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SOLVABLE GROUPS IN WHICH EVERY MAXIMAL PARTIAL ORDER IS ISOLATED

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It is shown that every maximal partial order in a torsion-free abelian-by-nilpotent group is isolated. The same is true for an ordered polycyclic group. Examples are given to show that maximal partial orders need not be isolated in torsion-free polycyclic groups, nor in ordered centre-by-metabelian groups.

1. Introduction. In ([3], p. 209, Problem 3) Fuchs suggested the problem of investigating those groups in which every maximal partial order is isolated. In this paper we show the dichotomies produced by groups having the above property in the class of solvable groups. We call P a partial order on a group G if it is a normal subsemigroup of G such that $P \cap P^{-1} = \{e\}$. P is a maximal partial order if it is a maximal normal subsemigroup subject to $P \cap P^{-1} = \{e\}$, and P is isolated if $g^n \in P$ implies $g \in P$ for all $g \in G$ and $n > 0$. Since every partial order on a group G can be extended to a maximal partial order using Zorn's lemma, the class of groups in which every maximal partial order is isolated is the same as that in which every partial order can be extended to an isolated partial order. We shall adopt the notation used by Hollister in [7] and denote this class of groups by I^*. Clearly, every I^* group is torsion-free since $\{e\}$ itself is a partial order. Our main results are as follows.

THEOREM 1. *Every torsion-free abelian-by-nilpotent group is in* I^*.

THEOREM 2.[1] *Every ordered polycyclic group is in* I^*.

THEOREM 3. *The group*

$$G = \langle a, b, t; a^t = b, b^t = (ba)^{-1}, t^9 = [b, a]\rangle$$

is not in I^*.

This group is one used in [2]. Theorem 3 shows that *torsion-free polycyclic groups need not be in* I^*, and has the following consequences.

COROLLARY 1. *An ordered centre-by-metabelian group need not be in* I^*.

[1] A stronger version of this result will appear in the Journal of Algebra.

COROLLARY 2. *An ordered abelian-by-polycyclic group need not be in* I^*.

The above results show that the class I^* produces a dichotomy in the class of torsion-free solvable groups, separating abelian-by-nilpotent groups from polycyclic groups and centre-by-metabelian groups, and also separating ordered polycyclic groups from ordered abelian-by-polycyclic groups.

Since a locally I^* group is in I^* ([7], Corollary 3), we can conclude, from Theorems 1 and 2, that torsion-free locally abelian-by-nilpotent groups and locally ordered polycyclic groups are in I^*.

2. Notations. The following list of symbols will be used to denote the various classes of groups that are mentioned in this paper; $\mathscr{N}\hat{c}$ for nilpotent groups of class at most c, $\mathscr{N}$ for nilpotent groups, $\mathscr{A}$ for abelian groups, $P\mathscr{C}_1$ for polycyclic groups, $\mathscr{A}\mathscr{N}$ for abelian-by-nilpotent groups, $\mathscr{F}$ for finite groups, $\mathscr{F}^{-s}$ for torsion-free groups, $\mathscr{O}$ for ordered groups.

For any subset X of a group G, we denote by $S(X)$ the normal semigroup generated by X in G. For a polycyclic group G, we denote by $h(G)$ the Hirsch length of G-the number of infinite cyclic factors in any cyclic series of G.

If x, y are elements of a group G, we write x^y to denote $y^{-1}xy$; $x^{p(y)}$ to denote $\prod_{i=0}^{n} x^{\alpha_i y^i}$ where $p(y) = \sum_{i=0}^{n} \alpha_i y^i$, α_i integers.

If P is a linear order on a group G and $x, y \in G$ then we write $x \ll y$ to mean that the convex subgroup generated by x does not contain y. Thus if $y \in P$ and $x \ll y$ then $x^n y \in P$ for all integers n.

3. Proofs. In order to prove Theorem 1 we require the following result which is interesting in itself.

LEMMA 3.1. *Let P be a maximal partial order on a group G and H a normal subgroup of G. If $H \in I^*$ then the restriction $P \cap H$ of P to H is an isolated partial order on H.*

Proof. Let A be the intersection of all maximal partial orders on H extending $P \cap H$. Clearly, A is isolated. A is normal in G for, if not, then $a^g \notin A$ for some $a \in A$, $g \in G$. Thus there is a maximal partial order Q on H such that $Q \supseteqq P \cap H$ and $a^g \notin Q$. This implies that $a \notin Q^{g^{-1}}$; but this is not possible since $Q^{g^{-1}}$ is also a partial order on H extending $P \cap H$. It is easy to check that AP is a part al order on G extending P and since P is maximal, $A = P \cap H$ is isolated.

