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CYCLIC MULTIPLICATION OPERATORS ON L_p-SPACES

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Let (X, Σ, μ) be a measure space. Suppose f is in $L_\infty(X, \Sigma, \mu)$. The operator M_f on $L_p(X, \Sigma, \mu)$ defined by $M_f(g) = f \cdot g$, for g in $L_p(X, \Sigma, \mu)$ is called a multiplication operator. The purpose of this paper is to characterize cyclic multiplication operators and to relate their structure to the properties of the measure space on which the underlying L_p-space is defined.

In $L_2(X, \Sigma, \mu)$ any maximal abelian self-adjoint algebra of bounded operators may be transformed isometrically to the algebra of all multiplication operators on some L_2-space (see, e.g. [9]). This is due to the fact that among the L_p-spaces, only L_2-possesses a sufficiently rich collection of orthogonal projections. In fact, if $p \neq 2$, the only "orthogonal" projections on L_p are multiplications by characteristic functions (shown by Sullivan [10] for real L_p). As a consequence, isometries between L_p-spaces are related to σ-isomorphisms between the underlying measure spaces when $p \neq 2$. These relationships may be exploited to characterize certain multiplication operators on L_p-spaces where $1 \leqq p < \infty$.

In § 1, we present Sullivan's theorem along with applications to direct sum decompositions of L_p-spaces and to surjective isometries between L_p-spaces.

Section 2 deals with the characterization of the operator M_z on $L_p(\nu)$, where ν is a finite Borel measure with compact support in the plane, defined by

$$M_z f(z) = z f(z)$$

for f in $L_p(\nu)$.

In § 3 the concept of a normal measure space (introduced by Halmos and von Neumann [3]) is used to relate the structure of certain measure spaces (X, Σ, μ) to the structure of cyclic multiplication operators on $L_p(X, \Sigma, \mu)$.

We mention that throughout, all measure spaces are assumed to be σ-finite. For notational ease, we denote $L_p(X, \Sigma, \mu)$ by $L_p(\mu)$ when no confusion arises. The algebra of all multiplication operators on $L_p(\mu)$ is denoted by $\mathscr{M}_\mu$. Also if f is a measurable function on (X, Σ), then $\operatorname{supp}(f) \equiv \{x \in X \mid |f(x)| > 0\}$ is called the support of f.

1. Structural and isometric properties of L_p-spaces.

DEFINITION 1.1. A closed subspace R of $L_p(\mu)$ is a *p-direct* *summand* of $L_p(\mu)$ if there exists a closed subspace S of $L_p(\mu)$ such that $L_p(\mu) = R \oplus S$ algebraically and if $r \in R$ and $s \in S$, then $\| r + s \|^p = \| r \|^p + \| s \|^p$. In this event we write $L_p(\mu) = R \oplus_p S$.

We recall a relation due to Hanner ([4], Theorem 1, p. 239). If f and g are in $L_p(X, \Sigma, \mu)$, $p \neq 2$, then

$$(1.1) \qquad \| f + g \|^p + \| f - g \|^p = 2(\| f \|^p + \| g \|^p)$$

if and only if $f \cdot g = 0$ a.e.

THEOREM 1.1 (*Sullivan*). *A closed subspace R of $L_p(X, \Sigma, \mu)$ is a p-direct summand of $L_p(\mu)$ if and only if $R = M_{\chi(\sigma)}(L_p(\mu))$ for some $\sigma \in \sum$.*

Proof. If $L_p(\mu) = R \oplus_p S$, then for $r \in R$ and $s \in S$, we have $\| r \pm s \|^p = \| r \|^p + \| s \|^p$. By Equation 1.1, it follows that $r \cdot s = 0$ a.e. There exists $f_0 \in L_p(\mu)$ such that supp $(f_0) = X$ a.e. Let $f_0 = r_0 + s_0$ where $r_0 \in R$ and $s_0 \in S$. Then there exists $\sigma_0 \in \Sigma$ such that $r_0 = f_0\chi(\sigma_0)$ a.e. and $s_0 = f_0\chi(X \backslash \sigma_0)$ a.e. It follows that if $r \in R$ and $s \in S$, then we have $r = r\chi(\sigma_0)$ a.e. and $s = s\chi(X \backslash \sigma_0)$ a.e. We conclude that $R = M_{\chi(\sigma)}(L_p(\mu))$. The converse is immediate.

Definition 1.1 extends in a natural way to any collection of closed subspaces $\{L_\alpha\}_{\alpha \in A}$ of $L_p(\mu)$ and we say that $L_p(\mu)$ is the *p-direct sum* of $\{L_\alpha\}_{\alpha \in A}$ (written $\oplus_{\alpha \in A\, p} L_\alpha$) if $L_p(\mu) = \oplus_{\alpha \in A} L_\alpha$ algebraically and $\| \sum_{\alpha \in A} f_\alpha \|^p = \sum_{\alpha \in A} \| f_\alpha \|^p$, where $f_\alpha \in L_\alpha$. Clearly in this event, each L_α is a p-direct summand and for each sum $\sum_{\alpha \in A} f_\alpha$ at most a countable number of summands are nonzero.

COROLLARY 1.1. *If $\{L_\alpha\}_{\alpha \in A}$ is a collection of nontrivial closed subspaces of $L_p(\mu)$, $p \neq 2$, such that $L_p(\mu) = \oplus_{\alpha \in A\, p} L_\alpha$, then* card $(A) \leqq \aleph_0$.

For a measure space (X, Σ, μ) we set Σ' equal to Σ / N_μ where N_μ is the collection of all sets of measure zero in Σ. Then $[\sigma] \in \Sigma'$ will be the class of all sets τ in Σ such that $\tau \Delta \sigma = \phi$ a.e.

For $[\sigma]$ and $[\tau]$, in Σ', let $[\sigma] \cup [\tau] \equiv [\sigma \cup \tau]$, $[\sigma] \cap [\tau] \equiv [\sigma \cap \tau]$, and $[\sigma] \backslash [\tau] \equiv [\sigma \backslash \tau] = [\sigma] \cap [X \backslash \tau]$. These operations are all well defined. We say that $[\tau] \subset [\sigma]$ if $\tau \subset \sigma$ a.e.

Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces.

DEFINITION 1.2. A map $\Gamma: \Sigma' \to \Phi'$ is *σ-isomorphism* if

(i) Γ is a bijection;

(ii) $\Gamma([\sigma \backslash \tau]) = \Gamma([\sigma]) \backslash \Gamma([\tau])$ for $[\sigma]$ and $[\tau]$ in Σ';

(iii) if $\{[\sigma_i]\}_{i=1}^\infty$ is a sequence of elements of Σ', then

$$\Gamma\Big(\bigcup_{i=1}^\infty [\sigma_i]\Big) = \bigcup_{i=1}^\infty \Gamma([\sigma_i]) .$$

It is convenient to regard Γ as a map from Σ to Φ by merely setting $\Gamma(\sigma) = \bar{\varphi}$ where $\sigma \in [\sigma]$ and $\bar{\varphi}$ is a fixed representative of $\Gamma([\sigma])$. Then this mapping is a σ-isomorphism if we identify sets equal almost everywhere.

DEFINITION 1.3. A surjective isometry $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$ *induces* a σ-isomorphism $\Gamma: \Sigma' \to \Phi'$ if $[\operatorname{supp}(J(f))] = \Gamma([\sigma])$ whenever f is in $L_p(\mu)$ and $\operatorname{supp}(f) = \sigma$ a.e. μ.

We now give a generalization of part of a theorem due to Lamperti ([7], Theorem 3.1, p. 361).

THEOREM 1.2. *Let $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$, $p \neq 2$, be a surjective isometry. Then J induces a σ-isomorphism $\Gamma: \Sigma' \to \Phi'$ and Γ preserves atoms.*

Proof. Since J is a surjective isometry, it preserves p-direct summands.

Define $\Gamma: \Sigma \to \Phi$ by $\Gamma(\sigma) = \varphi$ where $J(M_{\chi(\sigma)}(L_p(\mu))) = M_{\chi(\varphi)}(L_p(\nu))$. Then Γ is a σ-isomorphism and hence Γ preserves atoms.

Suppose $f \in L_p(\mu)$ and $\operatorname{supp}(f) = \sigma$ a.e. Clearly $\operatorname{supp}(J(f)) \subset \Gamma(\sigma)$ a.e. ν. If $\operatorname{supp}(J(f)) \neq \Gamma(\sigma)$ a.e., then there exists $h \neq 0$ in $M_{\chi(\Gamma(\sigma))}(L_p(\nu))$ such that $h \cdot J(f) = 0$ a.e. ν. By equation 1.1 it follows that $J^{-1}(h) \cdot f = 0$ a.e. μ and $J^{-1}(h) \in M_{\chi(\sigma)}(L_p(\mu))$. This is contradiction.

REMARK 1.1. Suppose (X, Σ, μ) and (Σ, Φ, ν) are such that there exists a σ-isomorphism $\Gamma: \Sigma' \to \Phi'$. The measure ω defined by $\omega(\varphi) = \mu(\Gamma^{-1}([\varphi]))$ for $\varphi \in \Phi$ is equivalent to ν and $h = d\omega/d\nu$. Define $J_1: L_p(\omega) \to L_p(\nu)$ by $J_1(f) = h^{1/p} \cdot f$. Let $J_2: L_p(\mu) \to L_p(\omega)$ be defined on characteristic functions by $J_2(\chi(\sigma)) = \chi(\Gamma(\sigma))$. Extend J_2 by linearity and continuity to all of $L_p(\mu)$. The map $J_\Gamma = J_1 J_2$ is called the *canonical surjective isometry inducing Γ*.

DEFINITION 1.4. A bounded operator T on $L_p(X, \Sigma, \mu)$ *corresponds to* a bounded operator U on $L_p(Y, \Phi, \nu)$ if there exists a surjective isometry $J: L_p(\mu) \to L_p(\nu)$ such that $T = J^{-1}UJ$.

THEOREM 1.3. *If $J: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$ is a surjective isometry inducing a σ-isomorphism $\Gamma: \Sigma' \to \Phi'$, then the algebra of multiplication operators $\mathscr{M}_\mu$ on $L_p(\nu)$ corresponds bijectively to the algebra of multiplication operators $\mathscr{M}_\nu$ on $L_p(\nu)$.*

Proof. Since J induces Γ, if $\sigma \in \Sigma$, then $M_{\chi(\sigma)} = J^{-1} M_{\chi(\Gamma(\sigma))} J$. Since J is linear, if $s \in L_\infty(\mu)$ is a simple function, say $s = \sum_{i=1}^n \tau_i \chi(\sigma_i)$, then $M_s = J^{-1} M_r J$ where $r = \sum_{i=1}^n \tau_i \chi(\Gamma(\sigma_i))$ and $\| M_s \| = \| M_r \|$. But the multiplications by simple functions form dense subsets in both $\mathscr{M}_\mu$

and $\mathscr{M}_\nu$. By continuity, it follows that $\mathscr{M}_\mu$ corresponds bijectively and isometrically to $\mathscr{M}_\nu$ under J.

DEFINITION 1.5. An algebra $\mathscr{A}$ of bounded operators on $L_p(X, \Sigma, \mu)$ is *maximal abelian* if

(i) $\mathscr{A}$ is commutative;

(ii) if T is a bounded operator which commutes with all the elements of $\mathscr{A}$, then T is in $\mathscr{A}$.

THEOREM 1.4. *A bounded operator T on $L_p(\mu)$, $p \neq 2$, is in $\mathscr{M}_\mu$ if and only if each p-direct summand of $L_p(\mu)$ is invariant for T.*

Proof. Let $R = M_{\chi(\sigma)}(L_p(\mu))$ be an arbitrary p-direct summand of $L_p(\mu)$. Suppose T is in $\mathscr{M}_\mu$. Then we have $TM_{\chi(\sigma)}(L_p(\mu)) = M_{\chi(\sigma)}TM_{\chi(\sigma)}(L_p(\mu))$, i.e., $TM_{\chi(\sigma)} = M_{\chi(\sigma)}TM_{\chi(\sigma)}$ and thus R is invariant for T.

