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GENERALIZED ω-$\mathscr{L}$-UNIPOTENT BISIMPLE SEMIGROUPS

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Let S be a bisimple semigroup and let $E(S)$ be the set of idempotents of S. If $E(S)$ is an ω-chain of rectangular bands (E_n: $n \in N$, the nonnegative integers) and $\mathscr{L}$, Green's equivalence relation, is a left congruence on $E(S)$, we term S a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup. We characterize S in terms of (I, o), an ω-chain of left zero semigroups (I_k: $k \in N$); $(J, *)$ an ω-chain of right groups (J_k: $k \in N$); a homomorphism $(n, r) \rightarrow \alpha_{(n,r)}$ of C, the bicyclic semigroup, into End (I, o), the semigroup of endomorphisms of (I, o) (iteration); a homomorphism $(n, r) \rightarrow \beta_{(n,r)}$ of C into End $(J, *)$; and an (upper) anti-homomorphism $j \rightarrow A_j$ of $(J, *)$ into T_I, the full transformation semigroup on I (A_j is "almost" an endomorphism). In fact, $S \cong ((i, (n, k), j)$: $i \in I_n$, $j \in J_k$, $n, k \in N)$ under the multiplication $(i, (n, k), j)(u, (r, s), v) = (i \circ (uA_j\alpha_{(k,n)})), (n+r-\min(k, r), k+s-\min(k, r)), j\beta_{(r,s)}*v)$ (Theorem 4.1). We then characterize $(J, *)$ as a semi-direct product of an ω-chain of right zero semigroups by an ω-chain of groups. Finally, we specialize Theorem 4.1 to obtain our previous characterization of ω-$\mathscr{L}$-unipotent bisimple semigroups $S(E(S)$ is an ω-chain of right zero semigroups).

We will use the definitions of Clifford and Preston [1] unless otherwise specified. In particular, $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's equivalence relations on a semigroup S, i.e., $((a, b) \in \mathscr{R}$ if $a \cup aS = b \cup bS$; $(a, b) \in \mathscr{L}$ if $a \cup Sa = b \cup Sb$; $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$; $\mathscr{D} = \mathscr{R} \circ \mathscr{L}((a, b) \in \mathscr{D}$ if there exists $x \in S$ such that $(a, x) \in \mathscr{R}$ and $(x, b) \in \mathscr{L})$. R_a will denote the $\mathscr{R}$-class containing $a \in S$. A semigroup consisting of a single $\mathscr{D}$-class is termed a bisimple semigroup. This bicyclic semigroup is $C = N \times N$ under the multiplication $(n, m)(p, q) = (n + p - \min(m, p), m + q - \min(m, p))$. A semigroup S which is a union of a collection of pairwise disjoint subsemigroups (S_y: $y \in Y$) where Y is a semilattice and $S_yS_t \subseteq S_{y \wedge t}$ for all $y, t \in Y$ is termed a semilattice Y of the semigroups (S_y: $y \in Y$).

If $Y = N$ with $n \wedge m = \max(n, m)$, S is termed an ω-chain of the semigroups (S_n: $n \in N$). A semigroup is termed regular if $a \in aSa$ for every $a \in S$. A rectangular band is the algebraic direct product of a left zero semigroup $U(x, y \in U$ implies $xy = x)$ and a right zero semigroup. A right group is a semigroup X such that $a, b \in X$ implies there exists a unique $x \in S$ such that $ax = b$. If V is a subset of a semigroup S, $E(V)$ will always denote the set of idempotents of V.

In [4], we defined a generalized $\mathscr{L}$-unipotent semigroup to be a regular semigroup S such that $E(S)$ satisfy the condition: $e, f \in E(S)$ and $ef = e$ imply that $gegfe = ge$ for all $g \in E(S)$. Combining [4, Lemma 1] and a result of Clifford and McLean [2, 1, p. 129, Exercise 1], a regular semigroup S is generalized $\mathscr{L}$-unipotent if and only if $E(S)$ is a semilattice Y of rectangular bands $(E_y: y \in Y)$ and $\mathscr{L}$ is a left congruence on $E(S)$. Since any bisimple semigroup containing an idempotent is regular by a result of Clifford and Miller [1, Theorem 2.11], the reason for the terminology "generalized ω-$\mathscr{L}$-unipotent bisimple semigroup" is clear. We introduced the term $\mathscr{L}$-unipotent in [3] to denote a semigroup in which each $\mathscr{L}$-class contains precisely one idempotent. By [3, Proposition 5], a semigroup S is $\mathscr{L}$-unipotent if and only if S is regular and $E(S)$ is a semilattice Y of right zero semigroups $(E_y: y \in Y)$. Hence, the terminology "ω-$\mathscr{L}$-unipotent bisimple semigroup" is also clear.

Let S be a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup. In §1, we define a congruence t on S such that $S/t = C$, the bicyclic semigroup, and give an explicit multiplication for $(E(C))t^{-1}$, the kernel of $t(\ker t)$. In §2, we describe S as an "extension" of $\ker t$ by S/t (the converse of Theorem 4.1). In §3, we prove the direct part of Theorem 4.1. In §4, we state Theorem 4.1 and characterize an ω-chain of right groups as a semi-direct product of an ω-chain of right zero semigroups by an ω-chain of groups (Theorem 4.3). Combining Theorem 4.1, Theorem 4.3, and Clifford's characterization of semilattices of groups [1; theorem 4.11], we have characterized generalized ω-$\mathscr{L}$-unipotent bisimple semigroups in terms of groups, ω-chains of left zero semigroups, ω-chains of right zero semigroups, and 'homomorphisms'. In §5, we obtain our characterization of ω-$\mathscr{L}$-unipotent bisimple semigroups [5, Theorem 7.11] as a corollary of Theorem 4.1.

1. The congruence t.

In this section, S will denote a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup, i.e., S is a bisimple semigroup such that $E(S)$ is an ω-chain of rectangular bands $(E_n: n \in N)$ and $\mathscr{L}$ is a left congruence on $E(S)$. Recall S is a regular semigroup. Thus, for each $a \in S$, there exists $y \in S$ such that $aya = a$ and $yay = y$ (for example, if $a = axa$, let $y = xax$ [1, Lemma 1.14]). The element y is termed an inverse of a. We will denote the set of all inverses of a by $\mathscr{I}(a)$.

Let $t = ((x, y) \in S^2: xx', yy' \in E_n$ and $x'x, y'y \in E_m$ for some $m, n \in N$, $x' \in \mathscr{I}(x)$, and $y' \in \mathscr{I}(y))$. We first show that t is a congruence on S and that $S/t \cong C$, the bicyclic semigroup. We also note that C may be taken as a set of representative elements for the t-classes of S and that $T = \ker t$ (the union of the collection of t-classes of S containing idempotents) is an ω-chain of rectangular groups.

Finally, we describe T in terms of (I, o), an ω-chain of left zero semigroups $(I_n: n \in N)$; $(J, {}^*)$, an ω-chain of right groups $(J_n: n \in N)$; and an anti-homomorphism $j \to A_j$ of J into T_I, the full transformation semigroups on I. In fact, $T \cong U(I_n \times J_n: n \in N)$ under the multiplication $(i, j)(p, q) = (i \circ pA_j, j^*q)$.

LEMMA 1.1. *If* $e_0 \in E_0$, R_{e_0} *is a semigroup.*

Proof. Lemma 1.1 is a special case of [5, Lemma 3.1].

REMARK. Immediately below, we write Theorem 1.2 [5, Theorem 3.3]). This means Theorem 1.2 is obtained by taking "Y" to be a one element set in [5, Theorem 3.3]. $(D_\delta: \delta \in Y)$ is the collection of $\mathscr{D}$-classes in the semigroup of [5, Theorem 3.3]. We do the same thing in Note 1.3, Propositions 1.4 and 1.5, and Lemmas 1.9–1.12.

THEOREM 1.2 [5, Theorem 3.3]. *t is a congruence on S and $S/t \cong C$.*

Note 1.3 [5, Note 3.4]. If we let $t_{(n,k)} = (n, k)t^{-1}$, the t-classes of S are $(t_{(n,k)}: n, k \in N)$ with $t_{(n,k)}t_{(r,s)} \subseteqq t_{(n+r-\min(k,r),k+s-\min(k,r))}$. We may write $E(S) = \bigcup (E_{(k,k)}: k \in N)$ where $E_{(k,k)}$ is a rectangular band and $E(t_{(k,k)}) = E_{(k,k)}$. Actually, $E_{(k,k)} = E_k$.

