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THE DIRICHLET PROBLEM FOR SOME OVERDETERMINED SYSTEMS ON THE UNIT BALL IN C^u

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A characterization is given of those functions on $\partial B^n = \{|z| = 1\}$ which can be extended to be analytic, pluriharmonic, or n-harmonic in $B^n = \{|z| < 1\}$.

1. **Introduction.** If f is a continuous function on $\partial B^n = \{z = (z_1, \cdots, z_n): |z| = 1\}$, then f can be extended to a harmonic function F in $B^n = \{z: |z| < 1\}$. That is, the Dirichlet problem is uniquely solvable. If we wish F, in addition, to be analytic, pluriharmonic, or n-harmonic, the extension is not always possible, and we must impose some restrictions on the function f. It is well-known that necessary and sufficient conditions for f to have an analytic extension are that f satisfy the tangential Cauchy-Riemann equation. In this paper we show that there are other systems that replace the tangential Cauchy-Riemann equations as consistency conditions. We also give the consistency conditions for a function to extend to be pluriharmonic or n-harmonic.

2. **Pluriharmonic extension.** Some important differential operators tangential to ∂B^n, $n \geqq 2$ are:

$$\mathscr{L}_{ij} = \bar{\zeta}_i \frac{\partial}{\partial \zeta_j} - \bar{\zeta}_j \frac{\partial}{\partial \zeta_i} \tag{1}$$

$$\overline{\mathscr{L}}_{ij} = \zeta_i \frac{\partial}{\partial \bar{\zeta}_j} - \zeta_j \frac{\partial}{\partial \bar{\zeta}_i} \tag{2}$$

where we take $1 \leqq i, j \leqq n$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \partial B^n$. A simple computation shows that the real and imaginary parts of these operators are tangent to ∂B^n. These operators extend naturally into the interior of B^n. The following lemma shows the interplay between the action of the $\mathscr{L}_{ij}$ on ∂B^n and in B^n.

LEMMA 1. *Let $\mathscr{L}$ be one of the operators* (1) *or* (2), *and let $u \in C^1(\partial B^n)$ be given. If $P(x, \zeta)$ is the Poisson kernel on B^n, we have:*

$$(\mathscr{L}_\zeta u) * P(z) = \mathscr{L}_z(u * P(z)) \tag{3}$$

for $\zeta \in \partial B^n$, $z \in B^n$.

Proof. The operator $\mathscr{L}$ satisfies the hypotheses of Lemma 2, and thus the right hand side of (3) is harmonic (the left hand side obviously is). Since (3) is valid for $|z| = 1$, it must hold for all $z \in B^n$.

LEMMA 2. *An operator* $\mathscr{D} = f(x, y)\partial/\partial y - g(x, y)\partial/\partial x$ *preserves harmonic functions if and only if the pair* (f, g) *satisfies the Cauchy-Riemann equations,*

$$f_x = g_y$$
$$f_y = -g_x \ .$$

Proof. It is a straightforward calculation that $(\mathscr{D}u)_{xx} + (\mathscr{D}u)_{yy} = 0$ for all harmonic u if and only if $f_x = g_y$ and $-f_y = g_x$.

COROLLARY 1. *If* $f \in L^1(\partial B^n)$, *and* $\mathscr{L}f = g$ *in the weak sense,* (i.e., $\int_{|\zeta|=1} f\mathscr{L}\varphi = -\int_{|\zeta|=1} g\varphi$ *for all* $\varphi \in C^\infty(\partial B^n)$, *then*

$$g * P(z) = \mathscr{L}_z(f * P(z)) \ .$$

Proof. Since the Poisson kernel on B^n is $P(\zeta, z) = 1 - |z|^2/|z - \zeta^{2n}|$, one can calculate that:

$$\mathscr{L}_z P(\zeta, z) = -\mathscr{L}_\zeta P(\zeta, z) \ .$$

Thus if dS is normalized surface area, we have:

$$\begin{aligned} \mathscr{L}_z(f * P(z)) &= \int_{|\zeta|=1} f(\zeta)\mathscr{L}_z P(\zeta, z)dS \\ &= -\int_{|\zeta|=1} f(\zeta)\mathscr{L}_\zeta P(\zeta, z)dS = \int_{|\zeta|=1} g(\zeta)P(\zeta, z)dS \\ &= g * P(z) \ . \end{aligned}$$