Conversely suppose that each $R = M_{\chi(\sigma)}(L_p(\mu))$ is invariant for T. Let $S = (I - M_{\chi(\sigma)})(L_p(\mu)) = M_{\chi(X\setminus\sigma)}(L_p(\mu))$. Then

$$M_{\chi(\sigma)}TM_{\chi(\sigma)} = TM_{\chi(\sigma)} \tag{1.2}$$

and

$$(I - M_{\chi(\sigma)})T(I - M_{\chi(\sigma)}) = T(I - M_{\chi(\sigma)}) . \tag{1.3}$$

Equations (1.2) and (1.3) imply that $TM_{\chi(\sigma)} = M_{\chi(\sigma)}T$. Thus T commutes with $\mathscr{P} = \{M_{\chi(\sigma)} \mid \sigma \in \Sigma\}$. Since $\mathscr{P}$ generates $\mathscr{M}_\mu$ which is maximal abelian, we conclude that $T \in \mathscr{M}_\mu$.

THEOREM 1.5. *Let $J_i: L_p(X, \Sigma, \mu) \to L_p(Y, \Phi, \nu)$, $i = 1, 2$, be two surjective isometries which induce the same σ-isomorphism $\Gamma: \Sigma' \to \Phi'$. Then there exists $f \in L_\infty(\nu)$ such that $|f| = 1$ a.e. ν and $J_2 = M_fJ_1$. If in addition $p \neq 2$, then the converse is also true.*

Proof. First assume (X, Σ, μ) is a finite measure space. Let $J_1(\chi(X)) = g$ and $J_2(\chi(X)) = h$. For $\sigma \in \Sigma$, it follows that $J_1(\chi(\sigma)) = g\chi(\Gamma(\sigma))$ a.e. ν and $J_2(\chi(\sigma)) = h\chi(\Gamma(\sigma))$ a.e. ν. We have $\int_\varphi |g|^p d\nu = \int_\varphi |h|^p d\nu$ for all $\varphi \in \Phi$. Thus $|g| = |h|$ a.e. ν. Let $f = h/g$. Since $\operatorname{supp}(g) = \operatorname{supp}(h) = Y$ a.e. ν, we have $|f| = 1$ a.e. ν, and if s is a simple function in $L_p(\mu)$, then $J_2(s) = M_fJ_1(s)$. By continuity, we conclude that $J_2 = M_fJ_1$.

In the standard way the theorem holds when (X, Σ, μ) is σ-finite.

If $p \neq 2$, then employing Theorem 1.2, the converse follows immediately.

2. Cyclicity and extended cyclicity.

DEFINITION 2.1. A bounded operator T on $L_p(X, \Sigma, \mu)$ is *cyclic* if there exists a function $f_0 \in L_p(\mu)$ such that $\{p(T)(f_0) \mid p$ is a polynomial in the variable $z\}$ is norm-dense in $L_p(\mu)$. The function f_0 is called a *cyclic function*.

DEFINITION 2.2. A multiplication operator M_f on $L_p(\mu)$ is *extended cyclic* if there exists a function g_0 in $L_p(\mu)$ such that $\{p(M_f, M_{\bar{f}})(g_0) \mid p(z, \bar{z})$ is a polynomial in the variables z and $\bar{z}\}$ is norm-dense in $L_p(\mu)$. The function g_0 is called an *extended cyclic function*. (Note that Definition 2.2 is merely a special case of the usual definition of the cyclicity of a normal operator on L_2.)

Throughout the sequel, the triple $(S, \mathscr{B}(S), \nu)$ shall be a finite measure space where S is a compact subset of C with Borel sets $\mathscr{B}(S)$ and ν is a finite measure (hence regular) on $\mathscr{B}(S)$.

REMARK 2.1. Let $z: S \to S$ be the identity function. The Stone-Weierstrass theorem implies that M_z is extended cyclic on $L_p(S, \mathscr{B}(S), \nu)$ with extended cyclic function $\chi(S)$.

The essential range of a function $f \in L_\infty(X, \Sigma, \mu)$ (written S_f) is defined in the usual way. It is easy to show that if M_f is a multiplication operator on $L_p(X, \Sigma, \mu)$, then S_f is the spectrum of M_f. Also, since C is a Lindelöf space, there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, such that $f(X \backslash X_0) \subset S_f$.

REMARK 2.2. Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces. Let $\Gamma: \Sigma' \to \Phi'$ be a σ-isomorphism. If there is a set $Y_0 \in \Phi$ with $\nu(Y_0) = 0$ and a measurable point mapping $\theta: Y \backslash Y_0 \to X$ such that $[\theta^{-1}(\sigma)] = \Gamma([\sigma])$ for $\sigma \in \Sigma$, then θ^{-1} *induces* Γ. In this event, $J_\Gamma : L_p(\mu) \to L_p(\nu)$ has the form $J_\Gamma(g)(y) = h^{1/p}(y)(g \circ \theta)(y)$ a.e. ν on $Y \backslash Y_0$ for $g \in L_p(\nu)$ and h as in Remark 1.1.

THEOREM 2.1. *An operator M_f of $L_p(X, \Sigma, \mu)$, $p \neq 2$, corresponds to M_z on $L_p(S, \mathscr{B}(S), \nu)$ if and only if f^{-1} induces a σ-isomorphism $\Gamma: \mathscr{B}'(S) \to \Sigma'$.*

Proof. Suppose there exists a surjective isometry $J: L_p(\nu) \to L_p(\mu)$ such that $M_f J = J M_z$ where J induces a σ-isomorphism $\Gamma: \mathscr{B}'(S) \to \Sigma'$. There exists $k \in L_\infty(\mu)$ such that $|k| = 1$ a.e. μ and $J = M_k J_\Gamma$. Also, there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, and a measurable point mapping $\theta: X \backslash X_0 \to S$ such that θ^{-1} induces Γ, (see, e.g., [8] Corollary 11, p. 272). By constructing J_Γ as in Remark 2.2 and noting that $\chi(S) \in L_p(\nu)$ we find that $M_f J(\chi(S)) = J M_z(\chi(S))$ implies that $f = \theta$ a.e. μ on $X \backslash X_0$.

Conversely if f^{-1} induces that σ-isomorphism Γ, it follows directly that if we construct J_Γ as in Remark 2.2 then $J_\Gamma M_z = M_f J_\Gamma$.

