PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 1, 1974

π-HOMOGENEITY AND π'-CLOSURE OF FINITE GROUPS

ZVI ARAD

The purpose of this paper is to present a proof, under additional conditions, of the following conjecture: Let π be a set of primes, and let all π-subgroups of G be 2-closed. (If $2 \notin \pi$, this condition is satisfied.) If G is π-homogeneous, then G is π'-closed.

All groups considered here are *finite*. If π is a set of prime numbers, we say that the element x of a group G is a *π-element* if $|x|$ is divisible only by primes in π. In particular, one may speak of a *p-element*, p a prime. Similarly, a group G is called a *π-group* if $|G|$ is divisible only by primes in π. In addition, $\pi(G)$ will denote the set of primes dividing $|G|$. The set of primes not in π will be denoted by π'. A group G is termed *π-closed,* if the subset of G consisting of π-elements is a subgroup of G. We say that a group G is *π-homogeneous* if $N_G(H)/C_G(H)$ is a π-group for every nonidentity π-subgroup H of G.

It is well known that π'-closed groups are π-homogeneous. The converse, in general, does not hold. For instance, A_5 is not 5-closed, but it is $5'$-homogeneous.

For $\pi = \{p\}$, p a prime, the conjecture reduces to Frobenius' theorem ([11], Theorem 7.4.5).

The conjecture is closely connected to other well known problems in group theory. The proof of the conjecture would imply the solution of Baer's problem [3] (see also [5], p. 117), the answer to which is not known.

Baer's Problem. Let $\pi \subsetneqq \pi(G)$. Suppose that G is π and π'-homogeneous. Is G a direct product of a π-group and a π'-group?

In order to show the connection with Frobenius' problem, we need some additional notation. For any prime p, we denote by $|G|_p$ the highest power of the prime p that divides $|G|$. Define G to be *weakly π-closed* if for every subgroup U of G the number of π-elements of U is exactly $\prod_{p \in \pi} |U|_p$.

Baer proved that if G is weakly π-closed then G is π'-homogeneous ([2], Lemma 2). Therefore, in the case that $2 \in \pi$, the proof of the above conjecture would imply also a solution of Frobenius' problem ([2], p. 325).

Frobenius' Problem. Let G be a weakly π-closed group. Is G π-closed?

Our first result is that the conjecture holds if $2 \in \pi$.

THEOREM A. *Let π be a set of primes which includes 2. Assume that all π-subgroups of G are 2-closed. Then G is π'-closed if and only if G is π-homogeneous.* (Compare with [2], Satze A, A^*.)

In the next omnibus theorem, $2 \notin \pi$. The proofs of Theorems B and C, as well as the proof of Corollary B, rely on the recent classification of simple $3'$-groups by J. Thompson.

THEOREM B. *Let π be a set of odd primes. Then G is π'-closed if G is π-homogeneous and any one of the following conditions holds:*

(i) $3 \notin \pi(G)$.

(ii) *The π'-subgroups of G are solvable (hence if $N_G(H)$ is π'-closed for every nonidentity π-subgroup of G and the π'-subgroups of G are solvable, then G is π'-closed).*

(iii) *G has dihedral or abelian S_2-subgroups.*

(iv) *Every chain of subgroups has length at most 7.*

A similar result holds if every 3rd maximal subgroup is nilpotent, or if every 2nd maximal subgroup is $2'$-closed.

Theorem B (ii) together with Burnside's $p^\alpha q^\beta$ Theorem yields:

COROLLARY A. *If $|G|$ has exactly 4 prime divisors and π is a set of odd primes, then G is π'-closed if and only if G is π-homogeneous.*

The proof of part (ii) of Theorem B uses the following lemma, which follows from a theorem of Baer ([11], Theorem 3.8.2).

LEMMA 2.6. *If a group G is $2'$-homogeneous then G is 2-closed.*

We shall say that G is a D_π-group if all the maximal π-subgroups of G are conjugate S_π-subgroups of G.

