

ON MATRIX MAPS OF ENTIRE SEQUENCES

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In this note the linear space E of entire sequences and various subspaces are considered. The fact that E represents the space of entire functions is emphasized by determining subspaces in terms of order and type. Matrix maps between the subspaces are characterized, and a related result and an example are also given.

1. Subspaces of E determined by order. First we recall that if $M(r) = \max |f(z)|$ on the circle $|z| = r$, then the order of the entire function f is $\rho = \limsup [(\log \log M(r))/\log r]$, and its type is $\tau = \limsup [(\log M(r))/r^\rho]$, assuming $\rho < \infty$. If $f(z) = \sum_0^\infty x_n z^n$ is an entire function, then it has finite order ρ if and only if $\mu = \limsup_n [n \log n / \log (1/|x_n|)]$ is finite, and then $\rho = \mu([1]$, p. 9).

DEFINITION 1.1. We say the complex sequence $x = \{x_n\}_0^\infty$ is analytic if the corresponding power series $\sum x_n z^n$ has radius of convergence $r(x) > 0$. x is an entire sequence if $r(x) = \infty$, and its order and type are those of the power series.

DEFINITION 1.2. For each $\rho \in [0, \infty)$, let $O(\rho)$ be the set of entire sequences of order not exceeding ρ . For each $\rho \in (0, \infty]$, let $O'(\rho)$ be the set of entire sequences of order less than ρ .

DEFINITION 1.3. If $0 \leq \rho < \infty$, let ρ^+ be the class of real sequences $\{\rho_n\}_1^\infty$ such that $\rho_n \searrow \rho$ and $\rho_n > \rho$. If $0 < \rho \leq \infty$, let ρ^- be similarly defined, but with $\rho_n \nearrow \rho$ and $0 < \rho_n < \rho$.

DEFINITION 1.4. Let $\alpha = \{\alpha_n\}_0^\infty$ be a complex sequence with no zero-terms, and let $s(\alpha) = \{\text{complex } x \mid \alpha_n x_n \rightarrow 0\}$.

If we define $\|x\|_\alpha = \sup |\alpha_n x_n|$, then $(s(\alpha), \|\cdot\|_\alpha)$ is a BK space ([8], Satz 5.4).

We will now characterize those matrices $A = (a_{nk})$ which map $s(\alpha) \rightarrow s(\beta)$. If f is a continuous linear functional on $s(\alpha)$, then f can be represented in the form $f(x) = \sum c_n \alpha_n x_n$, where $\sum |c_n| < \infty$ ([8], Satz 5.4). It is easily shown that the coefficients c_n in this representation are unique, and that $\|f\| = \sum |c_n|$. Suppose A maps $s(\alpha) \rightarrow s(\beta)$. Define $f_n(x) = \beta_n \sum_k a_{nk} x_k$. It is known ([7], Corollary 5, p. 204, or [8], Satz 4.4) that a matrix map between FK spaces is continuous, and $f_n = \beta_n P_n \circ A$ (where P_n is the n th projection map), so f_n is a continuous linear functional on $s(\alpha)$ with norm $\|f_n\| = \sum_k |\beta_n a_{nk} / \alpha_k|$.

$\beta_n y_n \rightarrow 0$ because $y = Ax \in s(\beta)$. It follows that for each x in $s(\alpha)$, $f_n(x) \rightarrow 0$ and $\sup_n |f_n(x)| < \infty$, so the uniform boundedness principle gives

$$(1.5) \quad \sup_n \sum_k |\beta_n a_{nk} / \alpha_k| = M \neq \infty .$$

Because e^k (the sequence having 1 in the k th coordinate and 0's elsewhere) is in $s(\alpha)$, we also have

$$(1.6) \quad Ae^k = \{a_{nk}\}_n = \text{the } k\text{th column of } A \text{ is in } s(\beta) \text{ for each } k .$$