THEOREM 2.2. *Let μ and ν be two σ-finite measures on the Borel sets of S, a compact subset of $\boldsymbol{C}$. Then M_z on $L_p(\mu)$ corresponds to M_z on $L_p(\nu)$ if and only if μ is equivalent to ν.*

Proof. If $p = 2$, this result is known and follows from the uniqueness of resolutions of the identity (see, e.g. [2], Theorem 1, p. 65).

Suppose $p \neq 2$ and M_z on $L_p(\mu)$ corresponds to M_z on $L_p(\nu)$ under a surjective isometry $J: L_p(\mu) \to L_p(\nu)$, inducing a σ-isomorphism $\Gamma: \mathscr{B}(S)/N_\mu \to \mathscr{B}(S)/N_\nu$. Then by Theorem 2.1, μ and ν are equivalent.

Conversely if μ and ν are equivalent, then the identity mapping $z: S \to S$ induces the identity σ-isomorphism $\Gamma_I: \mathscr{B}/N_\mu(S) \to \mathscr{B}/N_\nu(S)$. Constructing $J_{\Gamma_I}: L_p(\mu) \to L_p(\nu)$ as in Remark 2.2, we find that $M_z J_\Gamma = J_\Gamma M_z$.

LEMMA 2.1. *An operator M_f on $L_p(X, \Sigma, \mu)$ corresponds to M_z on $L_p(S, \mathscr{B}(S), \nu)$ if and only if M_f is extended cyclic.*

Proof. If $p = 2$, this is a special case of a well known theorem for cyclic normal operators ([2], Theorem 1, p. 95).

If $p \neq 2$ and if M_f corresponds to M_z, then there exists a surjective isometry $J: L_p(\nu) \to L_p(\mu)$ such that $M_f J = J M_z$ and J induces a σ-isomorphism $\Gamma: \mathscr{B}'(S) \to \sum'$. Let $\{s_n\}$ be a sequence of simple functions in $L_\infty(\mu)$ such that $s_n \to f$ as $n \to \infty$ in $L_\infty(\mu)$. Then $\{M_{s_n}\}$ converges to M_f in $\mathscr{M}_\mu$ and $\{J^{-1} M_{s_n} J\}$ converges to M_z. Therefore, the sequence $\{J^{-1} M_{\bar{s}_n} J\}$ converges to $M_{\bar{z}}$, i.e., $M_{\bar{f}} J = J M_{\bar{z}}$.

If q is a polynomial in the variables z and $\bar{z}$, then $J^{-1} q(M_f, M_{\bar{f}}) J$ equals $q(M_z, M_{\bar{z}})$. Let $g = J(\chi(S))$. Then it follows that

$$Jq(M_z, M_{\bar{z}})(J^{-1}(g)) = q(M_f, M_{\bar{f}})(g) .$$

Thus M_f is extended cyclic with extended cyclic function g.

Conversely let $A = \{p(M_f, M_{\bar{f}}) = M_{p(f,\bar{f})} \mid p(z, \bar{z})$ is a polynomial in z and $\bar{z}\}$. Let $\mathscr{A}$ be the operator norm closure of A. Then it follows that under the involution induced by complex-conjugation, $\mathscr{A}$ is isometrically $*$-isomorphic to $C(S_f)$, the continuous functions on S_f, under an algebra isomorphism $\tau: \mathscr{A} \to C(S_f)$ such that $\tau(I) = 1$, $\tau(M_f) = z$, and $\tau(M_{\bar{f}}) = \bar{z}$ where z is the identity mapping on S_f. Let $g \in L_p(\mu)$ be an extended cyclic function for M_f. Define a linear functional L on $\mathscr{P} = \{q(z, \bar{z}) \mid q$ is a polynomial in the variables z and $\bar{z}\}$ by

$$L(q) = \int q(M_f, M_{\bar{f}})(g) \, | g |^{p-1} \operatorname{sgn}(g) d\mu \equiv \int q(f, \bar{f}) \, | g |^p \, d\mu .$$

It follows from Hölder's inequality that the integral exists and $| L(q) | \leqq \| q \|_\infty \| g \|_p^p$. We extend L to all of $C(S_f)$ by continuity and observe that L is a positive linear functional. There exists a regular positive Borel measure $\nu \in C^*(S_f)$ such that $L(h) = \int h d\nu$ for $h \in C(S_f)$. Since $| h |^p \in C(S_f)$, it follows that

$$L(| h |^p) = \int | h |^p \, d\nu = \| h \|_p^p = \int | h(f) \cdot g |^p \, d\mu = \| h(f) \cdot g \|_p^p .$$

Let $W = \{q(M_f, M_{\bar{f}})(g) \mid q(M_f, M_{\bar{f}}) \in A\}$. Then $L_p(\mu)$ is the p-norm closure of W. Define $J\colon W \to L_p(\nu)$ by $J(q(M_f, M_{\bar{f}})(g)) = q(z, \bar{z}) \in \mathscr{P}$. J is an isometry on W. Extend J by continuity to a surjective isometry on all of $L_p(\mu)$. We conclude that $M_z J = J M_f$ since this is true on W.

The proof of this lemma follows that employed by Kalisch [6] in his proof of the above mentioned $p = 2$ cyclicity theorem of Halmos.

REMARK 2.3. By standard methods, one can show that a function $g \in L_p(\mu)$ is an extended cyclic function for an extended cyclic multiplication operator M_f on $L_p(\mu)$ if and only if $| g | > 0$ a.e.