DEFINITION. If α and β are multi-indices, then $z^\alpha \bar{z}^\beta = \prod_{j=1}^n z_j^{\alpha_j} \bar{z}_j^{\beta_j}$ has *type* (p, q) if $|\alpha| = p$ and $|\beta| = q$. If $h(z, \bar{z})$ is a sum of monomials of type (p, q), then h is of type (p, q).

Observe that if h is of type (p, q), then $\bar{\mathscr{L}}_{ij}h$ is either zero or of type $(p + 1, q - 1)$. Similarly, $\mathscr{L}_{ij}h$ is either of type $(p - 1, q + 1)$ or zero.

By L we will denote the matrix of operators $L = (\mathscr{L}_{ij})$.

If $K = (K_{rs})$ and $M = (M_{ij})$ are two matrices of operators, then KM will denote the tensor product of the two matrices:

$$KM(u) = K \otimes M(u) = (K_{rs}M_{ij}u) \ .$$

LEMMA 3. *Let* $F \in C^1(\bar{B}^n)$ *satisfy* $\Delta F = 0$. *If* $\bar{L}F(z) = 0$ *for all* $z \in B^n$, *then* F *is analytic.*

Proof. The system $\bar{L}F = 0$ is precisely the tangential Cauchy-

Riemann equations (see [1], [2]). Thus if f is the restriction of F to ∂B^n, then f has a holomorphic extension to B^n, which must coincide with F, since F is harmonic.

REMARK. The lemma may also be proved directly without mention of the tangential Cauchy-Riemann equations.

THEOREM 1. *If* $u \in C^3(\partial B^n)$, *then*

$$\bar{L}\bar{L}L(u) = 0 \tag{4}$$

if and only if u *extends to a pluriharmonic function* U *on* B^n.

Proof. If u extends to a pluriharmonic U, then we write $U(z, \bar{z}) = f(z) + g(\bar{z})$ where f and g are analytic. An entry of the matrix $\bar{L}\bar{L}LU$ looks like:

$$\begin{aligned}\bar{L}(\overline{\mathscr{L}}_{ij}\mathscr{L}_{kl}U) &= \bar{L}\overline{\mathscr{L}}_{ij}(\bar{z}_k f_{z_l} - \bar{z}_l f_{z_k}) \\ &= \bar{L}\Big(z_i\Big(\frac{\partial\bar{z}_k}{\partial\bar{z}_j}\Big)f_{z_l} - z_i\Big(\frac{\partial\bar{z}_l}{\partial\bar{z}_j}\Big)f_{z_k} \\ &\qquad - z_j\Big(\frac{\partial\bar{z}_k}{\partial\bar{z}_i}\Big)f_{z_l} + z_j\Big(\frac{\partial\bar{z}_l}{\partial\bar{z}_i}\Big)f_{z_k}\Big) \\ &= \bar{L}\,(\text{analytic}) = 0 \ .\end{aligned}$$

To prove the converse, we show that the harmonic extension U of u is pluriharmonic. Since U is harmonic, we may write, as before:

$$U(z, \bar{z}) = \sum_{p,q\geqq 0} F_{p,q} \ .$$

By Lemma 1, we have:

$$\bar{L}\bar{L}L(\sum F_{p,q}) = \sum_{p,q\geqq 0} \bar{L}\bar{L}LF_{p,q} = 0 \ .$$

Recall that $\bar{L}\bar{L}L$ takes a polynomial of type (p, q) into one of type $(p + 1, q - 1)$ or zero. Thus $\bar{L}\bar{L}LF_{p,q} = 0$ for each $p, q \geqq 0$.