PROPOSITION 1.4 [5, Proposition 3.5]. $t_{(n,k)} = (a \in S: aa' \in E_{(n,n)}$ *and* $a'a \in E_{(k,k)}$ *for some* $a' \in \mathscr{I}(a)) = \bigcup (R_e \cap L_f: e \in E_{(n,n)}$ *and* $f \in E_{(k,k)})$.

A rectangular group is the algebraic direct product of a group and a rectangular band.

PROPOSITION 1.5 [5, Proposition 3.6]. *For each $k \in N$, $t_{(k,k)}$ is a rectangular group. In fact, $t_{(k,k)} \cong G \times E_{(k,k)}$ where G is a fixed maximal subgroup of S. Furthermore, $t_{(k,k)}t_{(s,s)} \subseteqq t_{(\max(k,s),\max(k,s))}$.*

REMARK 1.6. If $b \in R_e \cap L_f (e, f \in E(S))$, there exists $x \in S$ such that $bx = e$. It is shown in the proof of [1, Theorem 2.18] that $b^{-1} = fxe$ is the unique inverse of b contained in $R_f \cap L_e$ and that $bb^{-1} = e$ and $b^{-1}b = f$.

Note 1.7. Let e_0 be a fixed element of E_0 and fix an element $e_1 \in E_{(1,1)}$ such that $e_1 < e_0$. For example, select any $f \in E_{(1,1)}$ and let $e_1 = e_0 f e_0$. Hence, $e_1 \in E_{(1,1)}$ by Note 1.3 and $e_1 < e_0$.

Note 1.8. Select and fix $a \in R_{e_0} \cap L_{e_1}$. By Remark 1.6, there exists a unique $a^{-1} \in \mathscr{I}(a) \cap R_{e_1} \cap L_{e_0}$ with $aa^{-1} = e_0$ and $a^{-1}a = e_1$. Define

$a^{-n} = (a^{-1})^n$ for all positive integers n and define $a^0 = e_0$. Utilizing Proposition 1.4 and Note 1.3, $a^{-n}a^k \in t_{(n,k)}$ for all $n, k \in N$.

LEMMA 1.9 [5, Lemma 3.9]. *$a^k a^{-k} = e_0$ for all $k \in N$.*

LEMMA 1.10 [5, Lemma 3.10].

$$a^k a^{-r} = \begin{cases} a^{k-r} & \text{if } k > r \\ a^{-(r-k)} & \text{if } r > k \\ e_0 & \text{if } r = k\,. \end{cases}$$

LEMMA 1.11 [5, Lemmas 3.11, 3.12].

(1) $a^{-k}a^p a^{-r}a^s = a^{-(k+r-\min(r,p))}a^{p+s-\min(r,p)}$ (2) $a^{-r}a^r \in E_{(r,r)}$ *for all* $r \in N$.

For brevity, let $T_k = t_{(k,k)}$ and let $T = \bigcup (T_k : k \in N)$. Hence, T is an ω-chain of the rectangular groups $(T_k : k \in N)$ by Proposition 1.5. Since $E(S) = E(T)$ by Note 1.3, T is generalized $\mathscr{L}$-unipotent. Utilizing Proposition 1.5, $T_k = G \times M_k \times N_k$ where G is a group, M_k is a left zero semigroup, and N_k is a right zero semigroup. By Lemma 1.11, $a^{-k}a^k \in E(T_k)$. Let I_k denote the set of idempotents of the $\mathscr{L}$-class of T_k containing $a^{-k}a^k$ and let J_k denote the $\mathscr{R}$-class of T_k containing $a^{-k}a^k$. We may suppose that $l_k \in M_k \cap N_k$, $a^{-k}a^k = (e, l_k, l_k)$ where e is the identity of G, $I_k = (e) \times M_k \times (l_k)$, and $J_k = G \times (l_k) \times N_k$. For brevity, let $e_k = (e, l_k, l_k)$. Hence, using Lemma 1.11, $e_m e_n = e_{\max(n,m)}$.

Let $I = \bigcup (I_n : n \in N)$ and let $J = \bigcup (J_n : n \in N)$.

LEMMA 1.12. *I is an ω-chain of left zero semigroups $(I_n : n \in N)$.*

Proof. By a direct calculation, I_n is a left zero semigroup for each $n \in N$. Let $x \in I_k$ and let $y \in I_n$. Hence, $x \mathscr{L} e_k$ and $y \mathscr{L} e_n$. Since T is generalized $\mathscr{L}$-unipotent, $xy \mathscr{L} xe_n$. Thus, since

$$xe_n \mathscr{L} e_k e_n,\ xy \mathscr{L} e_{\max(k,n)}\ .$$

Hence, $xy \in I_{\max(k,n)}$.

LEMMA 1.13. *For each $n \in N$, J_n is a right group. If $x \in J_n$, $y \in J_m$, and $n \geqq m$, $xy \in J_n$.*

Proof. By [1, Theorem 1.27], J_n is a right group for each $n \in N$. Let $x \in J_n$, $y \in J_m$, and $n \geqq m$. Hence, $y \mathscr{R} e_m$ implies $xy \mathscr{R} xe_m$. Since $e_n(xe_m) = (e_n x)e_m = xe_m$ and $xe_m \in T_n$, $xe_m \in J_n$ by a simple calculation.

Thus, $xy \in J_n$.

LEMMA 1.14. *Every element of T may be uniquely expressed in the form $x = ij$ with $i \in I_n$ and $j \in J_n$ for some $n \in N$.*

Proof. If $x = (g, i, j) \in T_n$, $x = (e, i, l_n)(g, l_n, j)$.

If X is a set, T_X will denote the semigroup (iteration) of mappings of X into X.

LEMMA 1.15. *There exists a mapping $j \to A_j$ of J into T_I and a mapping $p \to B_p$ of I into T_J such that $I_n A_j \subseteqq I_{\max(n,m)}$ for $j \in J_m$ and $J_n B_p \subseteqq J_{\max(n,m)}$ for $p \in I_m$. If $j \in J$ and $p \in I$, $jp = pA_j jB_p$. Furthermore, $jp \mathscr{R} pA_j (\in T)$ and $jp \mathscr{L} jB_p (\in T)$.*

Proof. Let $j \in J_m$ and $p \in I_n$. Thus, $jp \in T_{\max(m,n)}$. Hence, by Lemma 1.14, there exists a unique $u \in I_{\max(m,n)}$ and $v \in J_{\max(m,n)}$ such that $jp = uv$. Let $u = pA_j$ and $v = jB_p$. The last statement is valid by a simple calculation.

LEMMA 1.16. *If $j \in J$, $jB_{e_v} = e_v je_v$. If $j \in J_r$ and $r \geqq v$, $jB_{e_v} = je_v$.*

Proof. Let $j \in J_r$ and suppose that $v > r$. Thus, $je_v \in T_v$ and $(je_v)e_v = je_v$. Hence, $je_v = (g, i, l_v)$ for some $g \in G$ and $i \in M_v$. By Lemma 1.15, $je_v = e_v A_j jB_{e_v}$ with $jB_{e_v} \mathscr{L} je_v (\in T)$. Hence, $jB_{e_v} = (g, l_v, l_v)$. Thus, $jB_{e_v} = (e, l_v, l_v)(g, i, l_v) = e_v je_v$. Next, suppose that $r \geqq v$. Hence, $je_v \in J_r$ by Lemma 1.13. Thus, utilizing Lemma 1.15, $e_r(je_v) = je_v = e_v A_j jB_{e_v}$ where $e_v A_j \in I_r$ and $jB_{e_v} \in J_r$. Hence, $jB_{e_v} = je_v$ by Lemma 1.14. This establishes the second sentance of the lemma. Since, for $r \geqq v$, $e_v j = e_v e_r j = e_r j = j$, $jB_{e_v} = e_v je_v$ for $r \geqq v$.

LEMMA 1.17. *$(e, f) \in \mathscr{L} \cap (E(T))^2$ and $p \in T$ imply $(pe, pf) \in \mathscr{L} (\in T)$.*

Proof. Suppose $(e, f) \in \mathscr{L} \cap (E(T))^2$. Hence, for

$$p \in T,\ (p^{-1}pe, p^{-1}pf) \in \mathscr{L}$$

(p^{-1} is the group inverse of p in the group containing p). Thus, $p^{-1}pep^{-1}pf = p^{-1}pe$ and $p^{-1}pfp^{-1}pe = p^{-1}pf$. Hence, $(pep^{-1})pf = pe$ and $(pfp^{-1})pe = pf$. Thus, $(pe, pf) \in \mathscr{L} (\in T)$.