Bram ([1], Theorem 6, p. 85) has shown that for $p = 2$, M_z on $L_2(S, \mathscr{B}(S), \nu)$ is cyclic in the sense of Definition 2.1. An examination of the proof of this result shows that it is not dependent on the properties of Hilbert space. By use of Remark 2.1, Lemma 2.1 and an obvious modification of Bram's proof, one obtains:

THEOREM 2.3. *A multiplication operator on $L_p(\mu)$ is extended cyclic if and only if it is cyclic. (However, in general a cyclic multiplication operator has a larger collection of extended cyclic functions than cyclic functions.)*

We now obtain immediately:

THEOREM 2.4. *A bounded operator T on $L_p(X, \Sigma, \mu)$, $p \neq 2$ corresponds to M_z on $L_p(S, \mathscr{B}(S), \nu)$ if and only if each p-direct summand of $L_p(\mu)$ is invariant for T and T is cyclic.*

THEOREM 2.5. *An operator $M_f \in \mathscr{M}_\mu$ is cyclic on $L_p(\mu)$, $p \neq 2$, if and only if M_f is cyclic on $L_2(\mu)$.*

Proof. We may assume that (X, Σ, μ) is a finite measure space with $\mu(X) = 1$. Then if $1 \leqq p < p' < \infty$, we have $L'_p(\mu)$ is p-norm-dense in $L_p(\mu)$ and if $f \in L'_p(\mu)$, then $\| f \|_p \leqq \| f \|_{p'}$.

Suppose M_f is cyclic on $L_p(\mu)$, with cyclic function g and $p > 2$. The set $\mathscr{P} = \{q(M_f) \mid q$ is a polynomial in $z\}$ is norm-dense in $L_p(\mu)$. Thus it is norm-dense in $L_2(\mu)$.

If $p < 2$, there exists a surjective isometry $J: L_p(\mu) \to L_p(S, \mathscr{B}(S), \nu)$ such that $M_z J = J M_f$. Thus the dual mapping $J^*: L_p^*(\nu) \to L_p^*(\mu)$ is such that $J^* M_z^* = M_f^* J^*$. But $M_z^* = M_z$ on $L_p^*(\nu)$ and $M_f^* = M_f$ on $L_p^*(\mu)$. So M_f is cyclic on $L_p^*(\mu)$ and hence on $L_2(\mu)$.

The converse is proved similarly.

From Theorems 2.2 and 2.5 we conclude immediately that Theorem 2.1 holds for $p = 2$.

3. Cyclicity and univalence.

Let (X, Σ, μ) be a σ-finite measure space.

DEFINITION 3.1. A subset Λ of Σ is called a *σ-algebra contained in Σ* (written $\Lambda \subset \Sigma$) if Λ is itself a σ-algebra.

THEOREM 3.1. *Let $f \in L_\infty(\mu)$. There exists a σ-algebra $\Lambda_f \subset \Sigma$ depending on f such that f is in $L_\infty(X, \Lambda_f, \mu|_{\Lambda_f})$ and M_f is cyclic on $L_p(X, \Lambda_f, \mu|_{\Lambda_f})$.*

Proof. Consider $\Lambda_f = \{f^{-1}(\beta) \mid \beta \in \mathscr{B}(C)\}$. Then $\Lambda \subset \Sigma$ is a σ-algebra. Restrict μ to Λ_f. There exists $\beta_0 \in \Lambda_f$ such that $\mu(\beta_0) = 0$ and $f|_{X \setminus \beta_0} \subset S_f$. Define ν on $\mathscr{B}(S_f)$ by $\nu(\gamma) = \mu(f^{-1}(\gamma))$. Let ν' be a finite measure equivalent to ν. Then we see that f^{-1} induces a σ-isomorphism $\Gamma: \mathscr{B}(S_f)/N_{\nu'} \to \Lambda'_f$. Hence M_f on $L_p(X, \Lambda_f, \mu|_{\Lambda_f})$ corresponds to M_z on $L_p(S_f, \mathscr{B}(S_f), \nu')$ and thus M_f is cyclic.

DEFINITION 3.2. A measurable function f is *essentially univalent* if it is univalent on the complement of some set of measure zero.

We observe that M_z is cyclic on $L_p(S, \mathscr{B}(S), \nu)$ and z is a univalent function. It is reasonable to ask whether all cyclic multiplication operators on $L_p(X, \Sigma, \mu)$ arise from essentially univalent L_∞-functions and conversely. The answer in general is negative.

EXAMPLE 3.1. Consider $([0, 1], \mathscr{B}([0, 1]), \lambda)$, the usual Borel measure space on $[0, 1]$. Let f be defined by

$$f(t) = \begin{cases} t\,, & 0 \leqq t \leqq \dfrac{1}{2} \\ t - \dfrac{1}{2}\,, & \dfrac{1}{2} < t \leqq 1\,. \end{cases}$$

Then f is in $L_\infty(\lambda)$ and $S_f = [0, 1/2]$. There exists $\Lambda_f \subset \mathscr{B}([0, 1])$ such that M_f is cyclic on $L_p([0, 1], \Lambda_f, \lambda|_{\Lambda_f})$, but f is not essentially univalent on $([0, 1], \Lambda_f, \lambda|_{\Lambda_f})$.

We shall show that essential univalence of $f \in L_\infty(\mu)$ need not imply that M_f is cyclic on $L_p(\mu)$ after some preliminaries (see Example 3.2 ff.). In addition we will determine when essential univalence of an L_∞-function f is equivalent to the cyclicity of M_f on L_p.

DEFINITION 3.3. A σ-finite measure space (X, Σ, μ) is called *proper* if:

(i) it is *complete* and nonatomic;

(ii) there exists $\Lambda \subset \Sigma$ such that (X, Σ, μ) is *properly separable with respect to* Λ;

(iii) Λ has a *separating sequence.*

The σ-algebra Λ is called the *Borel sets* of Σ and a Λ-measurable function is called a *Baire function.* All italicized terms are defined as in [3]. We denote a proper measure space by $(X, \Sigma, \Lambda, \mu)$.

DEFINITION 3.4. A proper measure space $(X, \Sigma, \Lambda, \mu)$ is *normal* (c-normal) if to each real-valued (complex-valued) univalent Baire function f, there corresponds a set X_{0_f} in Σ depending on f such that $\mu(X_{0_f}) = 0$ and such that $f(X\backslash X_{0_f})$ is a Borel subset of $\boldsymbol{R}$(of $\boldsymbol{C}$).

REMARK 3.1. By duplicating the proofs of Lemmas 1-4 which Halmos and von Neumann proved for real-valued functions on proper and normal measure spaces ([3], pp. 337-339), we obtain:

THEOREM 3.2. *A proper measure space* $(X, \Sigma, \Lambda, \mu)$ *is normal if and only if it is c-normal.*

Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces.