We show that the necessary conditions (1.5) and (1.6) are also sufficient for A to map $s(\alpha)$ into $s(\beta)$. If $x \in s(\alpha)$, then $\alpha_k x_k \rightarrow 0$, so $\sum_k |a_{nk} x_k| = \mathcal{O}(1) \sum_k |a_{nk} / \alpha_k| < \infty$ by (1.5). Thus Ax is defined on $s(\alpha)$. Now let $\varepsilon > 0$ be given and choose N so that $k > N \Rightarrow |\alpha_k x_k| < \varepsilon / M$. Then

$$|y_n| \leq \sum_{k=0}^N |a_{nk} x_k| + (\varepsilon / M) \sum_{k=N+1}^{\infty} |a_{nk} / \alpha_k| ,$$

and (1.6) and (1.5) give

$$\limsup_n |\beta_n y_n| \leq 0 + (\varepsilon / M) M = \varepsilon .$$

It follows that $y = Ax \in s(\beta)$. We have proved

THEOREM 1.7. *In order that A maps $s(\alpha) \rightarrow s(\beta)$ it is necessary and sufficient that*

$$(1.5) \quad \sup_n \sum_k |\beta_n a_{nk} / \alpha_k| = M \neq \infty$$

and

$$(1.6) \quad \text{each column of } A \text{ is in } s(\beta)$$

be satisfied.

THEOREM 1.8. *Let $\{\alpha^j\}_{j=1}^{\infty}$ and $\{\beta^i\}_{i=1}^{\infty}$ be sequences of sequences $\alpha^j = \{\alpha_n^j\}_{n=0}^{\infty}$ and $\beta^i = \{\beta_n^i\}_{n=0}^{\infty}$ with the property that $k > j \Rightarrow s(\alpha^k) \subseteq s(\alpha^j)$ and $s(\beta^k) \subseteq s(\beta^j)$. Let $S = \bigcap s(\alpha^j)$ and $T = \bigcap s(\beta^i)$. Finally, we ask that for every j there exists $k > j$ such that $\sum |\alpha_n^j / \alpha_n^k| < \infty$, and that for every i there exists $\ell > i$ such that $\beta_n^i / \beta_n^\ell \rightarrow 0$. Then the matrix A maps $S \rightarrow T$ if and only if*

$$(1.9) \text{ for each } i \text{ there exist } j \text{ and } M \text{ such that } |\beta_n^i a_{nk} / \alpha_k^j| \leq M \text{ for all } n, k.$$

Proof. First, we remark that $s(\alpha) \subseteq s(\beta)$ if and only if $\limsup |\beta_n / \alpha_n| < \infty$. Clearly, the set $\{e^k, k = 0, 1, \dots\}$ is a Schauder

basis for $s(\alpha)$, so it follows ([9], Satz) that A maps $S \rightarrow T$ if and only if for every i there is a j such that A maps $s(\alpha^j) \rightarrow s(\beta^i)$, i.e., if and only if both of

$$(1.10) \quad \text{for each } i \text{ there is a } j \text{ such that } \sup_n \sum_k |\beta_n^i a_{nk} / \alpha_k^j| < \infty ,$$

$$(1.11) \quad \{a_{nk}\}_n \in T \text{ for each } k ,$$

hold. It is obvious that (1.10) implies (1.9). Conversely, let i be given and let j and M correspond to i as in (1.9). Choose $r > j$ so that $\sum_k |\alpha_k^j / \alpha_k^r| < \infty$. Then

$$\sum_k |\beta_n^i a_{nk} / \alpha_k^r| = \sum_k |\beta_n^i a_{nk} / \alpha_k^j| \cdot |\alpha_k^j / \alpha_k^r| \leq M \sum_k |\alpha_k^j / \alpha_k^r| < \infty ,$$

so (1.9) implies (1.10). To complete the proof we need only show that (1.9) implies (1.11). Let i be given and choose $\ell > i$ so that $\beta_n^i / \beta_n^\ell \rightarrow 0$. Let j and M correspond to ℓ as in (1.9). Then $|\beta_n^\ell a_{nk}| \leq M |\alpha_k^j|$ for every n , so $|\beta_n^i a_{nk}| \leq |\beta_n^i / \beta_n^\ell| M |\alpha_k^j| \rightarrow 0$ as $n \rightarrow \infty$, and $\{a_{nk}\}_n \in s(\beta^i)$.