LEMMA 1.18. *If $p \in I_r$, $B_p = B_{e_r}$.*

Proof. If $n, m \in N$, let $nm = \max(n, m)$ in this proof. Let $j \in J_s$

and $p \in I_r$. Hence $e_{rs}j = (g, l_{rs}, j')$ for some $g \in G$ and $j' \in N_{rs}$ by Lemma 1.13. By Lemma 1.12, $pe_{rs} = (e, n, l_{rs})$ for some $n \in M_{rs}$. Thus,

$$e_{rs}jpe_{rs} = (g, l_{rs}, l_{rs}) = e_{rs}je_re_{rs} .$$

Hence, if $jp = (w, m, n)$ and $je_r = (u, c, d)$, then $w = u$. Since $(p, e_r) \in \mathscr{L}$, $(jp, je_r) \in \mathscr{L}(\in T)$ by Lemma 1.17. Hence, $n = d$. Thus, $jp = (w, m, n) = (e, m, l_{rs})(w, l_{rs}, n)$ while $je_r = (w, c, n) = (e, c, l_{rs})(w, l_{rs}, n)$. Hence, utilizing Lemmas 1.14 and 1.15, $jB_p = jB_{e_r}$.

LEMMA 1.19. *Let $r \in J_u$, $s \in J_v$, $v \leqq u$, and $z \in N$. Then,* (a) $(rs)B_{e_z} = rB_{e_{\max(z,v)}}sB_{e_z}$ (b) *if* $x \in I_z$, $xA_{rs} = xA_sA_r$.

Proof. Let $r \in J_u$, $s \in J_v$, $u \geqq v$, and $x \in I_z$. Hence, utilizing Lemmas 1.13 and 1.15, $(rs)x = xA_{rs}(rs)B_x$ while

$$r(sx) = r(xA_ssB_x) = (r(xA_s))(sB_x) = xA_sA_r(rB_{xA_s}sB_x) .$$

Thus, utilizing Lemma 1.14, $xA_{rs} = xA_sA_r$ and $(rs)B_x = rB_{xA_s}\ sB_x$. Utilizing Lemmas 1.15 and 1.18, $(rs)B_{e_z} = rB_{e_{\max(z,v)}}sB_{e_z}$.

If $x \in J_u$ and $y \in J_v$, define $x^*y = xB_{e_v}y$.

LEMMA 1.20. *If $x \in J_u$ and $y \in J_v$, $x^*y = e_vxy$. If $u \geqq v$, $x^*y = xy$.*

Proof. Let $x \in J_u$ and $y \in J_v$. Hence, utilizing Lemma 1.16,

$$x^*y = xB_{e_v}y = (e_vxe_v)y = e_vx(e_vy) = e_vxy .$$

If $u \geqq v$, again utilizing Lemma 1.16, $x^*y = xB_{e_v}y = (xe_v)y = x(e_vy) = xy$.

LEMMA 1.21. *$(J, {}^*)$ is an ω-chain of right groups $(J_n\colon n \in N)$.*

Proof. Utilizing Lemmas 1.13 and 1.20, $(J_n, {}^*)$ is a right group for each $n \in N$ and $J_n{}^*J_m \subseteqq J_{\max(n,m)}$. We must just establish associativity. Let $i \in J_s$, $p \in J_y$, and $w \in J_z$. Hence, utilizing Lemmas 1.15 and 1.13, $i^*(p^*w) = i^*(pB_{e_z}w) = iB_{e_{\max(y,z)}}pB_{e_z}w$ while

$$(i^*p)^*w = (iB_{e_y}p)^*w = (iB_{e_y}p)B_{e_z}w .$$

Utilizing Lemma 1.19 (a) $(iB_{e_y}p)B_{e_z} = iB_{e_y}B_{e_{\max(y,z)}}pB_{e_z}$. However, utilizing Lemma 1.16,

$$iB_{e_y}B_{e_{\max(y,z)}} = e_{\max(y,z)}e_yie_ye_{\max(y,z)} = e_{\max(y,z)}ie_{\max(y,z)} = iB_{e_{\max(y,z)}}.$$

Hence, $(i^*p)^*w = iB_{e_{\max(y,z)}}pB_{e_z}w = i^*(p^*w)$.

DEFINITION 1.22. Let the semigroup X be an ω-chain of semigroups (X_n: $n \in N$) and let ϕ be a mapping of X into a semi-group Y. If $r \in X_n$, $s \in X_m$, and $n \geqq m$ imply $(rs)\phi = s\phi r\phi$, ϕ is termed an upper anti-homomorphism of X into Y.

LEMMA 1.23. *$r \to A_r$ is an upper anti-homomorphism of $(J, {}^*)$ into T_I.*

Proof. Combine Lemmas 1.20 and 1.19 (b).

LEMMA 1.24. *If $j \in J_v$ and $i \in I_z$, $ji = iA_jje_z = iA_jj^*e_z$.*

Proof. Let $j \in J_v$ and $i \in I_z$. Hence, $ji = iA_jjB_i$ by Lemma 1.15. However, utilizing Lemmas 1.18 and 1.16, $jB_i = jB_{e_z} = e_zje_z$. Since $iA_j \in I_{\max(v,z)}$, $iA_j = iA_je_{\max(v,z)}$. Hence, $ji = iA_je_{\max(v,z)}e_zje_z = iA_jje_z$. However, $e_zje_z = j^*e_z$ by Lemma 1.20. Hence, $ji = iA_jj^*e_z$.

LEMMA 1.25. *If $r, s \in I$ with $r \in I_u$, $(rs)A_x = rA_xsA_{x*e_u}$ for all $x \in J$.*

Proof. Let $r, s \in I$ with $r \in I_u$ and let $x \in J$. Hence, utilizing Lemmas 1.15 and 1.12, $x(rs) = (rs)A_xxB_{rs}$ while

$$(xr)s = (rA_xxB_r)s = rA_x(xB_rs) = rA_x(sA_{xB_r}xB_rB_s) = rA_xsA_{xB_r}xB_rB_s\ .$$

Hence, utilizing Lemmas 1.15, 1.12, and 1.14, $(rs)A_x = rA_xsA_{xB_r}$. Utilizing Lemmas 1.18, 1.16, and 1.20, $xB_r = xB_{e_u} = e_uxe_u = x^*e_u$.

REMARK 1.26. Results of [6] could have been applied to characterize T.

2. Structure theorem for generalized ω-$\mathscr{L}$-unipotent bisimple semigroups. (Proof of converse.) In this section, we complete the proof of the converse of our structure theorem for generalized ω-$\mathscr{L}$-unipotent bisimple semigroups (Theorem 2.21).

We will use a sequence of twenty entries to establish Theorem 2.21. S will denote a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup.

LEMMA 2.1. *Every element of S may be uniquely expressed in the form $x = ia^{-n}a^kj$ where $i \in I_n$ and $j \in J_k$.*

