PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

SETS GENERATED BY RECTANGLES

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For any family F of sets, let $\mathscr{B}(F)$ denote the smallest σ-algebra containing F. Throughout this paper X denotes a set and $\mathscr{R}$ the family of sets of the form $A \times B$, for $A \subseteqq X$ and $B \subseteqq X$. It is of interest to find conditions under which the following holds:

(1) Each subset of $X \times X$ is a member of $\mathscr{B}(\mathscr{R})$

The interesting case is when

$$\omega_1 < \text{Card } X \leqq c \text{ ,}$$

since results for other cases are known.

It is shown in Theorem 9 that (1) is equivalent to

(2) There is a countable ordinal α such that each subset of $X \times X$ can be generated from $\mathscr{R}$ is α Baire process steps .

It is also shown that the two-dimensional statements (1) and (2) are equivalent to the one-dimensional statement

(3) There is a countable ordinal α such that for each family H of subsets of X with $\text{Card } H = \text{Card } X$, there is a countable family G such that each member of H can be generated from G in α steps .

It is shown in Theorem 5 that the continuum hypothesis (CH) is equivalent to certain statements about rectangles of the form (1) and (2) with $\alpha = 2$.

Rao [7, 8] and Kunen [2] have shown that

THEOREM 1. *If* Card $X \leqq \omega_1$ (*the first uncountable cardinal*) *then* (1) *is true and if* Card $X > c$ *then* (1) *is false.*

The question of whether (1) is true (without the requirement Card $X \leqq \omega_1$) was raised by Johnson [1] and earlier by Erdös, Ulam, and others (see [8], p. 197). The arguments in Kunen's thesis actually showed that if Card $X \leqq \omega_1$ then

(4) Each subset of $X \times X$ can be generated from $\mathscr{R}$ in 2 steps (i.e., each subset is a member of $\mathscr{R}_{\sigma\delta}$. See definitions in § 2.) .

In Theorem 5 we generalize Theorem 1 and Kunen's result (4), and give a new characterization of CH by showing it to be equivalent to certain statements about rectangles of the form (1) and (4).

If CH is assumed the α appearing in statements (2) and (3) above is 2 (see Theorem 10). This raises the intriguing (but unanswered) question of whether α must *always* be 2 if (1) holds and CH is false.

It would also be interesting to know whether statements (1), (2), and (3) are equivalent to statement (5) below. Clearly (3) imples (5).

(5) If H is a family of subsets of X with Card $H =$ Card X, then there is a countable family G for which $H \subseteqq \mathscr{B}(G)$.

The equivalence of (1) and (2) means for example, (assuming CH), that there is a countable family G from which all real Borel sets (or analytic sets, or projective sets) can be generated in *two* steps (i.e., Borel sets $\subseteqq G_{\sigma\delta}$). This is remarkable in view of the well known result [4, 8] that if G is a countable basis for the real topology, then the Borel sets cannot be generated from G in less than ω_1 steps.

As a generalization of this well known result we show in Theorem 12 that any countable family G which is closed to complementation and which generates the Borel sets (i.e., Borel sets $\subseteqq \mathscr{B}(G)$) must have order ω_1. That is

$$\mathscr{B}(G) \nsubseteqq G_\alpha$$

for any countable ordinal α. Thus, even though G might generate the Borel sets in α steps (or 2 steps if CH is assumed), the process, nevertheless, continues to produce new members of $\mathscr{B}(G)$ until we reach G_{ω_1}.

We would like to point out in conjunction with our characterization of CH that Kunen [2] has proved that if Martin's Axiom A holds (see [6]) and Card $X \leqq c$ then (4) holds. He also proved that if $\omega_1 <$ Card $X \leqq c$ then (1) is independent of ZFC (Zermo-Frankel Axioms + the Axiom of Choice) together with the negation of CH.