Proof of Theorem 1. Since every partial order P on a free metabelian group F can be extended to a linear order ([1], Theorem 4 and [5], Theorem 1) every torsion-free quotient of F is in I^* ([7], Corollary 7). Assume, by way of induction, that $\mathscr{F}^{-s} \cap \mathscr{A}\mathscr{N}_c \subseteqq I^*$ and let $G \in \mathscr{F}^{-s} \cap \mathscr{A}\mathscr{N}_{c+1}$, P a maximal partial order on G and $g^n \in P$ for some $g \in G$, $n > 1$. Consider $H = \langle G', g \rangle$. $H \lhd G$ and $H \in \mathscr{F}^{-s} \cap \mathscr{A}\mathscr{N}_c$ ([6], Lemma 1.3). Thus $H \in I^*$. By Lemma 3.1, $P \cap H$ is isolated in H. Hence $g \in P \cap H \subseteqq P$.

LEMMA 3.2. *If $G \in \mathscr{O}$ and $H \in \mathscr{N}_c$ is a normal subgroup of G such that G/H is periodic then $G \in \mathscr{N}_c$.*

Proof. The above property is in fact true of groups in which $x^n = y^n$, $n \neq 0$ implies $x = y$. Such groups are called R-groups and contain the class $\mathscr{O}$. Using the fact that all terms of the upper central series of an R-group G are isolated in G (see [9], p. 245), one can easily check that $Z_i(G) \cap H = Z_i(H)$, $i \geqq 0$. In particular $Z_c(G) \cap H = Z_c(H) = H$ and hence $Z_c(G) = G$.

LEMMA 3.3. *Let P be an order on a nilpotent group G, and let $a, b \in G \backslash \{e\}$. Then $[a, b] \ll a$ and $[a, b] \ll b$.*

Proof. The result follows from the fact that under any order on a nilpotent group G, the convex jumps are central ([8], Lemma 1).

We shall need all the above results in the proof of Theorem 2. We shall also need the following property of ordered polycyclic groups.

REMARK. Let $G \in P\mathscr{G}_1 \cap \mathscr{O}$ and suppose that $C \prec D$ is a convex jump under some order on G. Then it is well known (see [3], pp. 50–54) that C, D are normal in G, D/C is order isomorphic to a subgroup of the reals under addition and for any $t \in G$, the action of t on D/C under conjugation is that of multiplication by a positive real when D/C is identified as the subgroup of the reals. Since D/C is finitely generated, the action of t on D/C is that of multiplication by a positive algebraic integer. Thus there is an integer monic polynomial $p(x)$, irreducible over the rational field such that $\bar{d}^{p(t)} = \bar{e}$ for all $\bar{d} \in D/C$. Moreover, at least one root of $p(x)$ is a positive real.

Proof of Theorem 2. We use induction on $h(G)$. If $h(G) = 1$, then by Lemma 3.2, $G \in \mathscr{F}^{-s} \cap \mathscr{A} \subseteqq I^*$. Assume the result when $h(G) < h$ and let $G \in P\mathscr{G}_1 \cap \mathscr{O}$, $h(G) = h$. Let P be a maximal partial order on G and supppse that $t^n \in P$ for some $t \in G$, $n > 1$, but $t \notin P$. Since P is maximal and $t \notin P \cup P^{-1}$, $t^{g_1} \cdots t^{g_r} \in P^{-1}$ for some choice of $g_1, \cdots, g_r$ in G and $r \geqq 1$. Thus $gt^{-r} \in P$ for some $g \in G'$, the commutator subgroup of G. We can clearly assume that r is a multiple of n so that t^r and gt^{-r} lie in P. Our next result, Lemma 3.4, shows that in this situation $h(T) < h(G)$ where $T = \langle t^G \rangle$. By induction hypothesis, $T \in I^*$ so that $t \in P$ by Lemma 3.1. This contradiction completes the proof.