Now let $\{\rho_j\} \in \rho^+$ and let $\alpha_n^j = n^{n \rho_j}$. It is readily verified that $O(\rho) = \bigcap s(\alpha^j)$.

THEOREM 1.12. *The matrix A maps $O(\rho) \rightarrow O(\mu)$ if and only if*

$$(1.13) \quad \text{for each } t > \mu \text{ there exist } r > \rho \text{ and } M \text{ such that} \\ n^{n t} |a_{nk}| (1/k^{k/r}) \leq M \text{ for all } n, k .$$

Proof. Let $\{\mu_i\} \in \mu^+$ and set $\beta_n^i = n^{n \mu_i}$, so that $O(\mu) = \bigcap s(\beta^i)$. The hypotheses of Theorem 1.8 are met, so A maps $O(\rho) \rightarrow O(\mu)$ if and only if (1.9) holds. But this is equivalent to (1.13).

We now consider $O'(\rho)$. If we choose $\{\rho_j\} \in \rho^-$, then $O'(\rho) = \bigcup O(\rho_j)$.

THEOREM 1.14. *The matrix A maps $O'(\rho) \rightarrow O'(\mu)$ if and only if*

$$(1.15) \quad \text{for each } r \in (0, \rho) \text{ there exist } t \in (0, \mu) \text{ and } M \text{ such that} \\ n^{n t} |a_{nk}| (1/k^{k/r}) \leq M \text{ for all } n, k .$$

Proof. We observe first that ([9], Satz, part (4)) remains true if the component spaces are merely FK spaces, nested or not, as long as their unions are linear spaces. (We will have occasion in the sequel to utilize this observation.) It follows that A maps $O'(\rho) \rightarrow O'(\mu)$ if and only if

for each j there is an i such that A maps $O(\rho_j) \rightarrow O(\mu_i)$.

By Theorem 1.12, this happens if and only if

for each j there is an i such that for every $t > \mu_i$, there exist $r > \rho_j$ and M such that $n^{n/t} |a_{nk}| (1/k^{k/r}) \leq M$ for all n, k .

But this is equivalent to (1.15).

Similar applications of ([9], Satz) give the next two theorems.

THEOREM 1.16. *The matrix A maps $O'(\rho) \rightarrow O(\mu)$ if and only if*

(1.17) *for each $t > \mu$ and $r \in (0, \rho)$ there is an M such that $n^{n/t} |a_{nk}| (1/k^{k/r}) \leq M$ for all n, k .*

THEOREM 1.18. *The matrix A maps $O(\rho) \rightarrow O'(\mu)$ if and only if*

(1.19) *there exist $t \in (0, \mu)$, $r > \rho$, and M such that $n^{n/t} |a_{nk}| (1/k^{k/r}) \leq M$ for all n, k .*

2. Subspaces of E determined by order and type. After the polynomials, the easiest class of entire functions to handle is the class (ρ, τ) , and the properties of its members have been investigated (see [1]). In our terminology, this subspace of E is defined below.

DEFINITION 2.1. Given $\rho \in (0, \infty)$ and $\tau \in [0, \infty)$, let (ρ, τ) be the set of entire sequences having order $< \rho$ or order ρ and type $\leq \tau$.

DEFINITION 2.2. For $\rho \in (0, \infty)$ and $\tau \in [0, \infty)$, define $G(\rho, \tau)$ to be the set of complex sequences x such that $\limsup n |x_n|^{\rho/n} \leq \tau e \rho$.