2. Notation and definitions.

If G is any family of sets, let G_0 be the family G, and for each ordinal α, $\alpha > 0$, let G_α be the family of all countable unions (intersections) of sets in $\bigcup_{\gamma<\alpha} G_\gamma$, if α is odd (even). Limit ordinals will be considered even. (Compare Kuratowski [3].) Thus we have

$$G_0 = G, G_1 = G_\sigma, G_2 = G_{\sigma\delta}, G_3 = G_{\sigma\delta\sigma}, \cdots, G_\alpha, \cdots .$$

Also $G_\alpha \subseteqq G_{\alpha+1}$ for each ordinal α and $G_{\omega_1} = G_{\omega_1+1}$, where ω_1 is the first uncountable ordinal. If $\alpha > 0$, then the family G_α is closed under countable unions (intersections) if α is odd (even).

We define the *order of* G to be the first ordinal α, $\alpha > 0$, such that $G_{\alpha+1} = G_\alpha$.

For each $A \subseteqq X$ (or $A \subseteqq X \times X$), let A' be the complement of A with respect to X (or $X \times X$), and for each family G of subsets of X (or $X \times X$) let $\mathscr{C}(G)$ be the family of complements of G. Note that if $\mathscr{C}(G) \subseteqq G$, or even if $\mathscr{C}(G) \subseteqq G_{\omega_1}$, then the family G_{ω_1} is the family $\mathscr{B}(G)$, the σ-algebra generated by G. Thus, since

$$(A \times B)' = A \times B' \cup A' \times X \in \mathscr{R}_1 ,$$

it follows that $\mathscr{R}_{\omega_1} = \mathscr{B}(\mathscr{R})$.

If G is a family of subsets of X, let $VG = \{A \times B\colon A \subseteqq X, B \in G\}$, and let $HG = \{A \times B\colon A \in G, B \subseteqq X\}$.

If $Z \subseteqq X \times X$ and $x \in X$, let Z_x denote the vertical section of Z at x, $Z_x = \{y\colon (x, y) \in Z\}$.

3. Results. The following lemma is easily proved by transfinite induction.

LEMMA 2. *If $1 \leqq \alpha < \omega_1$ and $A \in G_\alpha$, then there is a set B in G_1 such that $A \subseteqq B$.*

THEOREM 3. *If G is a countable family of subsets of X, $Z \subseteqq X \times X$, and $0 < \alpha < \omega_1$, then $Z \in (VG)_\alpha$ if and only if $Z_x \in G_\alpha$ for each $x \in$ domain Z.*

Proof. By considering the natural projections of the sets involved on the second coordinate axis, it is easily seen that

$$\text{if } Z \in (VG)_\alpha, \text{ then } Z_x \in G_\alpha \text{ for each } x \in \text{domain } Z .$$

Now suppose that $Z_x \in G_\alpha$, for each $x \in$ domain Z, and let $G = \{\theta_1, \theta_2, \theta_3, \cdots\}$. We complete the proof by transfinite induction on α.

Case 1. $\alpha = 1$.

For each n, let $A_n = \{x \in \text{domain } Z\colon \theta_n \subseteqq Z_x\}$, and let $Z_n = A_n \times \theta_n$. Then $Z_n \in VG$, for each n, and

$$Z = \bigcup_{n=1}^{\infty} Z_n \in (VG)_1 .$$

Now suppose $1 < \alpha < \omega_1$, and that the theorem holds for every γ, $0 < \gamma < \alpha$.

Case 2. α is even.

Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of odd ordinals less than α such that each odd ordinal less than α appears infinitely often in $\{\gamma_i\}_{n=1}^{\infty}$. For each $x \in$ domain Z, let

$$D_1(x),\ D_2(x),\ D_3(x),\ \cdots$$

be a sequence such that $D_i(x) \in G_{\gamma_i}$ for each i, and

$$Z_x = \bigcap_{i=1}^{\infty} D_i(x) .$$

This can be done in view of Lemma 2. For each i, let

$$Z^i = \bigcup_{x \in \text{domain } Z} \{x\} \times D_i(x) .$$

First note that $Z = \bigcap_{i=1}^{\infty} Z^i$. Also each nonempty section $(Z^i)_x$ of Z^i is equal to $D_i(x) \in G_{\gamma_i}$. Hence, by the induction hypothesis, $Z^i \in (VG)_{\gamma_i}$, for each i, and therefore

$$Z = \bigcap_{i=1}^{\infty} Z^i \in (VG)_\alpha ,$$

by the definition of the family $(VG)_\alpha$.