We conjecture that if π is a set of primes, then D_π and π-homogeneity imply π'-closure. (The alternating group A_5, for example, is $5'$-homogeneous, but it is not a $D_{5'}$-group ([12], p. 143) and it is not $5'$-closed.) The following theorem proves this conjecture under additional conditions.

THEOREM C. *If G is a D_π-group and π-homogeneous, then G is π'-closed if one of the following conditions holds:*

(i) $3 \notin \pi(G)$.

(ii) *The proper subgroups of G are π'-closed.*

Theorems A, B, and C imply the following corollary about groups all of whose proper subgroups are π'-closed.

COROLLARY B. *Let π be a set of primes. Let G be a finite group such that every proper subgroup of G is π'-closed, and assume that any one of the following conditions holds:*

(i) $2 \in \pi$ *and the π-subgroups of G are 2-closed.*

(ii) $2 \notin \pi$ *and* $3 \notin \pi(G)$.

(iii) $2 \notin \pi$ *and the π'-subgroups of G are solvable.*

(iv) $2 \notin \pi$ *and G has dihedral or abelian S_2-subgroups.*

(v) $2 \notin \pi$ *and every chain of subgroups has length at most 7.*

(vi) *G is a D_π-group.*

Then G is one of the following:

(a) *G is π'-closed, or*

(b) $\pi = \{p\}$, *p a prime, every proper subgroup of G is nilpotent, $|G| = p^a q^b$, q a prime, the S_q-subgroup of G are cyclic and G is p-closed.* (Compare this corollary with ([14], Chap. (iv), Satz 5.4.)

EXAMPLE. Let $\pi = \{2, 3\}$. Every proper subgroup of the alternating group A_5 is π'-closed. But A_5 is neither π'-closed nor solvable.

These results are part of the author's doctoral research at Tel-Aviv University. The author is extremely grateful to his thesis advisor, Professor M. Herzog, for his guidance and encouragement.

The author is also grateful to Dr. Avinoam Mann and Professor J. Muskat for their constructive remarks.

2. *Proofs.* We incorporate a portion of the proofs of Theorems A and B into independent lemmas.

LEMMA 2.1. *Let G be either* PSL$(2, r^t)$ *or $S_Z(q)$. Let π be a subset of $\pi(G)$ consisting of odd primes and assume $|\pi| \geqq 2$. Then G is not π-homogeneous. Moreover, if P is an S_p-subgroup of* PSL$(2, r^t)$ *where $p \in \pi$ and $p \neq r$, or P is an S_p-subgroup of $S_Z(q)$ where $p \in \pi$ then $2/|N_G(P)/C_G(P)|$.*

Proof. If P is an S_p-subgroup of PSL$(2, r^t)$, where $p \in \pi$ and $p \neq r$, then it is well known that $2/|N_G(P)/C_G(P)|$. Therefore, PSL$(2, r^t)$ is not π-homogeneous.

It follows by Theorem 4, Proposition 16, and Theorem 9 of [17] that in $S_Z(q)$, $2/|N_G(H)/C_G(H)|$ for every nonidentity subgroup H of $S_Z(q)$ of odd order.

The following four basic results concerning π-homogeneous groups were proved in [1].

LEMMA 2.2 ([1], Lemma 2.3). *Subgroups, direct products, and epimorphic images of π-homogeneous groups are π-homogeneous.*

LEMMA 2.3 ([1], Lemma 2.4). *If K is a normal subgroup of the π'-homogeneous group G, and if K and G/K are π-closed, then G is π-closed.*

LEMMA 2.4 ([1], Theorem 2.5). *The group G is π-closed if, and only if, G is π-separable and π'-homogeneous.*