$G(\rho, \tau) \subseteq O(\rho)$, and moreover ([1], Theorem 2.2.10) it is true that

$$(2.3) \quad (\rho, \tau) = G(\rho, \tau) \cup O'(\rho).$$

DEFINITION 2.4. Suppose $\rho \in (0, \infty)$ and $\tau \in [0, \infty)$. Let $\{\varepsilon_n\}_1^\infty \in 0^+$ and let $\alpha_n^\nu = [n/(\tau e \rho + \varepsilon_n)]^{n/\rho}$. Now set $G(\rho, \tau, \nu) = s(\alpha^\nu)$. (We set $\alpha_0^\nu = 1$.)

It follows from this definition that

$$(2.5) \quad G(\rho, \tau) = \bigcap G(\rho, \tau, \nu),$$

and that $\nu > \mu \Rightarrow G(\rho, \tau, \nu) \subseteq G(\rho, \tau, \mu)$.

We will now prove some general results which will allow us to characterize those matrices which map (ρ, τ) into (μ, σ) .

Let $B_j = \bigcap_\nu s(\alpha^\nu(j))$ and let $C_i = \bigcap_\nu s(\beta^\nu(i))$. Set $B = \bigcup B_j$ and $C = \bigcup C_i$. We shall assume the following:

$$(2.6) \quad B \text{ and } C \text{ are linear spaces,}$$

$$(2.7) \quad \text{for every } j \text{ and } i, \nu > \mu \text{ implies } s(\alpha^\nu(j)) \subseteq s(\alpha^\nu(i)) \text{ and}$$

$$s(\beta^\nu(i)) \subseteq s(\beta^\mu(i)) ,$$

(2.8) for every j and μ , there is a $\nu > \mu$ such that $\sum_n |\alpha_n^\nu(j)/\alpha_n^\mu(j)| < \infty$,

(2.9) for every i and μ there is a $\nu > \mu$ such that $\beta_n^\nu(i)/\beta_n^\mu(i) \rightarrow 0$ as $n \rightarrow \infty$.

LEMMA 2.10. *The matrix A maps $B_j \rightarrow C_i$ if and only if*

(2.11) *for every sufficiently large ν there exist μ and M such that*

$$|a_{nk}\beta_n^\nu(i)/\alpha_k^\mu(j)| \leq M$$
for all n, k .

Proof. (1.9), together with the observation that Theorem 1.8 remains true if in (1.9) “ i ” is replaced by “sufficiently large i ”.

LEMMA 2.12. *Suppose that for every i and sufficiently large ν it is true that $\beta_n^\nu(i_0) = \mathcal{O}(\beta_n^\nu(i))$ as $n \rightarrow \infty$. Then A maps $B \rightarrow C$ if and only if A maps $B \rightarrow C_{i_0}$.*

Proof. Suppose A maps $B \rightarrow C$. Then ([9], Satz, (4)) for each j there is an i such that (2.11) holds. Our hypothesis then implies that (2.11) holds with i_0 in place of i , whence Lemma 2.10 asserts that A maps each B_j into C_{i_0} .

LEMMA 2.13. *Suppose that for every j and μ there is a γ such that $\alpha_k^\gamma(j_0) = \mathcal{O}(\alpha_k^\mu(j))$ as $k \rightarrow \infty$. Then A maps $B \rightarrow C$ if and only if A maps $B_{j_0} \rightarrow C$.*

Proof. Suppose A maps $B_{j_0} \rightarrow C$. Then ([9], Satz, (4)) there is an i such that A maps $B_{j_0} \rightarrow C_i$. But then (2.11) is true with j_0 in place of j . Our hypothesis implies that if j is given there exists γ such that (2.11) holds with γ and j in place of μ and j_0 . Hence, each B_j is mapped into C_i .