Case 3. α is odd and greater than 1.

For each $x \in$ domain Z, let $\{D_i(x)\}_{i=1}^{\infty}$ be a sequence of members of $G_{\alpha-1}$ for which $Z_x = \bigcup_{i=1}^{\infty} D_i(x)$, and let $Z^i = \bigcup_{x \in \text{domain } (Z)} \{x\} \times D_i(x)$, for each i.

Again it follows that $Z^i \in G_{\alpha-1}$, for each i, and

$$Z = \bigcup_{i=1}^{\infty} Z^i \in (VG)_\alpha .$$

COROLLARY 4. *If* $Z \subseteq X \times X$ *is the graph of a function then* $Z \in \mathscr{R}_2 \subseteq \mathscr{B}(\mathscr{R})$.

Proof. Let G be a countable basis for the real topology and note that, for each $x \in X$, Z_x is a singleton and hence $Z_x \in G_2$. Thus by Theorem 3, $Z \in (VG)_2 \subseteq \mathscr{R}_2 \subseteq \mathscr{B}(\mathscr{R})$. Also see [7].

THEOREM 5. *Let* X *be the real numbers and let* G *be a countable base for the usual topology on* X. *The following three statements are equivalent*:

(1) CH *holds*

(2) *if* $Z \subseteq X \times X$, *then* $Z = A \cap B$, *where* $A \in (VG)_2$ *and* $B \in (HG)_2$ *and*

(3) *if* $E \subseteq X \times X$, *then* $E = C \cup D$, *where* $C \in \mathscr{B}(VG)$ *and* $D \in \mathscr{B}(HG)$.

Proof. First, assume CH and suppose $Z \subseteqq X \times X$. As is well known [7], the complement of Z is the union of two sets H and K such that each vertical section of H is countable and each horizontal section of K is countable.

Let A be the complement of H and let B be the complement of K. Then each vertical section of A is a G_2 set and by Theorem 3, $A \in (VG)_2$. Similarly, $B \in (HG)_2$. Of course, $Z = A \cap B$.

Since $A \in (VG)_2 \subseteqq \mathscr{R}_2$ and $B \in (HG)_2 \subseteqq \mathscr{R}_2$ and $\mathscr{R}_2$ is closed under finite intersections, $Z \in \mathscr{R}_2$. Thus, if CH holds, then the order of $\mathscr{R}$ is $\leqq 2$. Since the graph of the identity function, $f(x) = x$, is not in $\mathscr{R}_1$, it follows that the order of $\mathscr{R}$ is 2.

Now, suppose statement 2 holds and $E \subseteqq X \times X$. Then, the complement of E can be expressed as the intersection of sets A and B with $A \in (VG)_2$ and $B \in (HG)_2$. It follows that $A' \in (VG)_3 \subseteqq \mathscr{B}(VG)$ and $B' \in (HG)_3 \subseteqq \mathscr{B}(HG)$. Thus, E is the union of two sets C and D, where $C \in \mathscr{B}(VG)$ and $D \in \mathscr{B}(HG)$.

Finally, assume statement 3 holds. Let T be a totally imperfect subset of X of cardinality c. The existence of such a set can be proven without assuming CH [3, p. 514]. Let $E = T \times T$ and let $E = C \cup D$, with $C \in \mathscr{B}(VG)$ and $D \in \mathscr{B}(HG)$. Then each vertical section of C is a subset of T which is a Borel set. Since an uncountable Borel set contains a perfect set and T contains no perfect set, we have that each vertical section of C is countable. Similarly, each horizontal section of D is countable. But, as is well known [10] this implies CH.