LEMMA 2.5 ([1], Lemma 2.1). *π-closed groups are π'-homogeneous.*

We now obtain at once

LEMMA 2.6. *If a group G is $2'$-homogeneous then G is 2-closed.*

Proof. Let G be a minimal counterexample. Lemmas 2.2 and 2.3 imply that G is a nonabelian simple group. Let K be the conjugate class of an involution u of G; obviously $|K| > 1$. Then by Theorem 3.8.2 of [11] there exists $v \in K$, $v \neq u$, such that uv is not a 2-element. If $|uv| = 2^k m$, $m > 1$ odd, set $t = (uv)^{2^k}$; then $|t| = m > 1$ is odd. Now $t^u = t^{-1}$; therefore, $N_G(\langle t \rangle)/C_G(\langle t \rangle)$ is not a $2'$-group. Hence G is not $2'$-homogeneous, a contradiction.

Proof of Theorem A. If G is π'-closed, then without any assumption on π G is π-homogeneous by Lemma 2.5. Therefore, we will prove here that, under the assumptions of Theorem A, if G is π-homogeneous then G is π'-closed. Let $\pi_1 = \pi \cap \pi(G)$. If $2 \notin \pi(G)$ then Lemma 2.4 and [8] imply that G is π'-closed. If $\pi_1 = \{2\}$ this is Frobenius' theorem. Let G be a minimal counterexample. Then G has the following properties:

(a) G is π_1-homogeneous, $2 \in \pi_1$ and $|\pi_1| \geqq 2$.
(b) The π_1-subgroups of G are 2-closed.
(c) G is not π_1'-closed.

For the remainder of the proof we shall denote π_1 by π. Lemma 2.2 implies that subgroups and epimorphic images of G are π-homogeneous. Clearly π-subgroups of subgroups of G are 2-closed. Therefore we also have:

(d) Proper subgroups of G are π'-closed (hence solvable, by [8]). We want to prove

(e) G is simple.

Suppose not, and let N be a minimal normal subgroup of G. Since by (d) N is solvable, N is a p-group. If $p \in \pi$ and K/N is a π-subgroup of G/N, then K is a π-subgroup of G. Therefore, the π-subgroups of G/N are 2-closed. G/N is π'-closed, by induction. By Lemma 2.3, G is π'-closed, a contradiction. Assume now that $p \notin \pi$. If K/N is a π-subgroup of G/N, then by the Schur-Zassenhaus theorem $K = K_\pi N$ where K_π is an S_π-subgroup of K. Therefore, K/N has a normal S_2-subgroup. By induction G/N, and hence G, are π'-closed, a contradiction. Hence G is simple.

Moreover, by (d) G is a minimal simple group. By [21] G is one of the following:

(1) $PSL_2(2^p)$ where p is any prime.

(2) $PSL_2(3^p)$ where $p > 2$ is any prime.

(3) $PSL_2(p)$ where p is any prime with $p > 3$, and $p \equiv 2$ or 3 (mod 5).

(4) $S_Z(2^p)$ where p is any odd prime.

(5) $PSL_3(3)$.

If G is a group of type (1) or (4), then for $q \in \pi$, q odd ($|\pi| \geqq 2$), there exist Q, a q-subgroup of G, and a 2-element u of G, such that $u \in N_G(Q)$ but $u \notin C_G(Q)$, by Lemma 2.1. Now $T = \langle u \rangle Q$ is a non 2-closed π-group, a contradiction.

If G is $PSL_2(r^t)$ of type (2) or (3) and π contains a prime $u \neq r$, 2, then again Lemma 2.1 yields a contradiction. Hence $\pi = \{2, r\}$. Let R be an S_r-subgroup of G. It is well known that $C_G(R) = R$ and that $|N_G(R)| = 1/2(r^t - 1)|R|$. Since G is π-homogeneous we obtain that $1/2(r^t - 1) = 2^\alpha$ and therefore $N_G(R)$ is a π-subgroup of G. By assumption $N_G(R)$ is 2-closed, a contradiction.