LEMMA 3.4. *Let $G \in P\mathscr{G}_1 \cap \mathscr{O}$ and let P be a partial order on G such that $t^r, gt^{-r} \in P$ for some $e \neq t \in G, g \in G'$ and $r \geqq 1$. Then $h(G/T) \geqq 1$ where $T = \langle t^G \rangle$.*

Proof. Once again the proof is by induction on $h(G)$. If $h(G) = 1$ then by Lemma 3.2, $G \in \mathscr{A}$ so that $g = e$ and $t^r, t^{-r} \in P$. This is not possible. Assume that the result holds when $h(G) < h$ and assume that $h(G) = h$. Since $G \in \mathscr{O}$, amongst all possible orders on G pick one, say Q, such that the first convex subgroup $C_1 \neq \{e\}$ satisfies $h(C_1)$ minimal. Let

$$\{e\} < C_1 < \cdots < C_{k+1} = G$$

be the chain of convex subgroups of G under Q. Let N be the maximal normal nilpotent subgroup of G. Since $G \in P\mathscr{G}_1 \cap \mathscr{O}$, $G' \leqq N$ (see [3], p. 51), and $G/N \in \mathscr{A} \cap \mathscr{O}$. If $h(G/G') > 1$ or if $N = G$ or if $t \in N$, then $h(G/T) \geqq 1$ so that we can assume, without loss of generality, that $G = \langle N, t \rangle$, $h(G/G') = 1$, $N/G' \in \mathscr{T}$ and $N = C_k$.

Let $\bar{G} = G/C_1$ and look at $S(\bar{t}^r, \overline{gt}^{-r}) = \bar{S}$ in $\bar{G}$. Either $\bar{S}$ is a partial order on $\bar{G}$ or not. We discuss these two cases.

Case I. $\bar{S}$ is a partial order on $\bar{G}$. The hypotheses of the lemma are satisfied for the partial order $\bar{S}$ on $\bar{G}$ and $h(\bar{G}) < h$ so that $h(\bar{G}/\bar{T}) \geqq 1$ and hence $h(G/T) \geqq 1$.

Case II. $\bar{S}$ is not a partial order on $\bar{G}$. This implies that $S(t^r, gt^{-r}) \cap C_1 \neq \varnothing$. Thus $(t^r)^{x_1} \cdots (t^r)^{x_m} (gt^{-r})^{y_1} \cdots (gt^{-r})^{y_n} = c \in C_1$ for some choice of $x_1, \cdots, x_m, y_1, \cdots, y_n$ in G. Since $G/C_1 \in \mathscr{O}$, $S(gt^{-r}) \cap C_1 = \varnothing$ so that $m \neq 0$. Clearly we can assume that $x_1 = e$ by conjugating the expression by x_1^{-1} if necessary. Thus

$$t^{-r}c \in S(t^r, gt^{-r}) \subseteqq P$$

and $t^r, t^{-r}c \in P$ for some $c \in C_1$.

Note that $C_1 \leqq Z(N)$ for $\{e\} \prec C_1$ is a jump under the order $Q \cap N$ on N and $N \in \mathscr{N}$. Let $K = \langle C_1, t \rangle$. Then K is an ordered metabelian group and we can extend the partial order $P \cap K$ to a linear order R on K. Let $\{e\} < K_1 < \cdots < K_s = K$ form the chain of convex subgroups of K under R. Since $C_1 \leqq Z(N)$, $R \cap C_1$ is a G-invariant order on C_1 so that we can define an order on G with $\{e\} \leqq K_1 \cap C_1 \leqq$

$\cdots \leqq K_s \cap C_1 = C_1 < \cdots < C_k < G$ as convex subgroups. By our choice of C_1 we must have either (i) $K_1 \cap C_1 = \{e\}$ or (ii) $K_1 \cap C_1 = C_1$. If (i) holds then $K_1C_1 \lhd K$, $K_1C_1 \in \mathscr{A}$ and $K/K_1C_1 \in \mathscr{F}$ so that $K \in \mathscr{A}$. If (ii) holds then, since t^r and $t^{-r}c \in R$ and $c \in K_1$, we have $t^r \in K_1$ so that $K_1 = K$ and again $K \in \mathscr{A}$. Thus $C_1 \leqq Z(G)$.