THEOREM 2.14. *Suppose for each i and sufficiently large ν , $\beta_n^\nu(i_0) = \mathcal{O}(\beta_n^\nu(i))$ as $n \rightarrow \infty$, and moreover that for every j and μ there is a γ such that $\alpha_k^\gamma(j_0) = \mathcal{O}(\alpha_k^\mu(j))$ as $k \rightarrow \infty$. Then A maps $B \rightarrow C$ if and only if A maps $B_{j_0} \rightarrow C_{i_0}$.*

Proof. Lemmas 2.12 and 2.13.

COROLLARY 2.15. *Under the hypotheses of Theorem 2.14, A maps $B \rightarrow C$ if and only if*

(2.16) *for every sufficiently large ν there exist μ and M such that*

$$|a_{nk}\beta_n^\nu(i_0)/\alpha_k^\mu(j_0)| \leq M$$
for all n, k .

Proof. Lemma 2.10.

Let us specialize the situation. Suppose $\{\rho_j\} \in \rho^-$, $\{\gamma_\nu(j)\}_\nu \in \rho_j^+$, $\{\varepsilon_\nu\} \in 0^+$, $\alpha_\nu^\nu(0) = [n/(\tau\varepsilon\rho + \varepsilon_\nu)]^{n/\rho}$, and $\alpha_\nu^\nu(j) = n^{n/\tau\nu(j)}$ for $j > 0$. Set $B_0 = G(\rho, \tau) = \bigcap s(\alpha^\nu(0))$ and $B_j = O(\rho_j) = \bigcap s(\alpha^\nu(j))$ for $j > 0$. Then $(\rho, \tau) = B = \bigcup_0^\infty B_j$. Similarly, let $\{\mu_i\} \in \mu^-$, $\{\xi_\nu(i)\} \in \mu_i^+$, $\beta_\nu^\nu(0) = [n/(\sigma\varepsilon\mu + \varepsilon_\nu)]^{n/\mu}$, and $\beta_\nu^\nu(i) = n^{n/\xi_\nu(i)}$ for $i > 0$. Then if $C_0 = G(\mu, \sigma) = \bigcap s(\beta^\nu(0))$ and $C_i = O(\mu_i) = \bigcap s(\beta^\nu(i))$ for $i > 0$, it follows that $(\mu, \sigma) = C = \bigcup_0^\infty C_i$. Inasmuch as (2.6)–(2.9) and the hypotheses of Theorem 2.14, with $i_0 = 0 = j_0$, are met, we have

THEOREM 2.17. *The matrix A maps $(\rho, \tau) \rightarrow (\mu, \sigma)$ if and only if A maps $G(\rho, \tau) \rightarrow G(\mu, \sigma)$.*

THEOREM 2.18. *The matrix A maps $(\rho, \tau) \rightarrow (\mu, \sigma)$ if and only if*
 (2.19) *for every sufficiently large ν there exist ω and M such that*

$$|a_{nk}| [n/(\sigma\varepsilon\mu + \varepsilon_\nu)]^{n/\mu} [(\tau\varepsilon\rho + \varepsilon_\nu)/k]^{k/\rho} \leq M$$

for all n, k ,

where $\{\varepsilon_\nu\} \in 0^+$.

Proof. Corollary 2.15.

THEOREM 2.20. *The matrix A maps $(\rho, \tau) \rightarrow (\mu, \sigma)$ if and only if*
 (2.21) *for each $\beta \in (0, (\mu\sigma)^{-1})$ there exist $\alpha \in (0, (\rho\tau)^{-1})$ and M such*
that $|a_{nk}| (n!)^{1/\mu} (k!)^{-1/\rho} \beta^{n/\mu} \alpha^{-k/\rho} \leq M$ for all n, k .

