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AN ABEL-MAXIMAL ERGODIC THEOREM FOR SEMI-GROUPS

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The purpose of this paper is to prove a maximal ergodic theorem for Abel means of a strongly measurable semi-group $\Gamma = \{T_t; t \geqq 0\}$ of linear contractions on a complex L_1-space satisfying $|T_t f| \leqq c$ a.e. for any $t \geqq 0$ and any integrable f with $|f| \leqq c$ a.e. Applying the obtained maximal ergodic theorem, individual and dominated ergodic theorems for Abel means are also proved. These results extend results obtained by D. A. Edwards for sub-Markovian semi-groups.

2. The maximal ergodic theorem. Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space and $L_p(X) = L_p(X, \mathscr{B}, \mu)$, $1 \leqq p \leqq \infty$, the usual (complex) Banach spaces. Let $\Gamma = \{T_t; t \geqq 0\}$ be a strongly measurable semi-group of linear contractions on $L_1(X)$ with $\|T_t f\|_\infty \leqq \|f\|_\infty$ for any $f \in L_1(X) \cap L_\infty(X)$ and any $t \geqq 0$. By the Riesz convexity theorem Γ may be considered as a strongly measurable semi-group of linear contractions on $L_p(X)$ for each p with $1 \leqq p < \infty$. It is then known (cf. [4], p. 686) that for each $f \in L_p(X)$ with $1 \leqq p < \infty$, there exists a scalar function $T_t f(x)$, measurable with respect to the product of Lebesgue measure and μ, such that for almost all t, $T_t f(x)$, as a function of x, belongs to the equivalence class of $T_t f$ and such a measurable representation is uniquely determined except for a set of the product-measure zero. Moreover, since the integral $\int_0^\infty e^{-\lambda t} T_t f\, dt$ exists for any $\lambda > 0$, it follows from Theorem III.11.17 of [4] that there exists a μ-null set $E(f)$, dependent on f but independent of λ, such that if $x \notin E(f)$ then $e^{-\lambda t} T_t f(x)$ is integrable on $[0, \infty)$ for each $\lambda > 0$ and the integral $\int_0^\infty e^{-\lambda t} T_t f(x)\, dt$, as a function of x, belongs to the equivalence class of $\int_0^\infty e^{-\lambda t} T_t f\, dt$. Thus if we denote the integral $\int_0^\infty e^{-\lambda t} T_t f\, dt$ by $R_\lambda f$ then $\int_0^\infty e^{-\lambda t} T_t f(x)\, dt$ gives a representation of $R_\lambda f$, and hence, from now on, we shall write $R_\lambda f(x)$ for $\int_0^\infty e^{-\lambda t} T_t f(x)\, dt$.

Let $f \in L_p(X)$ and $a > 0$. Following Chacon [1], we define

$$f^{a-}(x) = [\operatorname{sgn} f(x)] \min(a, |f(x)|),$$

$$f^{a+}(x) = [\operatorname{sgn} f(x)](|f(x)| - \min(a, |f(x)|)),$$

$$f^*(x) = \sup_{0<\lambda<\infty} |\lambda R_\lambda f(x)|$$

and

$$E^*(a) = \{x; f^*(x) > a\},$$

where $\operatorname{sgn} f(x) = f(x)/|f(x)|$ if $f(x) \neq 0$ and $\operatorname{sgn} f(x) = 0$ if $f(x) = 0$.

We are now in a position to state the main theorem of this paper.

THEOREM 1. *If $f \in L_p(X)$, $1 \leqq p < \infty$, then for any $a > 0$ we have*

$$\int_{E^*(a)} (a - |f^{a-}|) d\mu \leqq \int |f^{a+}| d\mu.$$

For the proof of Theorem 1 we shall need the following lemma, whose proof is given in [7].