Let $B = [\langle t^r \rangle, N] = [\langle t^r \rangle, G]$. We now show that $B \cap \langle c \rangle = \{e\}$. For any $x, y \in G$, $(t^{-r}c)^x \in P$, $(t^r)^y \in P$ and $c^x = c$ so that $[g, t^r]c$ and $[t^r, g]c \in P$ for all $g \in G$. Since $N \in \mathscr{N}$, we can extend the partial order $P \cap N$ to a linear order L on N. Under this order $[g, t^r] \ll c$ for all $g \in N$. For if $[g, t^r]^n c^{-1} \in L$ for some $n > 1$, then $[g, t^r] \in L$ and $L \ni [g^{-n-1}, t^r]c = [g, t^r]^{-n-1}c\xi$ for some $\xi \in [[g, t^r], N]$ so that $[g, t^r]^{-1}\xi \in L$. But by Lemma 3.3, $\xi \ll [g, t^r]$. Thus we have a contradiction. If $[g, t^r]^n c^{-1} \in L$ for some $n < -1$, then replace $[g, t^r]$ by $[t^r, g]$ and use the above argument again.

Thus $h(N/B) \geqq 1$. Moreover $B \lhd G$. Let F be a maximal normal subgroup G subject to $B \leqq F < N$, $h(N/F) \geqq 1$. Let i be the smallest integer such that $C_i \nleqq F$. Clearly $C_i \leqq N$ since $C_k = N > F$. Also $N/FC_i \in \mathscr{F}$. Since C_i/C_{i-1} is a convex jump in G, there exists an integer monic polynomial $p(x)$, irreducible over the rationals and with at least one positive real root, such that $a^{p(t)} \in C_{i-1} \leqq F$ for all $a \in C_i$. Also $a^{t^r-1} \in F \geqq B = [\langle t^r \rangle, N]$. Thus $a^{t-1} \in F$ for all $a \in C_i$. This is equivalent to saying $[FC_i, \langle t \rangle] \leqq F$. Thus $FC_i/F \leqq Z(G/F)$. Now $F\langle t^r \rangle/F \leqq Z(G/F)$. Thus $Z(G/F)$ is of finite index in G/F. By Schur's lemma, $FG'/F \in \mathscr{F}$. Since $h(G/F) \geqq 2$ we have $h(G/G') \geqq 2$ so that $h(G/T) \geqq 1$. This completes the proof.

Proof of Theorem 3. Let $P = S(t^3)$. Since $t^3 \in Z(G)$, P is a partial order on G, and it can not be extended to an isolated partial order for t can not belong to any partial order on G since $t^5 t^{b^{-1}} t^2 t^b = e$. That $G \in \mathscr{F}^{-s}$ was shown in [2], and that $G \in P\mathscr{G}_1$ can be seen by the chain $\{e\} \lhd \langle t^3 \rangle \lhd \langle t^3, a \rangle \lhd \langle t^3, a, b \rangle \lhd G$.

Proof of Corollary 1. Let E be the free centre-by-metabelian group on two generators. Observe that $Z(G) = \langle t^3 \rangle$ and $G/\langle t^3 \rangle$ is abelian by cyclic of order 3. Since G can be generated by a and t, G is a torsion-free quotient of E, and therefore, by a result of Hollister ([7], Corollary 5), $E \notin I^*$. On the other hand, $E \in \mathscr{F}^{-s}$ (see [10]) and since $E/Z(E)$ is free metabelian and hence ordered ([1], Lemma 4.1), $E \in \mathscr{O}$ ([4], Theorem 4).

Proof of Corollary 2. Let

$$A = \langle a, b, t; a^t = b, b^t = (ba)^{-1}, [[a, b], b] = [[a, b], a] = e \rangle ,$$

and let F/K be a representation of A as a quotient of a free group.

Since A has the invariant chain $\{e\} \lhd \langle [a, b] \rangle \lhd \langle a, b \rangle < A$ with torsion-free abelian factors, $F/K' \in \mathscr{O}$ ([11], Theorem 2). Clearly F/K' is abelian-by-polycyclic and it is not in I^* since it has a quotient isomorphic to G.

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