Proof. Let $x \in t_{(n,k)}$. Hence, $(x, e) \in \mathscr{R}$ for some $e \in E_n$ by Proposition 1.4. Thus, $(x, i) \in \mathscr{R}$ for some $i \in I_n$. Thus, since $a^{-n}a^n \in I_n$, a left zero semigroup, $x = ix = (ia^{-n}a^n)x = ia^{-n}a^nx$. Since $a^n \in R_{e_0}$ by Note 1.8 and Lemma 1.1 and $a^ka^{-k} = e_0$ by Lemma 1.9, $a^ka^{-k}a^n =$

a^n. Hence $x = ia^{-n}(a^k a^{-k} a^n)x = (ia^{-n}a^k)(a^{-k}a^n x)$. However, $a^{-k}a^n x \in t_{(k,k)}$ by Notes 1.8 and 1.3. Thus, since $a^{-k}a^k(a^{-k}a^n x) = a^{-k}a^n x$ and $a^{-k}a^k \in J_k$, $a^{-k}a^n x \in J_k$ by Proposition 1.5. Hence, $x = ia^{-n}a^k j$ where $i \in I_n$ and $j \in J_k$. We next establish uniqueness. Suppose that $x = ia^{-n}a^k j = ua^{-r}a^s v (i \in I_n,\ j \in J_k,\ u \in I_r,$ and $v \in J_s)$. Thus, using Note 1.3, $x \in t_{(n,k)} \cap t_{(r,s)}$ and, hence, $n = r$ and $k = s$. Thus, $ia^{-n}a^k j = ua^{-n}a^k v$. Hence, $a^{-n}a^n ia^{-n}a^k j = a^{-n}a^n ua^{-n}a^k v$. Thus, since $a^{-n}a^n$, i, $u \in I_n$, a left zero semigroup, $a^{-n}a^n a^{-n}a^k j = a^{-n}a^n a^{-n}a^k v$. Hence, $a^{-n}a^k j = a^{-n}a^k v$. Thus, $a^{-k}a^n a^{-n}a^k j = a^{-k}a^n a^{-n}a^k v$. Hence, $a^{-k}a^k j = a^{-k}a^k v$. Since $a^{-k}a^k \in E(J_k)$ and j, $v \in J_k$, a right group, $j = v$. Thus, $ia^{-n}a^k j = ua^{-n}a^k j$. Since J_k is a right group, there exists $z \in J_k$ such that $jz = a^{-k}a^k$. Hence $ia^{-n}a^k jz = ua^{-n}a^k jz$ implies $ia^{-n}a^k a^{-k}a^k = ua^{-n}a^k a^{-k}a^k$. Thus, $ia^{-n}a^k = ua^{-n}a^k$. Hence $ia^{-n}a^k a^{-k}a^n = ua^{-n}a^k a^{-k}a^n$. Thus,

$$ia^{-n}a^n = ua^{-n}a^n .$$

Since i, u, $a^{-n}a^n \in I_n$, a left zero semigroup, $i = u$.

DEFINITION 2.2. If $u \in T$ and $n, k \in N$, define $u\nu_{(k,n)} = a^{-n}a^k u a^{-k}a^n$.

LEMMA 2.3. $T_r\nu_{(k,n)} \subseteqq T_{n+r-\min(k,r)}$.

Proof. Let $g \in T_r$. Hence, utilizing Note 1.3, $g\nu_{(k,n)} = a^{-n}a^k g a^{-k}a^n \in t_{(n,k)(r,r)(k,n)} = T_{n+r-\min(k,r)}$.

LEMMA 2.4. *Let $g_r \in T_r$ and $h_s \in T_s$. If $k \geqq r, s$ or $r = s \geqq k$, $(g_r h_s)\nu_{(k,n)} = g_r\nu_{(k,n)} h_s\nu_{(k,n)}$. In particular, $\nu_{(k,n)}$ is a homomorphism of T_r into $T_{n+r-\min(k,r)}$.*

Proof. Let $g_r \in T_r$ and $h_s \in T_s$ with $k \geqq r, s$. Hence,

$$(g_r h_s)\nu_{(k,n)} = a^{-n}a^k g_r h_s a^{-k}a^n = a^{-n}a^k(a^{-k}a^k g_r)u_k a^{-k}a^k f_k(h_s a^{-k}a^k)a^{-k}a^n$$

where $(u_k, a^{-k}a^k g_r) \in \mathscr{L}$ with $u_k \in E(J_k)$ and $(f_k, h_s a^{-k}a^k) \in \mathscr{R}$ with $f_k \in I_k$. Hence, $(g_r h_s)\nu_{(k,n)} = a^{-n}a^k g_r a^{-k}a^k h_s a^{-k}a^n = (a^{-n}a^k g_r a^{-k}a^n)(a^{-n}a^k h_s a^{-k}a^n) = g_r\nu_{(k,n)} h_s\nu_{(k,n)}$. Next suppose that $r = s \geqq k$. Then,

$$(g_r h_r)\nu_{(k,n)} = a^{-n}a^k g_r v_r a^{-k}a^k f_r h_r a^{-k}a^n$$

where $(v_r, g_r) \in \mathscr{L}$ with $v_r \in E(J_r)$ and $(f_r, h_r) \in \mathscr{R}$ with $f_r \in I_r$. Hence, $(g_r h_r)\nu_{(k,n)} = (a^{-n}a^k g_r a^{-k}a^n)(a^{-n}a^k h_r a^{-k}a^n) = g_r\nu_{(k,n)} h_r\nu_{(k,n)}$.

DEFINITION 2.5. Let $\nu_{(k,n)} | I = \alpha_{(k,n)}$ and $\nu_{(k,n)} | J = \beta_{(k,n)}$.

LEMMA 2.6. (a) $I_r\alpha_{(k,n)} \subseteqq I_{n+r-\min(k,r)}$ (b) $J_r\beta_{(k,n)} \subseteqq J_{n+r-\min(k,r)}$.

Proof. (a) By Lemma 2.3, $I_r\nu_{(k,n)} \subseteqq T_n$ if $k \geqq r$ and $I_r\nu_{(k,n)} \subseteqq T_{n+r-k}$

if $r \geqq k$. If $k \geqq r$, $\nu_{(k,n)}$ is a homomorphism of T_r into T_n by Lemma 2.4. Hence, $I_r\nu_{(k,n)} \subseteqq E(T_n)$. Let $g_r \in I_r$. Hence, $g_r \mathscr{L} a^{-r}a^r (\in T_r)$. Thus, $g_r\nu_{(k,n)} \mathscr{L} a^{-n}a^k a^{-r}a^r a^{-k}a^n (\in T_n)$. However,

$$a^{-n}a^k a^{-r}a^r a^{-k}a^n = a^{-n}a^n$$

by Lemma 1.11. Hence, $g_r\nu_{(k,n)} \in I_n$ if $k \geqq r$. The case $r \geqq k$ is treated similarly. To prove (b), just replace "I" by "J" and "$\mathscr{L}$" by "$\mathscr{R}$" in the proof of (a).

DEFINITION 2.7. If X is a semigroup End X will denote the semigroup of endomorphisms of X (iteration).

LEMMA 2.8. *$\alpha_{(k,n)} \in \operatorname{End} I$ for each $n, k \in N$.*

Proof. Let $i_r \in I_r$ and $i_s \in I_s$. If $r \geqq k$, $i_r a^{-k}a^k i_s = i_r i_s$. Hence,

$$\begin{aligned}(i_r i_s)\alpha_{(k,n)} &= a^{-n}a^k i_r i_s a^{-k}a^n = a^{-n}a^k i_r a^{-k}a^k i_s a^{-k}a^n \\ &= a^{-n}a^k i_r a^{-k}a^n a^{-n}a^k i_s a^{-k}a^n = i_r\alpha_{(k,n)} i_s\alpha_{(k,n)} .\end{aligned}$$

Next, suppose that $k > r$. Since S is generalized $\mathscr{L}$-unipotent, $i_r \mathscr{L} a^{-r}a^r$ implies $a^{-k}a^k i_r \mathscr{L} a^{-k}a^k a^{-r}a^r$. Thus, $a^{-k}a^k i_r \mathscr{L} a^{-k}a^k$ by Lemma 1.11. Hence, $a^{-k}a^k i_r \in I_k$. Thus,

$$\begin{aligned}(i_r i_s)\alpha_{(k,n)} &= a^{-n}a^k i_r i_s a^{-k}a^n = a^{-n}a^k (a^{-k}a^k i_r) a^{-k}a^k i_s a^{-k}a^n \\ &= (a^{-n}a^k i_r a^{-k}a^n)(a^{-n}a^k i_s a^{-k}a^n) = i_r\alpha_{(k,n)} i_s\alpha_{(k,n)} .\end{aligned}$$

LEMMA 2.9. *$(n, k) \to \alpha_{(n,k)}$ is a homomorphism of C into* End I.

Proof. Let $g \in I$. We will employ Lemma 1.11. Thus,

$$\begin{aligned}g\alpha_{(r,s)}\alpha_{(n,p)} &= a^{-p}a^n a^{-s}a^r g a^{-r}a^s a^{-n}a^p \\ &= a^{-(p+s-\min(n,s))}a^{n+r-\min(n,s)} g a^{-(r+n-\min(n,s))}a^{s+p-\min(n,s)} \\ &= g\alpha_{(r,s)(n,p)} .\end{aligned}$$

We next establish that $\beta_{(n,k)} \in \operatorname{End}(J, *)$. This will be accomplished by Lemmas 2.10–2.15.