LEMMA. *Let τ be a positive linear contraction on $L_1(X)$ satisfying $\|\tau f\|_\infty \leqq \|f\|_\infty$ for any $f \in L_1(X) \cap L_\infty(X)$, let $f \in L_p(X)$ with $1 \leqq p < \infty$ and let $a > 0$. Define*

$$e^*(a) = \left\{x; \sup_{0<r<1} \left|(1-r)\sum_{k=0}^{\infty} r^k \tau^k f(x)\right| > a\right\}.$$

Then we have

$$\int_{e^*(a)} (a - |f^{a-}|) d\mu \leqq \int |f^{a+}| d\mu.$$

Proof of Theorem 1. For each $\lambda > 0$ and each positive integer n, define

$$R_\lambda^{(n)} f = \frac{1}{n} \sum_{k=0}^{\infty} e^{-\lambda k/n} T_{k/n} f.$$

We shall first prove that for any fixed $\lambda > 0$,

$$\lim_n \|R_\lambda f - R_\lambda^{(n)} f\|_p = 0. \tag{1}$$

In fact, if $\varepsilon > 0$ then choose a positive real number a such that

$$\int_a^\infty \|e^{-\lambda t} T_t f\|_p \, dt < \varepsilon \quad \text{and} \quad \frac{e^{-\lambda a}}{\lambda} < \varepsilon. \tag{2}$$

Let $k(n)$ be the positive integer such that

$$\frac{k(n)}{n} \leqq a < \frac{k(n)+1}{n}. \tag{3}$$

Then

$$\|R_\lambda f - R_\lambda^{(n)} f\|_p \leqq \left\|\int_0^a e^{-\lambda t} T_t f \, dt - \frac{1}{n} \sum_{k=0}^{k(n)} e^{-\lambda k/n} T_{k/n} f\right\|_p + \varepsilon + \frac{1}{n} \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n}.$$

Since $1/n \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n} \leqq 1/n\, e^{-\lambda a}/(1 - e^{-\lambda/n})$ by (3) and $\lim_n n(1 - e^{-\lambda/n}) = \lambda$, it follows from (2) that for N_0 sufficiently large enough and $n \geqq N_0$ we have

$$(4) \qquad \frac{1}{n} \sum_{k=k(n)+1}^{\infty} e^{-\lambda k/n} < \varepsilon\,.$$

Let

$$g_n(t) = e^{-\lambda k/n} \quad \text{for} \quad \frac{k}{n} \leqq t < \frac{k+1}{n}$$

and

$$T_t^{(n)} f = T_{k/n} f \quad \text{for} \quad \frac{k}{n} \leqq t < \frac{k+1}{n}\,.$$

Since $\Gamma = \{T_t; t \geqq 0\}$ is strongly continuous on $(0, \infty)$ (cf. [4], Lemma VIII.1.3), $\lim_n \| g_n(t) T_t^{(n)} f - e^{-\lambda t} T_t f \|_p = 0$, from which it follows that $\lim_n \int_0^a \| e^{-\lambda t} T_t f - g_n(t) T_t^{(n)} f \|_p \, dt = 0$, and hence (1) follows.

Since $\lim_n n(1 - e^{-\lambda/n}) = \lambda$, (1) implies at once that $\lim_n \| \lambda R_\lambda f - (1 - e^{-\lambda/n}) \sum_{k=0}^{\infty} e^{-\lambda k/n} T_{k/n} f \|_p = 0$. Let Q be the set of all positive rational numbers. By the Cantor diagonal argument there exists a subsequence $\{n_i\}$ such that for any $\lambda \in Q$,

$$\lambda R_\lambda f(x) = \lim_i (1 - e^{-\lambda/n_i}) \sum_{k=0}^{\infty} e^{-\lambda k/n_i} T_{k/n_i} f(x) \quad \text{a.e.}$$

Hence if we let

$$f_i^*(x) = \sup_{0<\lambda<\infty} (1 - e^{-\lambda/n_i}) \sum_{k=0}^{\infty} e^{-\lambda k/n_i} \tau_i^k | f |(x)\,,$$

where τ_i denotes the linear modulus [2] of T_{1/n_i}, then $| \lambda R_\lambda f(x) | \leqq \liminf_i f_i^*(x)$ a.e. for any $\lambda \in Q$. Since the mapping $\lambda \to \lambda \int_0^\infty e^{-\lambda t} T_t f(x)\, dt$ is continuous for almost all $x \in X$, it follows that $\sup_{0<\lambda<\infty} | \lambda R_\lambda f(x) | = \sup_{\lambda \in Q} \left| \lambda \int_0^\infty e^{-\lambda t} T_t f(x)\, dt \right|$ a.e., and thus