By Lemma 3, the entries of the matrix $\bar{L}LF_{p,q}$ are analytic. But on the other hand, they must be of type (p, q) or zero. Thus if $q \geqq 1$, we conclude that $\bar{L}LF_{p,q} = 0$.

Again by Lemma 3, the entries of $LF_{p,q}$ are analytic if $q \geqq 1$. But since they will be type $(p - 1, q + 1)$ or zero, we conclude that $LF_{p,q} = 0$ for $q \geqq 1$. This means that $\bar{F}_{p,q} = 0$ is analytic if $q \geqq 1$. Thus if $p, q \geqq 1$, then $F_{p,q} = 0$.

Thus we may write

$$U(z, \bar{z}) = \sum_{j\geqq 1} (F_{j,0} + F_{0,j}) + F_{0,0} \ .$$

Hence U is pluriharmonic.

REMARK. It was observed by L. Nirenberg that there is no second order operator $\mathscr{D}$ which gives the consistency conditions for pluriharmonic functions ∂B^n.

COROLLARY 2. *Let $m \geqq 2$ and $u \in C^\infty(\partial B^n)$ be given. Then u can be extended to U pluriharmonic in B^n if and only if* (5) *or* (6) *holds*:

$$\bar{L}^2(L^2\bar{L}^2)^m Lu = 0 \tag{5}$$

$$(L^2\bar{L}^2)^m Lu = 0 \; . \tag{6}$$

Proof. If u can be extended, then the above equations are clearly valid.

We prove the other implication by induction. Line (5) holds for $m = 0$ (Theorem 1). We assume that (6) is valid for $m = k$ and show that (5) also holds for $m = k$. The other part, showing that (5) is valid for $m = k$ implies (6) valid for $m = k + 1$ is identical. If U is the harmonic extension of u, Lemma 1 applied to (5) yields:

$$\bar{L}^2 L^2(\bar{L}^2 L^2)^{k-1}\bar{L}(\bar{L}LU) = 0 \; .$$

Conjugating, we get:

$$(L^2\bar{L}^2)^k L(L\bar{L}\bar{U}) = 0 \; .$$

Thus the entries of $L\bar{L}\bar{U}$ are pluriharmonic. Thus if we write $U = \sum F_{p,q}$, we have $\bar{L}LF_{p,q} = 0$ for $p, q \geqq 1$, since $\bar{L}L$ preserves type. Thus $LF_{p,q}$ is analytic for $p, q \geqq 1$. Hence $F_{p,q} = 0$ for $p, q \geqq 1$. Hence $F_{p,q} = 0$ for $p, q \geqq 1$.

3. Cauchy-Riemann equations.

LEMMA 4. *If $f \in C^2(\bar{B}^n)$, then $\bar{\mathscr{L}}_{ij}f = 0$ if and only if*

$$\mathscr{L}_{ij}\bar{\mathscr{L}}_{ij}f = 0 \; .$$

Proof. If $\bar{L}f = 0$, then clearly $\mathscr{L}_{ij}\bar{\mathscr{L}}_{ij}f = 0$. To prove the converse, we fix all variables except z_i and z_j and restrict f to

$$C_r = \{|z_i|^2 + |z_j|^2 = r^2\} \; .$$

Let dS_r be the normalized surface area, and integrate by parts:

$$\int_{C_r} \bar{\mathscr{L}}_{ij}f(\overline{\bar{\mathscr{L}}_{ij}f})dS_r = -\int_{C_r} f(\overline{\mathscr{L}_{ij}\bar{\mathscr{L}}_{ij}f}) = 0 \; .$$

Thus $\bar{\mathscr{L}}_{ij}f = 0$ on C_r. Since this must hold for all r, $\bar{\mathscr{L}}_{ij}f = 0$.