LEMMA 2.10. *$\beta_{(1,0)} \in \operatorname{End}(J, *)$.*

Proof. Let $w \in J_p$ and $u_s \in J_s$. If $p = s$, $\beta_{(1,0)} \in \operatorname{End}(J, *)$ by Lemmas 2.4, 1.20, and 2.6(b), and Definition 2.5. Let us first suppose $s = 0$. Utilizing Lemmas 1.13, 1.11, Note 1.8, and Definition 2.5,

$$\begin{aligned}(wu_0)a^{-1} &= a^{-p}a^p a^{-1}a(wu_0)a^{-1} = a^{-p}a^p a^{-1}(wu_0)\beta_{(1,0)} \\ &= a^{-p}a^p a^{-1}a^0(wu_0)\beta_{(1,0)} = e_p a^{-p}a^{p-1}(wu_0)\beta_{(1,0)} .\end{aligned}$$

We note that $(wu_0)\beta_{(1,0)} \in J_{p-1}$ by Lemma 2.6(b). Utilizing Note 1.8, Lemmas 1.15, 1.16, and Definition 2.5, $u_0a^{-1} = u_0e_1a^{-1} = e_1A_{u_0}e_1u_0e_1a^{-1} = e_1A_{u_0}a^{-1}au_0a^{-1} = e_1A_{u_0}a^{-1}(u_0\beta_{(1,0)})$. Hence, utilizing Lemmas 1.15, 1.23, 1.18, 1.16, and 1.11, and Definition 2.5,

$$\begin{aligned} wu_0a^{-1} &= w(e_1A_{u_0})a^{-1}(u_0\beta_{(1,0)}) = e_1A_{wu_0}e_1we_1a^{-1}(u_0\beta_{(1,0)}) \\ &= e_1A_{wu_0}a^{-1}(awa^{-1}(u_0\beta_{(1,0)})) = e_1A_{wu_0}a^{-1}(w\beta_{(1,0)})u_0\beta_{(1,0)} \\ &= e_1A_{wu_0}a^{-p}a^pa^{-1}a^0(w\beta_{(1,0)})u_0\beta_{(1,0)} \\ &= e_1A_{wu_0}a^{-p}a^{p-1}(w\beta_{(1,0)})u_0\beta_{(1,0)} \,. \end{aligned}$$

Utilizing Lemmas 1.15, 2.6(b), and 1.13, $e_1A_{wu_0} \in I_p$ and $w\beta_{(1,0)}u_0\beta_{(1,0)} \in J_{p-1}$. Hence, $(wu_0)\beta_{(1,0)} = w\beta_{(1,0)}u_0\beta_{(1,0)}$ by Lemma 2.1. Thus, utilizing Lemma 1.20 and 2.6(b), $(w^*u_0)\beta_{(1,0)} = w\beta_{(1,0)}{}^*u_0\beta_{(1,0)}$. Next, we assume that $p \geqq s \geqq 1$. Hence, utilizing Lemmas 1.11, 2.6(b), and 1.20,

$$\begin{aligned} (w^*u_s)\beta_{(1,0)} &= (wu_s)\beta_{(1,0)} = awu_sa^{-1} = awa^{-s}a^su_sa^{-1} \\ &= awa^{-1}aa^{-s}a^su_sa^{-1} = (awa^{-1})(au_sa^{-1}) \\ &= w\beta_{(1,0)}u_s\beta_{(1,0)} = w\beta_{(1,0)}{}^*u_s\beta_{(1,0)} \,. \end{aligned}$$

Finally, we assume $s > p$. Utilizing Lemmas 1.20, 1.13, 1.10 and the case $(p \geqq s)$ just established,

$$\begin{aligned} (w^*u_s)\beta_{(1,0)} &= ((e_sw)u_s)\beta_{(1,0)} = (e_sw)\beta_{(1,0)}u_s\beta_{(1,0)} = e_s\beta_{(1,0)}w\beta_{(1,0)}u_s\beta_{(1,0)} \\ &= aa^{-s}a^sa^{-1}w\beta_{(1,0)}u_s\beta_{(1,0)} = e_{s-1}w\beta_{(1,0)}u_s\beta_{(1,0)} \,. \end{aligned}$$

Since $u_s\beta_{(1,0)} \in J_{s-1}$ by Lemma 2.6(b), $e_{s-1}w\beta_{(1,0)}u_s\beta_{(1,0)} = w\beta^*_{(1,0)}u_s\beta_{(1,0)}$ by Lemma 1.20. Hence, $(w^*u_s)\beta_{(1,0)} = w\beta^*_{(1,0)}u_s\beta_{(1,0)}$.

LEMMA 2.11. $\beta_{(0,0)} \in \operatorname{End}(J, *)$.

Proof. Let $u_r \in J_r$ and $v_s \in J_s$. Utilizing Lemmas 1.20 and 2.6(b),

$$\begin{aligned} (u_r^*v_s)\beta_{(0,0)} &= (e_su_rv_s)\beta_{(0,0)} = e_0e_su_rv_se_0 = e_se_0u_re_0e_sv_se_0 = e_se_0u_re_0e_0v_se_0 \\ &= e_s(u_r\beta_{(0,0)})v_s\beta_{(0,0)} = u_r\beta_{(0,0)}{}^*v_s\beta_{(0,0)} \,. \end{aligned}$$

LEMMA 2.12. *$(n, k) \to \beta_{(n,k)}$ is a homomorphism of C into T_J.*

Proof. Replace "I" by "J" and "α" by "β" in the proof of Lemma 2.9.

LEMMA 2.13. $\beta_{(k,0)} \in \operatorname{End}(J, *)$ *for all $k \in N$.*

Proof. We have shown that $\beta_{(0,0)}$ in $\operatorname{End}(J, *)$(Lemma 2.11) and that $\beta_{(1,0)} \in \operatorname{End}(J, *)$ (Lemma 2.10). Suppose that $\beta_{(n,0)} \in \operatorname{End}(J, *)$.

We show that $\beta_{(n+1,0)} \in \mathrm{End}(J, *)$. Let $g, h \in J$. Hence, utilizing Lemma 2.12,

$$\begin{aligned}(g^*h)\beta_{(n+1,0)} &= (g^*h)\beta_{(n,0)}\beta_{(1,0)} = (g\beta_{(n,0)}{}^*h\beta_{(n,0)})\beta_{(1,0)} \\ &= g\beta_{(n,0)}\beta_{(1,0)}{}^*h\beta_{(n,0)}\beta_{(1,0)} = g\beta_{(n+1,0)}{}^*h\beta_{(n+1,0)} .\end{aligned}$$

LEMMA 2.14. *$\beta_{(0,k)} \in \mathrm{End}(J, *)$ for all $k \in N$.*

Proof. Let $u_r \in J_r$ and $v_s \in J_s$. First, assume $s \geqq r$. Utilizing Lemma 1.20, $(u_r{}^*v_s)\beta_{(0,k)} = (e_s u_r v_s)\beta_{(0,k)}$. Since $u_r \mathscr{R} e_r$, $e_s u_r \mathscr{R} e_s e_r = e_s$. Hence, utilizing Lemma 2.4, $((e_s u_r)v_s)\beta_{(0,k)} = (e_s u_r)\beta_{(0,k)} v_s \beta_{(0,k)}$. Utilizing Definition 2.5, Note 1.8, Lemma 1.1, and Lemma 1.9, $(e_s u_r)\beta_{(0,k)} = a^{-k}e_0(e_s u_r)e_0 a^k = a^{-k}e_s u_r a^k = a^{-k}e_s a^k a^{-k} u_r a^k = (a^{-k}a^{-s}a^s a^k)(a^{-k}a^0 u_r a^{-0}a^k) = e_{s+k}(u_r\beta_{(0,k)})$. Since $v_s\beta_{(0,k)} \in J_{s+k}$ by Lemma 2.6(b), $e_{s+k}u_r\beta_{(0,k)}v_s\beta_{(0,k)} = u_r\beta_{(0,k)}{}^*v_s\beta_{(0,k)}$ by Lemma 1.20. Thus, $(u_r{}^*v_s)\beta_{(0,k)} = u_r\beta_{(0,k)}{}^*v_s\beta_{(0,k)}$. We utilize Lemmas 1.20 and 1.9, and Definition 2.5 for the case $r > s$.