This completes the proof of Theorem 5.

The following two lemmas are well known.

LEMMA 6. *If F is a family of sets, α is a countable ordinal, and $A \in F_\alpha$, then there is a countable subfamily J of F for which $A \in J_\alpha$.*

LEMMA 7. *If F is a family of sets, $\mathscr{C}(F) \subseteqq F$, and $A \in \mathscr{B}(F)$ then there is a countable subfamily J of F and a countable ordinal α for which $A \in J_\alpha$.*

THEOREM 8. (a) *The following two statements are equivalent:*

(i) *For each subset Z of $X \times X$ there is a countable ordinal α such that $Z \in \mathscr{R}_\alpha$.*

(ii) *If H is a family of subsets of X and* Card $H =$ Card X, *then there is a countable family G of subsets of X and a countable ordinal α for which $H \subseteqq G_\alpha$.*

(b) *If α is a countable ordinal, the following two statements are equivalent:*

(i) *Each subset of $X \times X$ is a member of $\mathscr{R}_\alpha$.*

(ii) *If H is a family of subsets of X and* Card $H =$ Card X *then there is a countable family G of subsets of X for which $H \subseteqq G_\alpha$.*

Proof. The proof of part (b) is similar to that of part (a) which is given below.

First suppose (i) holds, and suppose that H satisfies the hypotheses of (ii). Define the subset $Z \subseteqq X \times X$ by letting each member of H be a vertical section of Z. More precisely, let f be a 1-1 function from X to H and let

$$Z = \bigcup_{x \in X} \{x\} \times f(x) .$$

By (i) there is a countable ordinal α such that $Z \in \mathscr{R}_\alpha$ and hence by Lemma 6, there is a countable subfamily J of $\mathscr{R}$ for which $Z \in J_\alpha$. Let

$$G = \{B\colon A \times B \in J\} ,$$

note that $Z \in (VG)_\alpha$ and use Theorem 3 to conclude that $H \subseteqq G_\alpha$.

Now suppose (ii) holds, and that $Z \subseteqq X \times X$. Let H be the family of vertical sections of Z, and use (ii) to secure a countable family G and a countable ordinal α for which $H \subseteqq G_\alpha$. Thus $Z_x \in G_\alpha$ for each $x \in$ domain Z and by Theorem 3

$$Z \in (VG)_\alpha \subseteqq \mathscr{R}_\alpha .$$

THEOREM 9. *The following four statements are equivalent*:

(i) *Each subset of $X \times X$ is a member of $\mathscr{B}(\mathscr{R})$.*

(ii) *If H is a family of subsets of X and* Card $H =$ Card X *then there is a countable family G and a countable ordinal α for which $H \subseteqq G_\alpha$.*

(iii) *There is a countable ordinal α such that, for each family H of subsets of X with* Card $H =$ Card X, *there is a countable family G for which $H \subseteqq G_\alpha$.*

(iv) *There is a countable ordinal $\alpha \geqq 2$ such that each subset of $X \times X$ is a member of $\mathscr{R}_\alpha$.*

Proof. Statements (i) and (ii) are equivalent by Lemma 7 and Theorem 8a. Clearly (iii) implies (ii) and (iv) implies (i). Also by Theorem 8b it follows that (iii) implies (iv). α cannot be equal to 1 in (iv) because by (i) the identity function $f(x) = x$ is not in $\mathscr{R}_1$.

We complete the proof by showing that (ii) implies (iii). Since this result is immediate if X is countable we will assume that Card $X \geqq \omega_1$.

Suppose that (ii) holds and that (iii) does not. Then for each $\alpha < \omega_1$, there is a family $H(\alpha)$ of subsets of X for which Card $H(\alpha) =$

Card X and

(1) for each countable G, $H(\alpha) \not\subseteq G_\alpha$.