DEFINITION 3.5. A bijective mapping $\theta: X\backslash X_0 \rightarrow Y\backslash Y_0$, where $\mu(X_0) = \nu(Y_0) = 0$, is a *point isomorphism* if θ and θ^{-1} are measurable and θ induces a σ-isomorphism $\Gamma: \Sigma' \rightarrow \Phi'$. In this event we say that (X, Σ, μ) and (Y, Φ, ν) are *point isomorphic.* If in addition we have $\mu(\sigma) = \nu(\Gamma(\sigma))$ for $\sigma \in \Sigma$, then θ and Γ are said to be *measure preserving.*

Let (X, Σ, μ) be a measure space. Then we shall denote by $(X, \tilde{\Sigma}, \tilde{\mu})$ the measure space where $\tilde{\Sigma}$ is the completion of Σ and $\tilde{\mu}$ is the completion of μ.

Throughout the remainder, the usual Borel measure space $([0, 1], \mathscr{B}([0, 1]), \lambda)$ shall be denoted by $[0, 1]$ and the usual (normal) Lebesgue measure space $([0, 1], \mathscr{L}([0, 1]), \mathscr{B}([0, 1]), \tilde{\lambda})$ will be denoted by $[0, 1]^\sim$.

REMARK 3.2. Halmos and von Neumann have proved the following:

THEOREM. *A proper measure space* $(X, \Sigma, \Lambda, \mu)$ *with* $\mu(X) = 1$ *is normal if and only if it is measure preserving point isomorphic to* $[0, 1]^{\sim}$. (*See* [3], *Theorem* 2, *p.* 339.)

It follows from this and the fact that all measure preserving set automorphisms on $[0, 1]^{\sim}$ are induced by measure preserving point isomorphisms ([3], *p.* 340) that if $(X, \Sigma, \Lambda, \mu)$ and (Y, Φ, Ψ, ν) are normal measure spaces, then each σ-isomorphism $\Gamma: \Sigma' \to \Phi'$ is induced by a point isomorphism γ.

REMARK 3.3. It follows from a Theorem proved by Halmos and von Neumann ([3], Lemma 3, p. 338) that if $(X, \Sigma, \Lambda, \mu)$ and (Y, Φ, Ψ, ν) are normal measure spaces, then any point isomorphism between them may be so constructed as to take Borel sets to Borel sets.

DEFINITION 3.6. A measure space (X, Σ, μ) is *pre-normal* if there exists $\Lambda \subset \tilde{\Sigma}$ such that $(X, \tilde{\Sigma}, \Lambda, \tilde{\mu})$ is normal.

REMARK 3.4. The measure space $(S, \mathscr{B}(S), \nu)$, where ν is a non-atomic measure on $\mathscr{B}(S)$ with $\operatorname{supp}(\nu) = S$, is pre-normal and $\mathscr{B}(S)$ serves as the Borel sets ([5], Theorem XIII, p. 304).

LEMMA 3.1. *Let* (X, Σ, μ) *be pre-normal. Then* $f \in L_\infty(\mu)$ *is essentially univalent if and only if* M_f *is cyclic.*

Proof. Suppose f in $L_\infty(\mu)$ is essentially univalent. Since (X, Σ, μ) is pre-normal, there exists a σ-algebra $\Lambda \subset \tilde{\Sigma}$ such that $(X, \tilde{\Sigma}, \Lambda, \tilde{\mu})$ is normal; and thus clearly $\tilde{\Lambda} = \tilde{\Sigma}$. The function f is $\tilde{\Sigma}$-measurable. So there exists $X_0 \in \Lambda$ with $\tilde{\mu}(X_0) = 0$ and $f|_{X \setminus X_0}$ is Λ-measurable univalent, and $\operatorname{rge}(f|_{X\setminus X_0}) \subset S_f$. We conclude that there exists $X_1 \in \Lambda$ such that $X_1 \supset X_0$, $\tilde{\mu}(X_1) = 0$, and $f(\tau)$ is a Borel subset of S_f when $\tau \in \Lambda$ and τ is a subset of $X \setminus X_1$ ([3], Lemma 3, p. 338).

Define a function ν' on $\mathscr{B}(S_f)$ with range in the extended real numbers by

$$\nu'(\beta) = \tilde{\mu}(f^{-1}(\beta) \cap (X \setminus X_1)) = \tilde{\mu}(f^{-1}(\beta)) .$$

Then ν' is a measure on $\mathscr{B}(S_f)$. Let ν be a finite measure on $\mathscr{B}(S_f)$ equivalent to ν'. We see that f^{-1} induces a σ-isomorphism $\Gamma: \mathscr{B}'(S_f) \to \Lambda' = \tilde{\Sigma}'$. Thus M_f on $L_p(X, \tilde{\Sigma}, \tilde{\mu})$ corresponds to M_z on $L_p(S_f, \mathscr{B}(S_f), \nu)$. Hence M_f is cyclic on $L_p(X, \Sigma, \mu)$.

Conversely suppose M_f is cyclic on $L_p(X, \Sigma, \mu)$. Then M_f corresponds to M_z on $L_p(S, \mathscr{B}(S), \nu)$. So f^{-1} induces a σ-isomorphism

$\Gamma: \mathscr{B}'(S) \to \Sigma'$. Now Γ extends to a σ-isomorphism $\tilde{\Gamma}: \tilde{\mathscr{B}}'(S) \to \tilde{\Sigma}'$ defined by $\tilde{\Gamma}([\eta]) = \Gamma([\beta])$ where $\eta = \beta \cup \tau$ for $\beta \in \mathscr{B}(S)$ and τ a subset of a set of ν-measure zero. Since ν is nonatomic, it follows that $(S, \tilde{\mathscr{B}}(S), \mathscr{B}(S), \tilde{\nu})$ is normal. Since (X, Σ, μ) is pre-normal, there exist $X_0 \in \tilde{\Sigma}$ and $N_0 \in \mathscr{B}(S)$ with $\tilde{\mu}(X_0) = \tilde{\nu}(N_0) = 0$ and a point isomorphism $\theta: X \backslash X_0 \to S \backslash N_0$ such that θ^{-1} induces $\tilde{\Gamma}$.