If G is $PSL_3(3)$, then $\pi(G) = \{2, 3, 13\}$. If $\pi = \{2, 13\}$ then ([14], Satz 7.3, p. 187) implies that $3/|N_G(P)/C_G(P)|$, where P is an S_{13}-subgroup of G. Hence G is not π-homogeneous, a contradiction. If G is isomorphic to $PSL_3(3)$ and $\pi = \{2, 3\}$, then a study of the character table of $PSL_3(3)$ implies the existence of a subgroup K of order 54 in $PSL_3(3)$ which is not 2-closed, in contradiction to (b). The proof of Theorem A is now complete.

Before beginning the proof of Theorem B we need several definitions.

A chain of subgroups of G is a set of subgroups of G linearly ordered by inclusion:

$$G = G_0 \supset G_1 \supset \cdots \supset G_k \supset \cdots \supset 1 .$$

The length of a chain is the number of its distinct terms, minus 1.

A subgroup G_k of G is kth maximal if it is the kth term in some chain of proper subgroups, each of which is maximal in its predecessor and k is the smallest such integer.

Proof of Theorem B. Let G be a minimal counterexample.

Proof of (i). Lemmas 2.2 and 2.3 imply that G is simple. By Thompson's classification of simple 3′-groups G isomorphic to $S_Z(q)$. Therefore, Lemma 2.1 implies that G is not π-homogeneous, a contradiction.

Proof of (ii). G has the following properties:
(a) G is π-homogeneous, $2 \notin \pi$ and $|\pi \cap \pi(G)| \geqq 2$.
(b) The π'-subgroups of G are solvable.
(c) G is not π'-closed.

Lemma 2.2 implies that subgroups and epimorphic images of G are π-homogeneous. Clearly subgroups of G have solvable π'-subgroups. Therefore we also have:
(d) Proper subgroups of G are π-closed (hence solvable, by [8]).
We want to prove:
(e) G is simple.
Suppose not, and let N be a minimal normal subgroup of G. Since by (d) N is solvable, N is a p-group. If $p \in \pi'$ and K/N is a π'-subgroup of G/N, then K is a π'-subgroup, so that K is solvable, by hypothesis. Thus K/N is solvable. If $p \in \pi$ and K/N is a π'-subgroup of G/N, then by the Schur-Zassenhaus theorem $K = NK_{\pi'}$ where $K_{\pi'}$ is an $S_{\pi'}$-subgroup of K. By assumption K/N is solvable. Therefore, G/N has solvable π'-subgroups. By induction G/N, and hence G (by Lemma 2.3), are π'-closed, a contradiction. Hence G is simple. Moreover, by (d) G is a minimal simple group. By [21] G is of one of the 5 types mentioned in the proof of Theorem A.
Lemma 2.1 implies that G is not of type (1), (2), (3) or (4). Frobenius' theorem and Lemma 2.6 imply that G is not $\mathrm{PSL}_3(3)$, since $|\mathrm{PSL}_3(3)|$ has only 3 prime divisors, a contradiction.
Now, if $N = N_G(H)$ is π'-closed, for any π-subgroup $H \neq 1$ of G, then $N/C_G(H)$ is a π-group. Hence by the preceding paragraph G is π'-closed.
We now obtain at once

Proof of Corollary A. If $|G|$ has only 4 prime divisors; then Frobenius' theorem, Lemma 2.6, and Theorem B (ii), together with Burnside's $p^\alpha q^\beta$ theorem, yield that G is π'-closed.
We return to the proof of Theorem B.