LEMMA 2.15. *$\beta_{(n,k)} \in \mathrm{End}(J, *)$ for all $n, k \in N$.*

Proof. Let $g, h \in J$. Hence, utilizing Lemmas 2.12, 2.13, and 2.14,

$$\begin{aligned}(g^*h)\beta_{(n,k)} &= (g^*h)\beta_{(n,0)(0,k)} = (g^*h)\beta_{(n,0)}\beta_{(0,k)} = (g\beta_{(n,0)}{}^*h\beta_{(n,0)})\beta_{(0,k)} \\ &= g\beta_{(n,0)}\beta_{(0,k)}{}^*h\beta_{(n,0)}\beta_{(0,k)} = g\beta_{(n,k)}{}^*h\beta_{(n,k)} .\end{aligned}$$

LEMMA 2.16. *$(n, k) \rightarrow \beta_{(n,k)}$ is a homomorphism of C into* $\mathrm{End}(J, *)$.

Proof. Combine Lemmas 2.12 and 2.15.

If $a, b \in I$, define $a \circ b = ab$.

LEMMA 2.17. *$S \cong ((i, (n, k), j)$: $i \in I_n$, $j \in J_k$, $n, k \in N)$ under the multiplication $(i, (n, k), j)(u, (r, s), v) = (i \circ (uA_j\alpha_{(k,n)}), (n + r - \min(k, r), k + s - \min(k, r)), j\beta_{(r,s)}{}^*v)$.*

Proof. Let $i \in I_n$, $j \in J_k$, $u \in I_r$, and $v \in J_s$. Hence, utilizing Lemmas 1.24, 1.15, 1.11, 1.9, 2.6(b), 1.20, and Definition 2.5,

$$\begin{aligned}(ia^{-n}a^k j)(ua^{-r}a^s v) &= ia^{-n}a^k(ju)a^{-r}a^s v = ia^{-n}a^k uA_j j a^{-r}a^{-r}a^s v \\ &= ia^{-n}a^k uA_j a^{-k}a^k a^{-r}a^r j a^{-r}a^s v \\ &= i(a^{-n}a^k(uA_j)a^{-k}a^n)(a^{-n}a^k a^{-r}a^s)a^{-s}a^r j a^{-r}a^s v \\ &= i((uA_j)\alpha_{(k,n)})a^{-(n+r-\min(k,r))}a^{k+s-\min(k,r)}j\beta_{(r,s)}v \\ &= i\circ((uA_j)\alpha_{(k,n)})a^{-(n+r-\min(k,r))}a^{k+s-\min(k,r)}(j\beta^*_{(r,s)}v) .\end{aligned}$$

Utilizing Lemma 2.6, $i \circ ((uA_j)\alpha_{(k,n)}) \in I_{n+r-\min(k,r)}$ and

$$j\beta_{(r,s)}{}^*v \in J_{k+s-\min(k,r)} .$$

Hence, $((i, (n, k), j): i \in I_n, j \in J_k, n, k \in N)$ under the multiplication given in the statement of the lemma is a groupoid. The required isomorphism is given by the mapping $(ia^{-n}a^k j)\varphi = (i, (n, k), j)$ by virtue of the above and Lemma 2.1.

LEMMA 2.18. *$\alpha_{(r,s)}A_j\alpha_{(s,r)} = A_{j\beta(s,r)}$ for all $j \in J$ and $r, s \in N$.*

Proof. Let $j \in J_p$ and $w \in I_q$. Utilizing Definitions 2.2 and 2.5, and Lemmas 1.9, 1.12, 1.15, and 2.6,

$$\begin{aligned}(j(w\alpha_{(r,s)}))\nu_{(s,r)} &= a^{-r}a^s j(a^{-s}a^r w a^{-r}a^s)a^{-s}a^r \\ &= a^{-r}a^s j a^{-s}a^r(a^{-r}a^r w)a^{-r}a^r = a^{-r}a^s j a^{-s}a^r a^{-r}a^r w \\ &= a^{-r}a^s j a^{-s}a^r w = j\beta_{(s,r)}w = wA_{j\beta_{(s,r)}}j\beta_{(s,r)}B_w\,.\end{aligned}$$

Utilizing Lemmas 1.15 and 2.6, $wA_{j\beta(s,r)} \in I_{\max(q, r+p-\min(s,p))}$ and

$$j\beta_{(s,r)}B_w \in J_{\max(q, r+p-\min(s,p))}\,.$$

Utilizing Definitions 2.2 and 2.5, and Lemmas 1.15 and 2.6,

$$\begin{aligned}(j(w\alpha_{(r,s)}))\nu_{(s,r)} &= a^{-r}a^s j(w\alpha_{(r,s)})a^{-s}a^r \\ &= a^{-r}a^s(w\alpha_{(r,s)}A_j)(jB_{w\alpha_{(r,s)}})a^{-s}a^r \\ &= a^{-r}a^s(w\alpha_{(r,s)}A_j)a^{-s}a^s(jB_{w\alpha_{(r,s)}})a^{-s}a^r \\ &= a^{-r}a^s(w\alpha_{(r,s)}A_j)a^{-s}a^r(a^{-r}a^s jB_{w\alpha_{(r,s)}}a^{-s}a^r) \\ &= (w\alpha_{(r,s)}A_j)\alpha_{(s,r)}(jB_{w\alpha_{(r,s)}})\beta_{(s,r)}\,.\end{aligned}$$

Utilizing Lemmas 1.15 and 2.6, $w\alpha_{(r,s)}A_j\alpha_{(s,r)} \in I_{\max(p, s+q-\min(r,q))+r-s}$ and $(jB_{w\alpha_{(r,s)}})\beta_{(s,r)} \in J_{\max(p, s+q-\min(r,q))+r-s}$. Hence, $w\alpha_{(r,s)}A_j\alpha_{(s,r)} = wA_{j\beta(s,r)}$ by Lemma 1.14.

LEMMA 2.19. (a) *$g\alpha_{(s,s)} = e_s \circ g$ for all $g \in I$.* (b) *$g\beta_{(s,s)} = g^* e_s$ for all $g \in J$.*

Proof. (a) Let $g \in I$. Utilizing Lemma 1.12 $g\alpha_{(s,s)} = (e_s g)e_s = e_s g = e_s \circ g$. (b) Let $g \in J$. Utilizing Lemma 1.20, $g\beta_{(s,s)} = e_s g e_s = g^* e_s$.

In the following definition, we will describe the objects we will use to represent generallized ω-$\mathscr{L}$-unipotent bisimple semigroups.

DEFINITION 2.20. Let (I, o) be an ω-chain of left zero semigroups $(I_k: k \in N)$; let $(n, r) \rightarrow \alpha_{(n,r)}$ be a homomorphism of C into End (I, o); let $(J, {}^*)$ be an ω-chain of right groups $(J_k: k \in N)$; let $(n, r) \rightarrow \beta_{(n,r)}$ be a homomorphism of C into End $(J, {}^*)$; let $j \rightarrow A_j$ be an upper antihomomorphism of $(J, {}^*)$ into T_I; and let $I_k \cap J_k = (e_k)$, a single idempotent, for each $k \in N$ such that

(1) $g\beta_{(s,s)} = g^* e_s$ for all $g \in J$.
(2) $I_r \alpha_{(n,k)} \subseteqq I_{k+r-\min(n,r)}$ and $J_r \beta_{(n,k)} \subseteqq J_{k+r-\min(n,r)}$.
(3) $I_r A_j \subseteqq I_{\max(r,k)}$ if $j \in J_k$.
(4) $(r \circ s)A_x = rA_x \circ sA_{x*e_u}$ for $r, s \in I$ with $r \in I_u$ and $x \in J$.
(5) $\alpha_{(r,s)} A_j \alpha_{(s,r)} = A_{j\beta(s,r)}$ for all $j \in J$ and $r, s \in N$.

We denote $((i, (n, k), j): i \in I_n, j \in J_k)$ under the multiplication

(6) $(i, (n, k), j)(u, (r, s), v)$
$= (i \circ (uA_j \alpha_{(k,n)}), (n + r - \min(k, r), k + s - \min(k, r)), j\beta_{(r,s)}{}^* v)$

by (I, J, α, β, A).

THEOREM 2.21. *Let S be a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup. Then, S is isomorphic to some (I, J, α, β, A).*

Proof. The theorem is a consequence of the definition of "o", Lemmas 1.12, 2.9, 1.21, 2.16, 1.23, the choice of "e_k", Lemmas 2.19, 2.6, 1.15, 1.25, 2.18, and 2.17.

We thank the referee for the following remark.