$$f^*(x) \leqq \liminf_i f_i^*(x) \quad \text{a.e.}$$

Let $e_i^*(a) = \{x; f_i^*(x) > a\}$. It is clear that $E^*(a) \subset \liminf_i e_i^*(a)$, and hence Fatou's lemma and the above lemma imply that

$$\int_{E^*(a)} (a - | f^{a-} |) d\mu \leqq \liminf_i \int_{e_i^*(a)} (a - | f^{a-} |) d\mu \leqq \int | f^{a+} | d\mu\,,$$

and the theorem is proved.

3. Applications. It is known (cf. [3]) that (i) if $1 < p < \infty$ and $f \in L_p(X)$, then the function ${}^*f$ defined by

$$ {}^*f(x) = \sup_{0<b<\infty} \left| \frac{1}{b} \int_0^b T_t f(x)\, dt \right| $$

is in $L_p(X)$ and $\| {}^*f \|_p \leqq p/(p-1)\| f \|_p$; (ii) for every $f \in L_p(X)$ with $1 \leqq p < \infty$, the limit

$$ \lim_{b \uparrow \infty} \frac{1}{b} \int_0^b T_t f(x)\, dt $$

exists and is finite a.e. In this section we shall prove the exact analogues for Abel means.

THEOREM 2. *If* $1 \leqq p < \infty$ *and* $f \in L_p(X)$, *then* $f^* < \infty$ a.e. *In particular if* $1 < p < \infty$, *then* f^* *is in* $L_p(X)$ *and*

$$ \| f^* \|_p \leqq \frac{p}{p-1} \| f \|_p . $$

Proof. It follows easily from Theorem 1 that for any $a > 0$,

$$ \mu(E^*(a)) \leqq \frac{1}{a} \int_{E^*(a)} | f | d\mu < \infty , $$

from which we observe that $f^* < \infty$ a.e. The second half of the theorem follows from Theorem 2.2.3 of [6]. The proof is complete.

THEOREM 3. *For any* $f \in L_p(X)$ *with* $1 \leqq p < \infty$, *the limit*

$$ \lim_{\lambda \downarrow 0} \lambda R_\lambda f(x) \tag{5} $$

exists and is finite a.e.

Before the proof we note that if the semi-group $\Gamma = \{T_t; t \geqq 0\}$ is sub-Markovian (for definition, see [5]) and of type C_1, then the above theorem has been proved by Edwards [5].

Proof. For $1 < p < \infty$, $L_p(X)$ is reflexive and thus it follows from Corollary VIII.7.2 of [4] that the functions f of the form

$$ f = h + \sum_{i=1}^{n} (I - T_{t_i}) g_i , $$

where $T_t h = h$ for all $t \geqq 0$, is dense in $L_p(X)$ in the norm topology. Since

$$ \lambda \int_0^\infty e^{-\lambda t} T_t (I - T_{t_i}) g_i(x) dt = \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T_t g_i(x) dt $$
$$ + \lambda (1 - e^{\lambda t_i}) \int_0^\infty e^{-\lambda t} T_t g_i(x) dt \text{ a.e.} $$

for each i, and

$$\lim_{\lambda \downarrow 0} \lambda e^{\lambda t_i} \int_0^{t_i} e^{-\lambda t} T_t f(x) dt = 0 \text{ a.e.}$$

for each i, it follows from Theorem 2 that

$$\lim_{\lambda \downarrow 0} \lambda \int_0^{\infty} e^{-\lambda t} T_t (I - T_{t_i}) g_i(x) dt = 0 \text{ a.e.}$$

for each i. Thus we observe that the limit (5) exists and is finite a.e. for any function f in a dense subset of $L_p(X)$ in the norm topology. Hence the Banach convergence theorem [3] and Theorem 2 imply that the limit (5) exists and is finite a.e. for any $f \in L_p(X)$. Since $L_p(X) \cap L_1(X)$ is dense in $L_1(X)$ in the norm topology, the Banach theorem and Theorem 2 are also sufficient to prove that the limit (5) exists and is finite a.e. for any $f \in L_1(X)$. This completes the proof of Theorem 3.

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