Proof of (iii). Let G have a dihedral S_2-subgroup. If there exists $1 \neq N \lhd G$, then the S_2-subgroups of N are of one of the following types: dihedral, cyclic or trivial. In the first case N is π'-closed by induction, in the second case N is $2'$-closed and in the third N is solvable by [8]. Lemma 2.4 then implies that in every case N is π'-closed. Similarly G/N is also π'-closed. Therefore, Lemma 2.3 implies that G is π'-closed, a contradiction. Hence G is simple. By Theorem 16.3 of [11] G is isomorphic to either PSL$(2, q)$, q odd, $q > 3$, or to A_7. Lemma 2.1 implies that G is isomorphic to A_7. But $|A_7|$ has only 4 prime divisors, therefore, Corollary A implies that G is π'-closed, a contradiction.

Let G have abelian S_2-subgroups. Clearly G is simple. Walter [18, 19] proved that one of the following holds:

(1) G is isomorphic to $L_2(q)$, $q > 3$, $q \equiv 3, 5 \pmod 8$ or $q = 2^n$;

(2) G is isomorphic to $J(11)$; or

(3) G is of Ree type.

Lemma 2.1 eliminates the first possibility. Now $J(11)$ is of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. If P is an S_p-subgroup of $J(11)$ for $p = 3, 5, 7, 11, 19$, then $2/|N(P)/C(P)|$ by [15]. Hence $J(11)$ is not π-homogeneous, so that G must be of Ree type. Then G is of order $q^3(q-1)(q+1)(q^2-q+1)$ where $q = 3^{2k+1}$, $k \geqq 1$. If $3 \in \pi$ and P is an S_3-subgroup of G, then $N(P) = PW$, where W is cyclic of order $q-1$. Now if J is the involution of W, then $J \notin C(P)$. Hence if $3 \in \pi$ then G is not π-homogeneous. We know also [20] that G possesses Abelian Hall subgroups M^+ and M^- of orders $q+1+3m$ and $q+1-3m$, where $m = 3^k$ and $q^2-q+1 = (q+1+3m)(q+1-3m)$. If t is a prime such that either $t/|M^+|$ or $t/|M^-|$ and T is an S_t-subgroup of $M^{\pm}$, then $N(T) \supseteqq N(M^{\pm}) = M^{\pm}W^{\pm}$, where $W^{\pm}$ are cyclic of order 6. But $C(T) = M^{\pm}$. Hence if $t \in \pi$ then G is not π-homogeneous. Now by the definition of G [20] there exist cyclic subgroups $R^{\pm}$ of order $1/2(q \pm 1)$. The normalizer $N_G(R_0)$ of any subgroup $R_0 \neq 1$ of $R^{\pm}$ is contained in $\langle J \rangle \times L_2(q)$, where J is an involution of G. If R_0 is of odd order then $R_0 \subseteqq L_2(q)$ and $2/|N_G(R_0)/C_G(R_0)|$. Therefore, if π contains of primes dividing either $q+1$ or $q-1$, then G is not π-homogeneous. Since $|G| = q^3(q-1)(q+1)(q^2-q+1)$ where $q = 3^{2k+1}$, $k \geqq 1$, (iii) follows.

Proof of (iv). Lemmas 2.2 and 2.3 imply that G is simple. Gagen's theorem [9] and Harada's theorem [13] imply that G is isomorphic to one of the following groups: PSU$_3(3)$, PSU$_3(5)$, A_7, M_{11}, $J(11)$, or PSL$(2, q)$, for certain values of q. The last possibility is eliminated by Lemma 2.1. In the proof of (iii) we found that $J(11)$ is not π-homogeneous. Since the remaining groups have orders with at most 4 prime divisors, they are π'-closed, by Corollary A and

Lemma 2.6.

Proof of Theorem C. Let G be a minimal counterexample. In both cases Lemmas 2.2, 2.3, and ([14], Chap. (iv), Hilf. 7.2, p. 444) imply that G is simple. Therefore, if (i) $3 \notin \pi(G)$ then, assuming Thompson's classification of simple 3'-groups, G is isomorphic to $S_Z(q)$. If in addition $2 \notin \pi$ then Theorem B implies that G is π'-closed, a contradiction. If $2 \in \pi$ then Theorem 9 of [17] implies that G is not a D_π-group, again a contradiction. In case (ii) Theorem 3.1 of [7] implies that G is π'-closed. This contradiction completes the proof of Theorem C.

