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A CHARACTERIZATION OF QF-3 RINGS

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Let R be a ring with minimum condition on left or right ideals. It is shown that R is a QF-3 ring if and only if each finitely generated submodule of the injective hull of R, regarded as a left R-module, is torsionless. The same approach yields a simplified proof that R is quasi-Frobenius if and only if every finitely generated left R-module is torsionless.

A ring with identity is called a *left QF-3 ring* if it has a (unique) minimal faithful left module, and a QF-3 ring means a ring which is both left and right QF-3. This class of rings originated with Thrall [9] as a generalization of quasi-Frobenius or QF algebras and has been studied extensively in recent years. Quasi-Frobenius rings have many interesting characterizations and in most instances there exists an analogous characterization of QF-3 rings at least in the case of rings with minimum condition and often for a much larger class of rings. It is well known that a ring with minimum condition on left or right ideals is a left QF-3 ring if and only if the injective hull $E({}_RR)$ of the ring R regarded as a left R-module is projective. Moreover, in this case R is a QF-3 ring (cf. [6] and [8]). For semi-primary or perfect rings; however, the situation is somewhat different. Namely, a perfect ring is a left QF-3 ring if and only if $E({}_RR)$ is torsionless. A module is called *torsionless* if it can be embedded in a direct product of copies of the ring regarded as a module over itself. In this case $E({}_RR)$ need not be projective and R need not be right QF-3 (cf. [3] and [8]). However, a perfect ring is QF-3 if and only if both $E({}_RR)$ and $E(R_R)$ are projective (see [8]). In this note, it is shown that if R is left perfect ring, $E({}_RR)$ is projective if and only if each finitely generated submodule of $E({}_RR)$ can be embedded in a free R-module. For a ring with minimum condition on left or right ideals this latter condition is equivalent to each finitely generated submodule of $E({}_RR)$ being torsionless. Thus in that case QF-3 rings may be characterized by this weaker condition. The technique of proof also yields a much simplified proof of a characterization of QF rings given by the present author in [7]. Namely, a ring with minimum condition on left or right ideals is QF if and only if each finitely generated left module is torsionless. Indeed, the characterization of QF-3 rings given here may be regarded as the analog of that result.

THEOREM 1. *Let R be a left perfect ring. $E({}_RR)$ is projective if and only if each finitely generated submodule of $E({}_RR)$ can be embedded* *in a free R-module.*

Since flat modules over a left perfect ring are projective, this result is immediate from the following lemma. For a discussion of left perfect rings see [1].

LEMMA 2. *Let I be an injective left R-module. If each finitely generated submodule of I can be embedded in a flat R-module, then I is flat.*

Proof. By [2, Exercise 6, p. 123] it suffices to show that for any $a_1, \cdots, a_m \in I$ satisfying a linear relation $\sum_{i=1}^{m} r_i a_i = 0$ with $r_i \in R$, there exists a positive integer n and elements $b_j \in I$, $s_{ij} \in R$ such that for each $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$

$$(*) \qquad a_i = \sum_{j=1}^{n} s_{ij} b_j \,, \qquad \sum_{i=1}^{m} r_i s_{ij} = 0 \,.$$

Let A be the submodule of I generated by $a_1, \cdots, a_m$. By hypothesis A is a submodule of a flat R-module F and so by [2, Exercise 6, p. 123] there exists an integer n and elements $c_j \in F$, $s_{ij} \in R$ such that for all $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$

$$(**) \qquad a_i = \sum_{j=1}^{m} s_{ij} c_j \,, \qquad \sum_{i=1}^{m} r_i s_{ij} = 0 \,.$$

Since I is injective the inclusion map of A into F can be extended to an R-homomorphism α of F into I such that $(a)\alpha = a$ for all $a \in A$. Setting $b_j = (c_j)\alpha$ and applying α to the first half of (**) shows that (*) can be satisfied for any such choice of $a_1, \cdots, a_m$. Thus I is flat.

COROLLARY 3. *If R is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

(a) *R is a QF-3 ring.*

(b) *Every finitely generated submodule of* $E({}_RR)$ *can be embedded in a free R-module.*

(c) *Every finitely generated submodule of* $E({}_RR)$ *is torsionless.*

Proof. In view of the introductory remarks and Theorem 1 it suffices to show that (c) implies (b). Let M be a finitely generated submodule of $E({}_RR)$ and $M^* = \mathrm{Hom}_R(M, R)$. It suffices to find $f_1, \cdots, f_n \in M^*$ such that $\bigcap_{i=1}^{n} \mathrm{Ker}\, f_i = (0)$ since the map $f: M \rightarrow \bigoplus_{i=1}^{n} {}_RR$ via $m \rightarrow (f_1(m), \cdots, f_n(m))$ will then give the desired embedding. Since M is torsionless, $(0) = \bigcap_f \mathrm{Ker}\, f$ with $f \in M^*$. If R satisfies the minimum condition on left ideals such $f_1, \cdots, f_n$ exist since M being finitely generated satisfies the descending chain condition on

R-submodules. If R satisfies the minimum condition on right ideals then since M is finitely generated M^* is isomorphic to a submodule of a finitely generated free right R-module and hence is finitely generated. (See [5, p. 66].) If $f_1, \cdots, f_n$ generate M^*, they have the desired property since M is torsionless.

REMARK. Condition (b) does not imply condition (a) for rings with maximum condition since any commutative integral domain which is not a field satisfies (b) but is not QF-3.

COROLLARY 4. *If R is a left and right perfect ring then R is a QF-3 ring if and only if every finitely generated submodule of $E({}_RR)$ and $E(R_R)$ is isomorphic to a submodule of a free R-module.*

Proof. In view of the introductory remarks this result is immediate from Theorem 1 and its right hand analog.

The next theorem and its corollary were proved in [7].

THEOREM 5. *Let R be a left perfect ring. R is a quasi-Frobenius ring if and only if every finitely generated left R-module is isomorphic to a submodule of a free R-module.*

Proof. This result follows from Lemma 2 and the fact that QF rings are characterized by the property that every injective module is projective [4, Theorem 5.3].

The next corollary follows from Theorem 5 in exactly the same manner that Corollary 3 follows from Theorem 1.

COROLLARY 6. *If R is a ring with minimum condition on left or right ideals, the following conditions are equivalent.*

(a) *R is quasi-Frobenius.*

(b) *Every finitely generated left R-module is isomorphic to a submodule of a free R-module.*

(c) *Every finitely generated left R-module is torsionless.*

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