REMARK. If $\Omega = \{\rho = 0\}$ is a smooth domain, grad $\rho \neq 0$ on $\partial\Omega$, then we set $\overline{\mathscr{L}}_{ij} = \rho_{\bar{z}_i}(\partial/\partial\bar{z}_j) - \rho_{\bar{z}_j}(\partial/\partial\bar{z}_i)$. The proof above shows that for $f \in C^2(\partial\Omega)$, $\overline{\mathscr{L}}_{ij}f = 0$ on $\partial\Omega$ if and only if $\mathscr{L}_{ij}\overline{\mathscr{L}}_{ij}f = 0$ on $\partial\Omega$.

THEOREM 2. *Let $m \geqq 1$ and $u \in C^m(\partial B^n)$ be given. Then u can be extended to an analytic function on B^n if and only if*:

$$\overline{\mathscr{L}}_{ij}(\mathscr{L}_{ij}\overline{\mathscr{L}}_{ij})^{(m-1)/2}u(\zeta) = 0 \qquad (m \text{ odd}) \tag{7}$$

$$\mathscr{L}(\mathscr{L}_{ij}\overline{\mathscr{L}}_{ij})^{m/2}u(\zeta) = 0 \qquad (m \text{ even}) \tag{8}$$

for all $\zeta \in \partial B^n$ and $1 \leqq i, j \leqq n$.

Proof. In Lemma 4 we have shown that Range $(\mathscr{L}_{ij}) \cap$ Null$(\overline{\mathscr{L}}_{ij}) = 0$. Similarly, Range $(\overline{\mathscr{L}}_{ij}) \cap$ Null $(\mathscr{L}_{ij}) = 0$. Thus equations (7) and (8) will hold if and only if $\overline{\mathscr{L}}_{ij}u = 0$. Since $\bar{L}u$ is the tangential Cauchy-Riemann system, (7) and (8) will hold if and only if u can be extended to an analytic function.

REMARK. The above theorem remains valid for $f \in C^\infty(\partial\Omega)$, as in the remark following Lemma 4.

4. N-Harmonic functions.

DEFINITION. Let Γ be the set of subsets of $\{1, 2, \cdots, n\}$. For $\gamma \in \Gamma$, we say that u is *γ-regular* if $\partial u/\partial\bar{z}_k = 0$ when $k \in \gamma$ and $\partial u/\partial z_k = 0$ when $k \notin \gamma$. We define a new operator $T = (\mathscr{L}_{ij}\overline{\mathscr{L}}_{ij})$. For $\gamma \in \Gamma$, we define T^γ(resp. L^γ) to be T(resp. L) with the variables z_k and $\bar{z}_k$ interchanged whenever $k \notin \gamma$.

The function z_1, for instance, is γ-regular for many γ, but $z_1\bar{z}_1$ is not γ-regular for any γ. Note that every γ-regular function is n-harmonic.

LEMMA 5. *If f is harmonic on B^n, then $T^\gamma f = 0$ if and only if f is γ-regular.*

Proof. We have established in Lemma 4 that $Tg = 0$ if and only if g is analytic. Consider the real linear map $\gamma\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$

$$\gamma(x_1, y_1, \cdots, x_n, y_n) = (\zeta_1, \cdots, \zeta_n)$$

where

$$\zeta_k = x_k + iy_k \qquad \text{if } k \in \gamma$$

$$\zeta_k = x_k - iy_k \qquad \text{if } k \notin \gamma\,.$$

Any γ-regular function f can be obtained from some analytic g by composition:

$$f = g \circ \gamma .$$

Hence $T^\gamma f = Tg = 0$ if and only if f is γ-regular.