Let $H' = \bigcup_{\alpha<\omega_1} H(\alpha)$. Thus Card H' = Card X and hence by (ii) there is a countable family G' and a countable ordinal α' for which $H' \subseteq G'_{\alpha'}$. But then $H(\alpha') \subseteq H' \subseteq G'_{\alpha'}$ in contradiction of (1).

Therefore (ii) implies (iii).

In part (ii) above the family G can be chosen so that G_{ω_1} is closed to complementation (i.e., is a σ-algebra).

In view of condition (ii) of Theorem 9, it is interesting to note that R. Mansfield has shown that if G is a countable family of Lebesgue measurable sets, then $B(G)$ does not contain all analytic sets [5].

As was mentioned in the introduction it would be interesting to know whether the formula "$H \subseteq G_\alpha$" in Theorem 9 could be replaced by $H \subseteq \mathscr{B}(G)$. We do not know the answer to this question.

THEOREM 10. *If* CH *holds,* Card $X = c$, *H is a family of subsets of X, and* Card $H = c$, *then there is a countable family G for which* $H \subseteq G_2$.

Proof. By Theorem 5 each subset Z of $X \times X$ is a member of $\mathscr{R}_2$. The desired conclusion now follows from Theorem 8b.

4. **Generating Borel sets.** Let R be the set of reals, and let H be the family of all Borel subsets of R. This family has cardinality c. Suppose G is a countable family of subsets of R such that $H \subseteq G_{\omega_1}$ and G_{ω_1} is closed to complementation. The next two theorems show that, even if the family G generates all the Borel sets at an early stage, the order of G is ω_1. This is a generalization of the well known result [4, 9] that if G is a countable basis for the real topology then G has order ω_1. Our proof which is a usual "diagonal" type argument, parallels somewhat Lebesgue's proof of that result [3, p. 368].

Let $G = \{V_1, V_2, V_3, \cdots\}$, let N be the set of irrational numbers between 0 and 1 and let K be the family $\{\theta_1, \theta_2, \theta_3, \cdots\}$ of all intersections of the members of G with N,

$$\theta_i = V_i \cap N .$$

It will be shown that the order of K is ω_1. It then follows that the order of G is ω_1.

For each $z \in N$, let $(z_1, z_2, z_3, \cdots)$ be the sequence of integers appearing in the continued fraction expansion of z. This defines a reversible transformation from N onto the set of all sequences of positive integers. Let

$$\begin{aligned} z^1 &= (z_1, z_3, z_5, \cdots) \qquad \text{(odd indices)} \\ z^2 &= (z_2, z_6, z_{10}, \cdots) \\ z^3 &= (z_4, z_{12}, z_{20}, \cdots) \\ &\vdots \\ z^n &= (z_{2^n-1}, z_{3\cdot 2^{n-1}}, z_{5\cdot 2^{n-1}}, \cdots) \\ &\vdots \end{aligned} \tag{$*$}$$

This defines a homeomorphism between N and $N^{\aleph_0}$ (see Kuratowski [3], p. 369). Also note that if f is a continuous function from N into N, then the functions f_n from N into the space of positive integers are continuous, where

$$f(z) = (f_1(z), f_2(z), f_3(z), \cdots)\,, \quad \text{or} \quad (f_n(z) = f(z)_n)\,.$$

Recall that $K = \{\theta_1, \theta_2, \theta_3, \cdots\}$. The family K_α which appears in Theorem 11 is defined in §2.

THEOREM 11. *For each countable ordinal α, $\alpha > 0$, there is a function U_α from N onto K_α such that if f is a continuous function from N into N, then the set*

$$A_f = \{z: z \in U_\alpha(f(z))\}$$

is in $\mathscr{B}(K)$.

Proof. Let $U_1(z) = \bigcup_{n=1}^\infty \theta_{z_n}$, for each $z \in N$. Clearly U_1 maps N onto K_1.