REMARK 2.22. In Definition 2.20, the middle component (m, n) of $(i, (m, n), j)$ serves only as a marker. Hence, S is actually represented by the cartesian product $I \times J$ under the multiplication

$$(i, j)(u, v) = (i \circ (uA_j \alpha_{(k,n)}), j\beta_{(r,s)}{}^* v)$$

where $i \in I_n$, $j \in J_k$, $u \in I_r$, and $v \in J_s$.

3. Structure theorem for generalized ω-$\mathscr{L}$-unipotent bisimple semigroups (proof of direct half).

In this section, we show that (I, J, α, β, A) is a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup.

LEMMA 3.1. *(I, J, α, β, A) is a semigroup.*

Proof. We use (2) and (3) of Definition 2.20 to establish closure. We next establish associativity. Let $(i, (n, k), j)_1 = i$ and $(i, (n, k), j)_{23} = ((n, k), j)$. Let $a = (i, (n, k), j)$, $b = (u, (r, s), v)$, and $c = (z, (p, q), w) \in (I, J, \alpha, \beta, A)$. Utilizing the fact that $(n, r) \rightarrow \beta_{(n,r)}$ is a homomorphism,

$$\begin{aligned}((ab)c)_{23} &= ((i \circ (uA_j \alpha_{(k,n)}), (n, k)(r, s), j\beta_{(r,s)}{}^* v)(z, (p, q), w))_{23} \\ &= ((n, k)(r, s)(p, q), (j\beta_{(r,s)}{}^* v)\beta_{(p,q)}{}^* w) \\ &= ((n, k)(r, s)(p, q), j\beta_{(r,s)(p,q)}{}^* v\beta_{(p,q)}{}^* w)\end{aligned}$$

while

$$(a(bc))_{23} = ((i, (n, k), j)(u \circ (zA_v\alpha_{(s,r)}), (r, s)(p, q), v\beta_{(p,q)}{}^*w))_{23}$$
$$= ((n, k)(r, s)(p, q), j\beta^*_{(r,s)(p,q)}{}^*v\beta_{(p,q)}{}^*w) .$$

Hence, $((ab)c)_{23} = (a(bc))_{23}$. Utilizing the fact $(k, n) \rightarrow \alpha_{(k,n)}$ is a homomorphism of C into End (I, o), the fact $j \rightarrow A_j$ is an upper anti-homomorphism of $(J, ^*)$ into T_I, (5), (1), and (4),

$$\begin{aligned}((ab)c)_1 &= i \circ uA_j\alpha_{(k,n)} \circ zA_{j\beta_{(r,s)}*v}\alpha_{(s,r)(k,n)} \\ &= i \circ ((uA_j \circ zA_{j\beta_{(r,s)}*v}\alpha_{(s,r)})\alpha_{(k,n)}) \\ &= i \circ ((uA_j \circ zA_vA_{j\beta_{(r,s)}}\alpha_{(s,r)})\alpha_{(k,n)}) \\ &= i \circ ((uA_j \circ zA_v\alpha_{(s,r)}A_j\alpha_{(r,s)}\alpha_{(s,r)})\alpha_{(k,n)}) \\ &= i \circ ((uA_j \circ zA_v\alpha_{(s,r)}\alpha_{(r,r)}A_j\alpha_{(r,r)})\alpha_{(k,n)}) \\ &= i \circ ((uA_j \circ zA_v\alpha_{(s,r)}A_{j\beta_{(r,r)}})\alpha_{(k,n)}) \\ &= i \circ ((uA_j \circ zA_v\alpha_{(s,r)}A_{j*e_r})\alpha_{(k,n)}) \\ &= i \circ ((u \circ zA_v\alpha_{(s,r)})A_j\alpha_{(k,n)}) \\ &= (a(bc))_1 .\end{aligned}$$

Hence, $(ab)c = a(bc)$.

LEMMA 3.2. *Let* $(i, (n, k), j), (w, (p, q), z) \in (I, J, \alpha, \beta, A)$.

(a) $(i, (n, k), j)\mathscr{R}(w, (p, q), z)$ *if and only if* $i = w$ *and* $n = p$.

(b) $(i, (n, k), j)\mathscr{L}(w, (p, q), z)$ *if and only if* $k = q$ *and* $(j, z) \in \mathscr{H}(\in J_q)$.

Proof. (a) Let us show that $(i, (n, k), j)\mathscr{R}(i, (n, q), z)$. Let $u \in I_k$. Hence, $uA_j\alpha_{(k,n)} \in I_n$ by (2) and (3). Thus, since (I_n, o) is a left zero semigroup, $i \circ uA_j\alpha_{(k,n)} = i$. By (2), $j\beta_{(k,q)} \in J_q$. Hence, since $(J_q, ^*)$ is a right group, there exists $v \in J_q$ such that $j\beta_{(k,q)}{}^*v = z$. Hence, utilizing (6), $(i, (n, k), j)(u, (k, q), v) = (i, (n, q), z)$. Similarly, there exists $a \in I_q$ and $b \in J_k$ such that $(i, (n, q), z)(a, (q, k), b) = (i, (n, k), j)$. Utilizing (6), the converse follows from the fact that $\mathscr{R}$ is the dentity on $(I, 0)$ and $(n, k)\mathscr{R}(p, q)$ in C implies $n = p$. Let us show that $(i, (n, k), j)\mathscr{L}(w, (p, k), z)$ if $(j, z) \in \mathscr{H}(\in J_k)$. Since $(j, z) \in \mathscr{H}(\in J_k)$, there exists $u \in J_k$ such that $u^*j = z$. By (2), $u\beta_{(k,n)} \in J_n$. Utilizing (1) and the fact $(n, k) \rightarrow \beta_{(n,k)}$ is a homomorphism of C into End $(J, ^*)$, $u\beta_{(k,n)}\beta_{(n.k)} = u\beta_{(k,k)} = u^*e_k$. Hence, $(u\beta_{(k,n)})\beta_{(n,k)}{}^*j = u^*e_k^*j = u^*j = z$. Thus, utilizing (2), (3), and (6), $(w, (p, n), u\beta_{(k,n)})(i, (n, k), j) = (w, (p, k), z)$. Similarly, there exists $v \in J_k$ such that $(i, (n, p), v\beta_{(k,p)})(w, (p, k), z) = (i, (n, k), j)$. Utilizing (6), the converse follows from the fact that $\mathscr{H} = \mathscr{L}$ in $(J, ^*)$ and $(n, k)\mathscr{L}(p, q)$ in C implies $k = q$.

LEMMA 3.3. *(I, J, α, β, A) is a bisimple semigroup.*

Proof. Let $(i, (n, k), j), (u, (r, s), v) \in (I, J, \alpha, \beta, A)$. Hence, utilizing Lemma 3.2, $(i, (n, k), j) \mathscr{R}(i, (n, s), v)\mathscr{L}(u, (r, s), v)$. (I, J, α, β, A) is a semigroup by Lemma 3.1.

LEMMA 3.4. $E(I, J, \alpha, \beta, A) = ((i, (n, n), j): j \in E(J_n), n \in N)$.

Proof. Let $(i, (n, k), j) \in E(I, J, \alpha, \beta, A)$. Hence, $(i, (n, k), j)(i, (n, k), j) = (i, (n, k), j)$. Using (6), $n = k$ since $(n, k)^2 = (n, k)$ in C. Hence, using (6) and (1), $j = j\beta_{(n,n)}{}^*j = j^*e_n^*j = j^2$. Utilizing (6), (2), (3), and (1), $j \in E(J_n)$ implies $(i, (n, n), j) \in E(I, J, \alpha, \beta, A)$ for $n \in N$ and $i \in I_n$.

LEMMA 3.5. *(I, J, α, β, A) is a regular bisimple semigroup.*

Proof. It follows from a result of Clifford and Miller [1, Theorem 2.11] that any bisimple semigroup containing an idempotent is regular. Hence, we just apply Lemmas 3.3 and 3.4.

LEMMA 3.6. *$E(I, J, \alpha, \beta, A)$ is a semigroup.*

Proof. We will utilize Lemma 3.4. Let $a = (i, (n, n), j)$, $b = (u, (s, s), v) \in E(I, J, \alpha, \beta, A)$. Hence, $j \in E(J_n)$ and $v \in E(J_s)$. Thus, using (1), $j\beta_{(s,s)}{}^*v = j^*e_s^*v = j^*v$. However, $E(T)$ is a semigroup for any chain of right groups T. Thus, it follows that $j^*v \in E(J_{\max(n,s)})$. Hence, $ab \in E(I, J, \alpha, \beta, A)$ by Lemma 3.4.