Now θ is $\tilde{\Sigma}$-measurable on $X \backslash X_0$. There exists $X_1 \in \Sigma$ with $X_1 \supset X_0$ and $\mu(X_1) = 0$ such that $\theta|_{X \backslash X_1}$ is Σ-measurable, univalent and $(\theta|_{X \backslash X_1})^{-1}$ induces $\tilde{\Gamma}$ and Γ.

There exists a surjective isometry $J: L_p(\nu) \to L_p(\mu)$ inducing Γ such that $M_f J = J M_z$. There exists $k \in L_\infty(\mu)$ with $|k| = 1$ a.e. such that $J = M_k J_\Gamma$. Thus for $g \in L_p(S, \mathscr{B}(S), \nu)$, we have $J(g)(x) = k(x)h(x)^{1/p} g(f(x)) = k(x)h^{1/p}(x)g(\theta(x))$ a.e. μ on $X \backslash X_2$ where $X_2 \supset X_1$, $\mu(X_2) = 0$ and $\operatorname{rge}(f|_{X \backslash X_2}) \subset S$. In particular, with $g = z$, we conclude that $f(x) = \theta(x)$ a.e. μ on $X \backslash X_2$.

Let (X, Σ, μ) be a measure space.

DEFINITION 3.7. Suppose Y is a (not necessarily measurable) subset of X with inner measure $\mu_*(X \backslash Y) = 0$. Let $\Sigma_Y = \{\tau \mid \tau = \sigma \cap Y$ for some $\sigma \in \Sigma\}$. Then Σ_Y is a σ-algebra and the extended real-valued function μ_y on Σ_y defined by $\mu_y(\tau) = \mu(\sigma)$, where $\sigma \in \Sigma$ and $\sigma \cap Y = \tau$, is a well defined measure on Σ_y. The triple (Y, Σ_y, μ_y) is called the *induced measure space* on Y.

DEFINITION 3.8. A (not necessarily measurable) subset Y of X with $\mu_*(X \backslash Y) = 0$ is *restrictive* if each essentially univalent function $f \in L_\infty(Y, \Sigma_y, \mu_y)$ is the restriction of an essentially univalent function $f' \in L_\infty(X, \Sigma, \mu)$.

REMARK 3.5. If Y is as in Definition 3.7, then the mapping $\Gamma: \Sigma' \to \Sigma'_y$ defined by $\Gamma([\sigma]) = [\sigma \cap Y]$ is a σ-isomorphism. Hence the canonical mapping $J_\Gamma: L_p(\mu) \to L_p(\mu_y)$ is a surjective isometry and under J_Γ we see that if $M_g \in \mathscr{M}_\mu$, then $M_{g|_Y} \in \mathscr{M}_{\mu_y}$ corresponds to M_g.

EXAMPLE 3.2. In the measure space $[0, 1]$ it is known that there exists a non-Lebesgue measurable subset σ such that $\lambda^*(\sigma) = 1$ and $\lambda_*(\sigma) = 0$ (see e.g. [3], Lemma 10, p. 342). Thus we see that

$$\lambda^*(([0, 1]) \backslash \sigma) = 1 \text{ and } \lambda_*(([0, 1]) \backslash \sigma) = 0 .$$

Let $\tau = [0, 1] \backslash \sigma$. The map $\varphi: [0, 1] \to [0, 1/2]$ defined by $\varphi(t) = t/2$ is a homeomorphism which preserves Borel and Lebesgue measurability. Thus $\varphi(\sigma)$ and $\varphi(\tau)$ are non-Lebesgue measurable subsets of $([0, 1/2], \mathscr{B}([0, 1/2]), \lambda)$ with $\varphi(\sigma) = [0, 1/2] \backslash \varphi(\tau)$, and $\lambda^*(\varphi(\sigma)) = \lambda^*(\varphi(\tau)) = 1/2$

while $\lambda_*(\varphi(\sigma)) = \lambda_*(\varphi(\tau)) = 0$. The map $\omega: [0, 1] \to [0, 1]$ defined by $\omega(t) = 1 - t$ is a homeomorphism which preserves Borel and Lebesgue measurability. Let $Y = \varphi(\sigma) \cup \omega(\varphi(\tau))$. Then we see that $Y \subset [0, 1]$ with $\lambda^*(Y) = 1$ and $\lambda_*(Y) = 0$. In addition, it follows from the construction of Y that if $t \in [0, 1] \cap Y$, then $1 - t \in [0, 1] \setminus Y$, for $t \neq 1/2$.

Let

$$f(t) = \begin{cases} 2t\,, & 0 \leqq t \leqq \dfrac{1}{2} \\ 2 - 2t\,, & \dfrac{1}{2} \leqq t \leqq 1\,. \end{cases}$$

Then $f(t)$ is a bounded $\mathscr{B}([0, 1])$-measurable function on $[0, 1]$ which is not essentially univalent. However, the function $f|_Y$ is bounded, univalent, and $\mathscr{B}([0, 1])_y$-measurable.

Since $[0, 1]$ is pre-normal, M_f is not cyclic on $L_p([0, 1])$. Thus $M_{f|_y}$ is not cyclic on $L_p(Y, \mathscr{B}([0, 1])_y, \lambda_y)$.

DEFINITION 3.9. Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces. Then (X, Σ, μ) is *almost point isomorphic to* (Y, Φ, ν) if there exists $X_0 \in \Sigma$ with $\mu(X_0) = 0$, and an injective measurable map $\rho: X \setminus X_0 \to Y$ such that ρ^{-1} induces a σ-isomorphism $\Gamma: \Phi' \to \Sigma'$, and if $X_1 \in \Sigma$, $X_1 \supset X_0$, and $\mu(X_1) = 0$, then $\rho(X \setminus X_1)$ is a restrictive subset of Y. The map ρ is called an *a-point isomorphism.*

We observe that if ρ is a point isomorphism between (X, Σ, μ) and (Y, Φ, ν) then ρ is also an a-point isomorphism, because ρ preserves measurable sets.