Proof. Suppose (2.19) holds and let $\beta = (\mu\sigma + \delta)^{-1}$ be given. Choose ν so large that $\varepsilon_\nu < \delta\varepsilon$ and let ω and M correspond to ν as in (2.19). Choose η so that $0 < \eta < \varepsilon_\omega e^{-1}$ and let $\alpha = (\rho\tau + \eta)^{-1}$. Then routine calculation, using Stirling's formula, shows that (2.21) is true. Conversely, suppose (2.21) is true, and let ν be given. Let $\beta = (\sigma\mu + \varepsilon_\nu e^{-1})^{-1}$ and let α and M correspond to β as in (2.21). Choose ω so that if $\alpha = (\rho\tau + \delta)^{-1}$, then $2\varepsilon_\omega < \delta\varepsilon$. Then (2.19) holds.

3. An example. We give an example of a matrix A which satisfies (2.21) with $\rho = \mu = 1$. Let $\sigma, \tau \in (0, \infty)$. Suppose f is analytic at the origin, and moreover on the closed disc of radius $R > \sigma^{-1}$. Suppose further that $f(0) = 0$ and that $|f(z)| \leq M \leq R\sigma\tau^{-1}$ on the disc. Let $C = (c_{nk})$ be the Sonnenschein matrix generated by f (if $f^n(z) = \sum_k c_{nk} z^k$ for $n \geq 0$, with $f^0(z) \equiv 1$, then $C = (c_{nk})$). Cauchy's estimate gives $|c_{nk}| \leq M^n R^{-k}$, so, inasmuch as our restrictions insure that for each $\beta \in (0, \tau)$ there is an $\alpha \in (0, \sigma)$ such that $\alpha R \geq 1$ and $\beta M \leq \alpha R$, it follows that $|c_{nk}| \beta^n \alpha^{-k} \leq (\beta M)^n (\alpha R)^{-k} = (\beta M / \alpha R)^n / (\alpha R)^{k-n} \leq$

1 (because $f(0) = 0$ makes C upper triangular, whence we may assume $k \geq n$). Now set $a_{nk} = (k!/n!)c_{nk}$. Then $A = (a_{nk})$ maps $(1, \sigma^{-1})$ into $(1, \tau^{-1})$.

4. The spaces E and \mathcal{A} . Suppose \mathcal{A} is the space of analytic sequences. In [2] and [3], those matrices which map $E \rightarrow E$ are characterized, while in [5] and [6] those which map $\mathcal{A} \rightarrow \mathcal{A}$ are determined. We state these results.

THEOREM 4.1. *The matrix A maps $E \rightarrow E$ if and only if*

(4.2) *for every $\beta \in (0, \infty)$ there exist $\alpha \in (0, \infty)$ and M such that*

$$|a_{nk}| \beta^n \alpha^{-k} \leq M \text{ for all } n, k.$$

THEOREM 4.3. *The matrix A maps $\mathcal{A} \rightarrow \mathcal{A}$ if and only if*

(4.4) *for every $\alpha \in (0, \infty)$ there exist $\beta \in (0, \infty)$ and M such that*

$$|a_{nk}| \beta^n \alpha^{-k} \leq M \text{ for all } n, k.$$

The symmetry between (4.2) and (4.4) is unmistakable, and allows an especially easy direct proof of

THEOREM 4.5. *The matrix A maps $\mathcal{A} \rightarrow \mathcal{A}$ if and only if the transposed matrix A^T maps $E \rightarrow E$.*

Proof. By (4.4), A maps $\mathcal{A} \rightarrow \mathcal{A}$ if and only if for each $\alpha > 0$ there exist $\beta > 0$ and M such that $|a_{kn}| \beta^k \alpha^{-n} \leq M$ for all k, n . Let $\alpha = \gamma^{-1}$, $\beta = \delta^{-1}$. Then this condition is equivalent to: for each $\gamma > 0$ there exist $\delta > 0$ and M such that $|a_{kn}| \delta^{-k} \gamma^n \leq M$ for all n, k . But this is (4.2) for A^T .

We note that if Theorem 4.5 is known, then each of Theorems 4.1 and 4.3 follows from the other in the manner of proof of Theorem 4.5. We note further that this theorem allows us to extract from Theorems 7, 8, and 9 of [4] information about A , rather than just about A^T .

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