3(n+1) < \infty$,
(iii) $\sum_{n=1}^{\infty} |\Delta\varepsilon_n|/n \log(n+1) < \infty$,
(iv) $\sum_{n=1}^{\infty} |\varepsilon_n|/n \log(n+1) = \infty$.
Then

$$\varphi(t) \log k/t \in AC[0, \pi] \Longrightarrow \sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin \left|N, \frac{1}{n+1}\right| .$$

Proof. Since the Fourier series of the even periodic function $(\log k/|t|)^{-1}$ is absolutely convergent (see Mohanty [7]) we get that

$$\sum_{n=1}^{\infty} \left| \int_0^{\pi} \frac{\cos nu}{\log k/u} du \right| < \infty . \tag{15}$$

It may be observed that if we take $\varepsilon_n = 1$, $p_n = 1/(n+1)$ in Corollary 1, then we get Theorem P.

COROLLARY 2. *Let* $\varphi(t) \in BV[0, \pi]$ *and let conditions* (5), (6), *and* (9) *hold. Then* $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \notin |N, p|$.

Take $\chi(t) \equiv 1$, $E = [0, \pi]$ in Theorem 2. In this case $g(n, \pi) = 0$.

REMARK. Corollary 2 in the case $\varepsilon_n = 1$ gives that

$$\varphi(t) \in BV[0, \pi] \Longrightarrow \sum_{n=1}^{\infty} A_n(x) \notin \left|N, \frac{1}{n+1}\right| .$$

This interalia establishes the result that $\varphi(t) \in BV[0, \pi]$ is *not sufficient* to guarantee the absolute convergence of the series $\sum_{n=1}^{\infty} A_n(x)$. See Bosanquet (1) who showed this by taking an example.

5. Throughout this section we consider the case $p_n = 1/(n+1)$ only. We write t_n and τ_n respectively for the $(N, 1/(n+1))$ means of the sequences $\{\sum_{v=1}^{n} \varepsilon_v A_v(x)\}$ and $\{n\varepsilon_n A_n(x)\}$. It follows from a result of Das [4] Theorem 6 on general infinite series that

$$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(1) \text{ if and only if } \sum_{n=1}^{m} |t_n - t_{n-1}| = O(1) . \tag{16}$$

Proceeding as in the proof of above result we in fact get that for any positive nondecreasing sequence $\{\lambda_n\}$

(17) $$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(\lambda_m) \text{ if and only if } \sum_{n=1}^{m} |t_n - t_{n-1}| = O(\lambda_m) .$$

Since Theorem P implies that the variation of $\{t_n\}$ is of unbounded order, we are immediately confronted with the problem of determining the order of variation of $\{t_n\}$. Because of relation (17) this problem simplifies to determining the order of $\sum_{n=1}^{m} |\tau_n|/n$ and this is achieved in

THEOREM 3. *If* $g(t) \equiv \varphi(t) \log k/t \in BV[0, \pi]$. *Then*

$$\sum_{n=1}^{m} \frac{|\tau_n|}{n} = O(\log \log m) .$$

Proof. We have

$$\tau_n = \frac{2}{\pi P_n} \sum_{\nu=1}^{n} p_{n-\nu}\nu \int_0^\pi \varphi(t) \cos \nu t dt .$$

Since

$$\int_0^\pi \varphi(t) \cos \nu t dt = g(0) \int_0^\pi \frac{\cos \nu t}{\log k/t} dt + \int_0^\pi dg(t) \int_t^\pi \frac{\cos \nu u}{\log k/u} du ,$$

we get

$$\sum_{n=1}^{m} \frac{|\tau_n|}{n} \leqq \frac{2}{\pi} |g(0)| \sum_{n-1}^{m} \frac{1}{nP_n} \left| \int_0^\pi \frac{dt}{\log k/t} \left(\sum_{\nu=1}^{n} p_{n-\nu}\nu \cos \nu t \right) \right|$$
$$+ \frac{2}{\pi} \int_0^\pi |dg(t)| \sum_{n=1}^{m} \frac{1}{nP_n} \left| \int_t^\pi \frac{dt}{\log k/t} \left(\sum_{\nu=1}^{n} p_{n-\nu}\nu \cos \nu t \right) \right|$$
$$= \frac{2}{\pi} \{ |g(0)| G_m + H_m \} .$$

Since the series $\sum_{n=1}^{\infty} \int_0^\pi \cos nu/\log k/u \, du$ is absolutely convergent (see (15)) and therefore it is absolutely $(N, 1/(n+1))$ summable, we get that $G_m = O(1)$ by using relation (16).

Since $\int_0^\pi |dg(t)| < \infty$, using Lemma 2 with $\log k/t$ in place of $\chi(t)$ we get that

$$H_m = O(1) \sum_{n=1}^{m} \frac{1}{n \log (n+1)} \left| \sum_{\nu=1}^{n} \frac{\sin \nu t}{n - \nu + 1} \right|$$
$$+ O(1) \sum_{n=1}^{m} \frac{1}{n \log (n+1)} \sum_{\nu=1}^{n} \frac{1}{(n - \nu + 1) \log^2 (\nu + 1)} = H_m^{(1)} + H_m^{(2)} .$$

By a result of McFadden ([6], Lemma 5.10) we get

$$\sum_{\nu=1}^{n} \frac{\sin \nu t}{n - \nu + 1} = O(\log \tau), \; (\tau = [k/t])$$

and consequently

$$H_m^{(1)} = O(1)\frac{\log \tau}{\log k/t} \sum_{n=1}^{m} \frac{1}{n \log (n+1)} = O(\log \log m) \ .$$

On change of order of summation in $H_m^{(2)}$ and by use of the fact that

$$\sum_{n=\nu}^{m} \frac{1}{(n-\nu+1)n \log (n+1)} = O\Big(\frac{1}{\nu+1}\Big) ,$$

we get

$$H_m^{(2)} = O(1) \sum_{\nu=1}^{m} \frac{1}{\nu \log^2 (\nu+1)} = O(1) \qquad (m \longrightarrow \infty) \ ;$$

and this completes the proof.

REMARKS. In view of Corollary 1, one is naturally led to determine suitable sequences $\{\varepsilon_n\}$ such that $g(t) \in BV[0, \pi] \Rightarrow \sum \varepsilon_n A_n(x) \in |N, 1/(n+1)|$. But in view of Theorem 3 it is enough to determine the sequence of factors $\{\varepsilon_n\}$ such that $\sum_{n=1}^{\infty} \varepsilon_n A_n(x) \in |N, 1/(n+1)|$ whenever $\sum_{n=1}^{m} |\tau_n|/n = O(\log \log m)$. Such a result is contained in the more general result of Das [5].

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