LEMMA 3.2. *Let (X, Σ, μ) and (Y, Φ, ν) be measure spaces. Suppose that f an essentially univalent function in $L_\infty(\mu)$ implies that M_f is cyclic on $L_p(\mu)$, and suppose that M_g cyclic on $L_p(\nu)$ implies that $g \in L_\infty(\nu)$ is essentially univalent. If there exists $X_0 \in \Sigma$, with $\mu(X_0) = 0$, and a measurable injective mapping $\rho: X \setminus X_0 \to Y$ such that ρ^{-1} induces a σ-isomorphism $\Gamma: \Phi' \to \Sigma'$, then ρ is an a-point isomorphism.*

Proof. Let $X_1 \in \Sigma$ be such that $X_1 \supset X_0$ and $\mu(X_1) = 0$. Let $W = \rho(X \setminus X_1)$ and let (W, Φ_w, ν_w) be the induced measure space on W. We observe that $\rho|_{X \setminus X_1}$ is measurable and $(\rho|_{X \setminus X_1})^{-1}$ induces the σ-isomorphism $\Gamma_w: \Phi'_w \to \Sigma'$ defined by $\Gamma_w([\tau]) = \Gamma([\varphi])$ where $\varphi \in \Phi$ and $\tau = \varphi \cap W$.

Let $g \in L_\infty(\nu_w)$ be essentially univalent. The composition $g \circ \rho$ is a Σ-measurable function on $X \setminus X_1$.

Define the function

$$h(x) = \begin{cases} g \circ \rho(x) , & x \in X \backslash X_1 \\ 0 , & x \in X_1 , \end{cases}$$

an essentially univalent function in $L_\infty(\mu)$. Thus M_h is cyclic on $L_p(\mu)$. It follows that M_h corresponds to M_z on $L_p(S_h, \mathscr{B}(S_h), \omega)$. In addition h^{-1} induces a σ-isomorphism $\Lambda: \mathscr{B}'(S_h) \to \Sigma'$. It follows that g^{-1} induces the σ-isomorphism $\Gamma_w^{-1}\Lambda: \mathscr{B}'(S_h) \to \Phi'_w$. Thus M_g is cyclic on $L_p(\nu_w)$. There exists $g' \in L_\infty(\nu)$ such that $g'|_W = g$ a.e. ν_w and $M_{g'}$ corresponds to M_g under the surjective isometry $J: L_p(\nu) \to L_p(\nu_w)$ as constructed in Remark 3.5. Thus $M_{g'}$ is cyclic on $L_p(\nu)$. It follows that g' is an essentially univalent $L_\infty(\nu)$ function. Thus W is restrictive subset of Y.

THEOREM 3.3. *Let (X, Σ, μ) be a separable, nonatomic measure space. The following are equivalent*:

(i) *a function $f \in L_\infty(\mu)$ is essentially univalent if and only if M_f is cyclic on $L_p(\mu)$*;

(ii) *(X, Σ, μ) is almost point isomorphic to* $[0, 1]$.

Proof. ((i) $\Rightarrow$ (ii)) There exists a function $f \in L_\infty(\mu)$ such that M_f is cyclic on $L_p(\mu)$. Thus f is essentially univalent and M_f corresponds to M_z on $L_p(S_f, \mathscr{B}(S_f), \nu)$. In addition f^{-1} induces a σ-isomorphism $\Gamma: \mathscr{B}'(S_f) \to \Sigma'$ and there exists $X_0 \in \Sigma$ such that $\mu(X_0) = 0$ and $f|_{X \backslash X_0}$ has range in S_f.

Since $(S_f, \mathscr{B}(S_f), \nu)$ is nonatomic, the measure space $(S_f, \widetilde{\mathscr{B}}(S_f), \mathscr{B}(S_f), \tilde{\nu})$ is normal. There exists a point isomorphism $\rho: S_f \backslash B_0 \to [0, 1] \backslash N_0$ where $\nu(\beta_0) = \lambda(N_0) = 0$, ρ preserves Borel sets, and ρ^{-1} induces a σ-isomorphism $\Delta: \mathscr{B}'([0, 1]) \to \mathscr{B}'(S_f)$. There exists $X_1 \supset X_0$ such that $\mu(X_1) = 0$ and $\rho \circ f|_{X \backslash X_1}$ is defined, univalent, and $(\rho \circ f|_{X \backslash X_1})^{-1}$ induces the σ-isomorphism $\Gamma\Delta: \mathscr{B}'([0, 1]) \to \Sigma'$. Since $[0, 1]$ is pre-normal, it follows from Lemmas 3.1 and 3.2 that (X, Σ, μ) is almost point isomorphic to $[0, 1]$.

((ii) $\Rightarrow$ (i)) There exists $X_0 \in \Sigma$ with $\mu(X_0) = 0$ and an a-point isomorphism $\theta: X \backslash X_0 \to [0, 1]$ such that θ^{-1} induces a σ-isomorphism $\Gamma: \mathscr{B}'([0, 1]) \to \Sigma'$. So we construct $J_\Gamma: L_p([0, 1]) \to L_p(\mu)$, as in Remark 2.2. Then under J_Γ, for each $M_f \in \mathscr{M}_\mu$, there exists $M_k \in \mathscr{M}_\lambda$ such that $M_f J_\Gamma = J_\Gamma M_k$. The function $\chi([0, 1])$ is in $L_p([0, 1])$ and we conclude that $M_f J_\Gamma(\chi([0, 1])) = J_\Gamma M_k(\chi([0, 1])$ a.e. μ. Thus there exists $X_1 \supset X_0$, $\mu(X_1) = 0$, and $f(x) = k(\theta(x))$ on $X \backslash X_1$.

Suppose M_f is cyclic on $L_p(\mu)$. Then M_k is cyclic on $L_p([0, 1])$. Since $[0, 1]$ is pre-normal, k is essentially univalent. Thus $f = k \circ \theta$ is essentially univalent.

Conversely suppose that f is essentially univalent. There exists $X_2 \in \Sigma$ such that $X_2 \supset X_1 \supset X_0$, $\mu(X_2) = 0$, and $f = k \circ \theta$ is univalent on

$X \backslash X_2$. Since $\theta(X \backslash X_2) \equiv Y$ is a restrictive subset of $[0, 1]$, and $k|_Y$ is univalent, it follows that k is an essentially univalent $L_\infty([0, 1])$ function. Since $[0, 1]$ is pre-normal, M_k is cyclic on $L_p([0, 1])$. Thus M_f is cyclic on $L_p(\mu)$.

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