Let f be a continuous function from N onto N.
We have

$$\begin{aligned} A_f &= \{z: z \in U_1(f(z))\} \\ &= \left\{z: z \in \bigcup_{n=1}^\infty \theta_{f_n(z)}\right\} \\ &= \bigcup_{n=1}^\infty \{z: z \in \theta_{f_n(z)}\}\,. \end{aligned}$$

For each n,

$$\{z: z \in \theta_{f_n(z)}\} = \bigcup_{i=1}^\infty \{J_{n_i} \cap \theta_i\}$$

where $J_{n_i} = \{z: f_n(z) = i\}$. Since each f_n is continuous it follows that each J_{n_i} is open and therefore the set A_f belongs to G_{ω_1}.

Suppose $1 < \alpha < \omega_1$ and suppose that the function U_γ has been defined for each ordinal γ with $1 \leqq \gamma < \alpha$. (Induction hypothesis.)

If α is odd, let

$$U_\alpha(z) = \bigcup_{n=1}^{\infty} U_{\alpha-1}(z^n), \quad \text{for} \quad z \in N .$$

Clearly U_α maps N onto K_α.

If α is even, let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of odd ordinals less than α such that each odd ordinal less than α appears infinitely often in $\{\gamma_i\}_{n=1}^{\infty}$ and let

$$U_\alpha(z) = \bigcap_{n=1}^{\infty} U_{\gamma_n}(z^n) , \quad \text{for} \quad z \in N .$$

If $A \in K_\alpha$ (α is still even), then

$$A = \bigcap_{n=1}^{\infty} D_n ,$$

where $D_n \in K_{\gamma_n}$, for each n. For each n, let y_n be a point of N such that

$$D_n = U_{\gamma_n}(y_n) .$$

And let z be the point mapped by the transformation described by (*) to the point $(y_1, y_2, y_3, \cdots)$ of $N^{\aleph_0}$. Thus

$$U_\alpha(z) = A$$

and U_α maps N onto K_α.

This completes the definition of the functions U_α. Now let f be a continuous function from N into N. It will be shown that if α is even the set

$$A_f = \{z\colon z \in U_\alpha(f(z))\}$$

is in G_{ω_1}. The argument for the case α is odd is similar.

We have

$$\begin{aligned} A_f &= \left\{z\colon z \in \bigcap_{n=1}^{\infty} U_{\gamma_n}((f(z))^n)\right\} \\ &= \bigcap_{n=1}^{\infty} \{z\colon z \in U_{\gamma_n}((f(z))^n)\} . \end{aligned}$$

But, for each n, the function $z \to (f(z))^n$, being the composition of two continuous functions, is a continuous function from N to N.

Thus by the induction hypothesis, the sets $\{z\colon z \in U_{\gamma_n}((f(z))^n)\}$ are in the family G_{ω_1}. Therefore $A_f \in G_{\omega_1}$.

THEOREM 12. *If G is a countable family of subsets of real numbers with $\mathscr{C}(G) \subseteqq G$, and each Borel set is a member of $\mathscr{B}(G)$ then G has order ω_1.*

Proof. Let α be any countable ordinal, and let

$$I_\alpha = \{z \colon z \notin U_\alpha(z)\} .$$

Suppose $I_\alpha \in K_\alpha$, and let $U_\alpha(z) = I_\alpha$. If $z \in I_\alpha$ then $z \in U_\alpha(z)$. But this contradicts the definition of I_α. If $z \notin I_\alpha$, then $z \in U_\alpha(z) = I_\alpha$, $z \in I_\alpha$. This contradiction shows that $I_\alpha \notin K_\alpha$.

Since $\mathscr{B}(G) = G_{\omega_1}$ (because $\mathscr{C}(G) \subseteqq G$), and $I'_\alpha = \{z \colon z \in U_\alpha(z)\} \in G_{\omega_1}$ by Theorem 11, it follows that $I_\alpha \in G_{\omega_1} - G_\alpha$. Thus $G_\alpha \neq G_{\omega_1}$, and hence G has order ω_1 [3, p. 371].

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Received November 14, 1972 and in revised form July 12, 1973. This work was partially supported by NSF Grant GJ-32269.

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