LEMMA 3.7. *$\mathscr{L}$ is a congruence on the semigroup $E(I, J, \alpha, \beta, A)$.*

Proof. Let X be any semigroup such that $E(X)$ is a semigroup. Then, it is easily seen that if $e, f \in E(X)$, $(e, f) \in \mathscr{L}(\in X)$ if and only if $(e, f) \in \mathscr{L}(\in E(X))$. Let $j \in E(J_n)$ and $v \in E(J_s)$. Hence, utilizing Lemmas 3.4 and 3.2(b), $(i, (n, n), j)\mathscr{L}(u, (s, s), v)(\in E(I, J, \alpha, \beta, A))$ if and only if $n = s$ and $j = v$. Thus, using (6), $\mathscr{L}$ is a left congruence on $E(I, J, \alpha, \beta, A)$ by a routine calculation.

LEMMA 3.8. *$E(I, J, \alpha, \beta, A)$ is an ω-chain of rectangular bands $(E_n: n \in N)$ where $E_n = ((i, (n, n), j): i \in I_n, j \in E(J_n))$.*

Proof. Let $(i, (n, n), j), (u, (n, n), v) \in E_n$. Utilizing (6), (2), (3), and a routine calculation, $(i, (n, n), j)(u, (n, n), v) = (i, (n, n), v)$. Hence, E_n is a rectangular band. Again, utilizing (6), (2), (3), and a routine calculation, $E_nE_k \subseteqq E_{\max(n,k)}$.

THEOREM 3.9. *(I, J, α, β, A) is a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup.*

Proof. Combine Lemmas 3.5–3.8.

4. Structure of generalized ω-$\mathscr{L}$-unipotent bisimple semigroups.

Combining Theorems 4.1 and 4.3 (below) will give a description of generalized ω-$\mathscr{L}$-unipotent bisimple semigroups in terms of groups, ω-chains of left zero semigroups, and ω-chains of right zero semigroups.

THEOREM 4.1. *(I, J, α, β, A) is a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup, and conversely every such semigroup is isomorphic to some (I, J, α, β, A).*

Proof. Combine Theorems 3.9 and 2.21.

REMARK. In contrast to the structure theorem for generalized $\mathscr{L}$-unipotent semigroups given in [4], no factor systems are required in Theorem 4.1.

We will next characterize an ω-chain J of right groups $(J_n: n \in N)$ as a semi-direct product of an ω-chain X of right zero semigroups $(X_n: n \in N)$ by an ω-chain G of groups $(G_n: n \in N)$.

We first need a definition.

DEFINITION 4.2. Let the semigroup U be an ω-chain of semigroups $(U_n \in N)$ and let θ be a mapping of U into a semigroup V such that $r \in U_n$, $s \in U_m$, and $m \geqq n$ imply $(rs)\theta = r\theta s\theta$. We term θ a lower homomorphism of U into V.

Let (G, o) be an ω-chain of groups $(G_n: n \in N)$ and let $(X, {}^*)$ be an ω-chain of right zero semigroups $(X_n: n \in N)$ such that $G_n \cap X_n = (e_n)$, a single idempotent element, for each $n \in N$. Let $g \rightarrow B_g$ be a lower homomorphism of G into T_X subject to the conditions (1) $X_n B_g \subseteqq X_{\max(n,m)}$ if $g \in G_m$ (2) if $r \in X_m$, $s \in X_n$ and $m \geqq n$, $(r^* s)B_g = rB_{e_n \circ g}{}^* B_g$. Let (G, X, B) denote $\bigcup (G_n \times X_n: n \in N)$ under the multiplication $(i, j)(p, q) = (i \circ p, jB_p^* q)$.

THEOREM 4.3. *J is an ω-chain of right groups if and only if $J \cong (G, X, B)$ for some collection G, X, B.*

Proof. We just specialize [6, Theorem 7.2].

Note 4.4. The structure of G is known mod groups and homomorphisms by a well known result of Clifford [1, Theorem 4.11].

5. **ω-$\mathscr{L}$-unipotent bisimple semigroups.** In this section, we specialize Theorem 4.1 to obtain [5, Theorem 7.11] (our previous structure theorem for ω-$\mathscr{L}$-unipotent bisimple semigroups).

A bisimple semigroup S is termed ω-$\mathscr{L}$-unipotent if $E(S)$ is an ω-chain of right zero semigroups.

THEOREM 5.1. *Let S be an ω-$\mathscr{L}$-unipotent bisimple semigroup. Then, there exists an ω-chain $(J, *)$ of right groups $(J_n: n \in N)$ and a homomorphism $(n, r) \rightarrow \beta_{(n,r)}$ of C into* End $(J, *)$ *such that for each $k \in N$ there exists $e_k \in E(J_k)$ and*

(1) *$g\beta_{(k,k)} = g^*e_k$ for all $g \in J$.*

(2) *$J_r\beta_{(n,k)} \subseteqq J_{k+r-\min(n,r)}$. Furthermore, $S \cong (((n, k), j): j \in J_k, n, k \in N)$ under the multiplication.*

(3) *$((n, k), j)((r, s), v) = ((n, k)(r, s), j\beta_{(r,s)}{}^*v)$ where juxtaposition denotes multiplication in C.*

Conversely, let $(J, *)$ be an ω-chain of right groups and let $(n, r) \rightarrow \beta_{(n,r)}$ be a homomorphism of C into End $(J, *)$ *such that* (1) *and* (2) *are valid. Then, $S = (((n, k), j): j \in J_k, n, k \in N)$ under* (3) *is an ω-$\mathscr{L}$-unipotent bisimple semigroup.*

Proof. We first establish the converse. We employ Theorem 4.1 and its notation. Let $I_v = (e_v)$ for each $v \in N$ and define $e_u \circ e_v = e_{\max(u,v)}$. Let $I = \bigcup (I_v: v \in N)$. Then, (I, o) is an ω-chain of left zero semigroups $(I_n: n \in N)$. Define $e_n\alpha_{(r,s)} = e_{s+n-\min(n,r)}$ and $e_nA_v = e_{\max(n,m)}$ if $v \in J_m$. By a routine calculation, $(n, r) \rightarrow \alpha_{(n,r)}$ is a homomorphism of C into End (I, o) and $p \rightarrow A_p$ is an upper anti-homomorphism of $(J, *)$ into T_I such that (2)-(5) of Theorem 4.1 is valid. The multiplication (6) of Theorem 4.1 becomes (6′) $(e_n, (n, k), j)(e_r, (r, s), v) = (e_{n+r-\min(k,r)}, (n, k)(r, s), j\beta_{(r,s)}{}^*v)$ where juxtaposition is multiplication in C. Hence, $U = (I, J, \alpha, \beta, A)$ (notation of §3) is a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup by Theorem 4.1. Utilizing Lemma 3.4, $E(U) = ((e_n, (n, n), j): j \in E(J_n), n \in N)$. Utilizing Lemma 3.2, $(e_n, (n, n), j)\mathscr{L}(e_k, (k, k), u)(j \in E(J_n)$ and $u \in E(J_k))$ implies $n = k$ and $j = u$. Hence, $E(U)$ is an ω-chain of right zero semigroups and, thus, U is an ω-$\mathscr{L}$-unipotent bisimple semigroup. Since $(e_n, (n, k), j)\varphi = ((n, k), j)$ define an isomorphism of $(U, (6'))$ onto $(S, (3))$. S is an ω-$\mathscr{L}$-unipotent bisimple semigroup.

Next, let T be an ω-$\mathscr{L}$-unipotent bisimple semigroup. Hence, T is a generalized ω-$\mathscr{L}$-unipotent bisimple semigroup and the structure of T is given by Theorem 4.1. Thus, utilizing Lemmas 3.8 and 3.2, $I_n = (e_n)$ for each $n \in N$. Hence, utilizing (2) and (3) of Theorem 4.1, $e_r\alpha_{(n,k)} = e_{k+r-\min(n,r)}$ and $e_rA_j = e_{\max(r,k)}$ if $j \in J_k$. Thus, (6) of Theorem 4.1 becomes (6′) and $(U, (6')) \cong (S, (3))$. The conditions of Theorem 5.1 are given by Theorem 4.1 ((1) and (2)).

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