THEOREM 3. *A function $u \in C^\infty(\partial B^n)$ can be extended to a function U which is n-harmonic in B^n if and only if*:

$$(\prod_{\gamma \in \Gamma} T^\gamma)u = 0 . \tag{9}$$

(*Since the T^γ's do not commute, the product* (9) *is taken in an arbitrary but fixed order.*)

Proof. We shall show that the harmonic extension U of u is n-harmonic if and only if (9) holds. The function U is n-harmonic if and only if we may write:

$$U = \sum_{\gamma \in \Gamma} u^\gamma \text{ where } u^\gamma \text{ is } \gamma\text{-regular} .$$

The "if" is clear since each u^γ is n-harmonic. The "only if" follows because we may use the Cauchy integral formula in z_1 to write:

$$u(z, \bar{z}) = f(z_1, w) + g(\bar{z}_1, w) \quad w = (z_2, \bar{z}_2, \cdots, z_n, \bar{z}_n)$$

where f and g are n-harmonic. If we continue and split each part in a similar fashion we obtain the desired representation.

Now we show that if f is γ-regular, then so is Tf. We compute:

$$\begin{aligned} \mathscr{L}_{ij}\overline{\mathscr{L}}_{ij} f &= z_i\bar{z}_i f_{z_j\bar{z}_j} - z_i\bar{z}_j f_{z_i\bar{z}_j} \\ &- z_j\bar{z}_i f_{z_j\bar{z}_i} + z_j\bar{z}_j f_{z_i\bar{z}_i} - \bar{z}_j f_{\bar{z}_j} - \bar{z}_i f_{\bar{z}_j} . \end{aligned} \tag{10}$$

In expression (10), f will be multiplied by the variable ξ only if $f_\xi \neq 0$. Thus if f is γ-regular so is Tf.

If we perform the analogous computation for T^σ, we can use the same argument to show that if f is γ-regular then so is $T^\sigma f$.

Now if U is n-harmonic, then $U = \sum_{\sigma \in \Gamma} u^\sigma$; and

$$\begin{aligned} \prod_{\gamma \in \Gamma} T^\gamma u^\sigma &= \prod_{\Gamma_1} T^\gamma T^\sigma \prod_{\Gamma_2} T^\gamma u^\sigma \\ &= 0 . \end{aligned}$$

This is because $\prod T^\gamma u^\sigma$ is σ-regular and will be annihilated by T^σ.

To prove the converse we establish the following result:

LEMMA 6. *Let $v, v_1, \cdots, v_k$ be harmonic. If v_j is γ_j-regular and*

$$T^\gamma v = v_1 + \cdots + v_k , \tag{11}$$

then we may write $v = u + u_1 + \cdots + u_k$ *where* u_j *is* γ_j*-regular, and* u *is* γ*-regular.*

Proof of lemma. Let $u_0 = u_1 + \cdots + u_k$ be the sum of all monomials of v that are γ_j-regular for some $j = 1, 2, \cdots, k$. Thus u_0 is harmonic and so is $v - u_0$. We now claim that $T^\gamma(v - u_0)$ is zero.

By the construction of u_0, every monomial $z^\alpha \bar{z}^\beta$ of $v - u_0$ is not γ_j-regular for any $j = 1, 2, \cdots, k$. From an inspection of (10), one can see that if $T^\gamma(v - u_0)$ is nonzero, then it will be a sum of monomials, none of which is γ_j-regular for any $j = 1, 2, \cdots, k$.

On the other hand, from (11) and the construction of u_0, it is clear that $T^\gamma(v) - T^\gamma u_0$ is a sum of γ_j-regular functions. Hence $T^\gamma(v - u_0)$ must vanish. By Lemma 5, we conclude that $v - u_0 = u$ is γ-regular, concluding the proof of this lemma.

Proof of theorem. We iterate Lemma 6 several times and find that if (8) is valid, then

$$U = \sum_{\gamma \in \Gamma} u^\gamma \, , \quad \text{as desired} \, .$$

COROLLARY 3. *A function* $u \in C^\infty(\partial B^n)$ *can be extended to a function* $U = \sum_{j=1}^k u_j$, *where* u_j *is* γ_j*-regular if and only if*

$$\left(\prod_{j=1}^{k} T^{\gamma_j}\right) u = 0 \, .$$

Proof. This follows easily from Lemma 6.

REMARK. All of the above results remain valid if the boundary differential operators are interpreted in the weak sense of Corollary 1.

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