It is well known that if every proper subgroup of G is p'-closed but G is not p'-closed, then every proper subgroup of G is nilpotent, $|G| = p^\alpha q^\beta$, q a prime, and the S_q-subgroups of G are cyclic (see [14], Chap. (iv), Satz 5.4, p. 434).

Theorems A, B, and C imply the same conclusion under additional conditions for groups every proper subgroup of which is π'-closed.

Proof of Corollary B. Let G be a minimal counterexample. If G is not π'-closed, then Theorems A, B, and C imply that there exist S, a π-subgroup of G, and x, a π'-element of G, such that $x \in N_G(S)$ but $x \notin C_G(S)$. Therefore, Theorem 6.2.2 of [11] implies that there exists a prime p in π and P, an S_p-subgroup of S, such that $x \in N_G(P)$ but $x \notin C_G(P)$. Set $T = P\langle x\rangle$. If $T \subset G$, then by hypothesis $T = P \times \langle x\rangle$ and $x \in C_G(P)$, a contradiction. If $T = G = P\langle x\rangle$, then every proper subgroup of G is by hypothesis p'-closed, but G itself is not p'-closed. Hence ([14], Chap. (iv), Satz 5.4, p. 434) implies (b).

References

1. R. Baer, *Closure and dispersion of finite groups*, Illinois J. Math., **2** (1958), 619-640.
2. ———, *Kriterien für die abgeschlossenheit endlicher gruppen*, Math. Z., **71** (1959), 325-334.
3. ———, *Direkte product von gruppen teilerfremder ordnung*, Math. Z., **71** (1959), 454-457.
4. Ja. G. Berkovic, *Finite groups with dispersive second maximal subgroups*, Dokl. Akad. Nauk SSSR, **158** (1964), 1007-1009.
5. W. Feit, *Characters of Finite Groups*, W. A. Benjamin, New York, 1967.
6. ———, *The current situation in the theory of finite simple groups*, Actes, Congres intern. Math. Tome, **1** (1970), 55-93.
7. ———, *On a conjecture of Frobenius*, Proc. Amer. Math. Soc., **7** (1956), 177-187.
8. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math., **13** (1963), 775-1029.
9. T. M. Gagen, *A characterization of Janko's simple group*, Proc. Amer. Math. Soc., **19** (1968), 1393-5.
10. T. M. Gagen and Z. Janko, *Finite simple groups with nilpotent third maximal subgroups*, J. Austral. Math. Soc., **6** (1966), 466-469.

11. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
12. M. Hall, *The Theory of Groups*, Macmillan Company, New York 1959.
13. K. Harada, *Finite simple groups with short chains of subgroups*, J. Math. Soc. Japan, **20** (1968), 655-672.
14. B. Huppert, *Endlich Gruppen I*, Springer-Verlag, New York, 1967.
15. Z. Janko, *A new finite simple group with abelian 2-Sylow subgroups and its characterization*, J. of Algebra, **3** (1966), 147-186.
16. ———, *Endlich gruppen mit lauter nilpotenten zweitmaximalen untergruppen*, Math. Z., **79** (1962), 422-424.
17. M. Suzuki, *On a class of doubly transitive groups*, Ann. of Math., **75** (1962), 105-145.
18. J. H. Walter, *Finite groups with abelian Sylow 2-subgroups of order 8*, Invent. Math., **2** (1967), 332-376.
19. ———, *The characterization of finite groups with abelian Sylow 2-subgroups*, Ann. of Math., **89** (1969), 405-514.
20. H. N. Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc., **121** (1966), 62-89.
21. J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc., **74** (1968), 383-437.

Received December 12, 1972 and in revised form August 23, 1973.

TEL-AVIV UNIVERSITY, ISRAEL