PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

APPROXIMATING POLYHEDRA IN CODIMENSION ONE SPHERES EMBEDDED IN S^n BY TAME POLYHEDRA

ROBERT J. DAVERMAN

We investigate properties of an $(n-1)$-sphere Σ topologically embedded in the n-sphere S^n $(n \geqq 6)$ implying that each $(n-3)$-dimensional polyhedron in Σ can be homeomorphically approximated by polyhedra in Σ that are tame in S^n. In case Σ bounds an n-cell, we relate these properties and the existence of homeomorphic approximations to Σ by locally flat spheres "mostly" outside this n-cell. This leads to a negative result eliminating a natural generalization to Bing's Side Approximation Theorem.

1. **Definitions and notation.** For any point p in a metric space S and any positive number δ, $N_\delta(p)$ denotes the set of points in S whose distance from p is less than δ.

The symbol Δ^2 denotes a 2-simplex fixed throughout this paper, $\partial\Delta^2$ its boundary, and Int Δ^2 its interior.

Let A denote a subset of a metric space X and p a limit point of A. We say that A is *locally simply connected at* p, written 1-LC at p, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ can be extended to a map of Δ^2 into $A \cap N_\varepsilon(p)$. Furthermore, we say that A is *uniformly locally simply connected*, written 1-ULC, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into a δ-subset of A can be extended to a map of Δ^2 into an ε-subset of A. Similarly, we say that A is *locally simply connected in X at* p, written 1-LC in X at p, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ extends to a map of Δ^2 into $N_\varepsilon(p)$, and we say that A is *uniformly locally simply connected in* X (1-ULC in X) if the corresponding uniform property is satisfied.

We use ρ to denote a metric on S^n. In case f and g are maps of a space Y into S^n, then $\rho(f, g)$ denotes the least upper bound of $\{\rho(f(y), g(y)) \mid y \in Y\}$. If Y is a subset of S^n and f maps Y into S^n, we call f an *ε-map* if $\rho(f, i) < \varepsilon$, where i denotes the inclusion map; in addition, Y is called an *ε-set* if the diameter of Y, written diam Y, is less than ε.

Let Σ be a closed $(n-1)$-manifold topologically embedded in S^n and T a (curvilinear) triangulation of Σ. For $i = 0, 1, \cdots, n-1$ $T^{(i)}$ denotes the i-skeleton of T and mesh T the maximum diameter of the simplexes in T. For a subset X of Σ, the *star of X in T*, written St (X, T), is the collections of all simplexes τ of T for which there exists a simplex γ of T such that τ is a face of γ and $\gamma \cap X \neq \emptyset$.

A compact 0-dimensional subset X of Σ is said to *tame* (relative to Σ) if there exists a homeomorphism h of Σ onto itself such that $h(X)$ is contained in a tame arc in Σ, and a 0-dimensional F_σ set F in Σ is said to be *tame* (*relative to* Σ) if F can be expressed as the countable union of tame (relative to Σ) compact subsets.

We use the symbols Cl, Bd, Int to denote the topological closure, boundary, and interior, respectively.

Other relevant terms are defined in [4].

2. Approximations to polyhedra and tame 0-dimensional sets.

LEMMA 1. *Suppose Σ is an $(n-1)$-sphere in $S^n (n \geqq 6)$ and W is a component of $S^n - \Sigma$ such that there exist triangulations R of Σ of arbitrarily small mesh for which W is* 1-ULC *in $W \cup (\Sigma - R^{(2)})$. Let f be a map of Δ^2 into* Cl *W such that $f(\partial\Delta^2) \subset W$ and ε be a positive number. Then there exist a map $g: \Delta^2 \to \mathrm{Cl}\, W$ and a triangulation T of Σ satisfying* (i) $\rho(g, f) < \varepsilon$, (ii) $g|\partial\Delta^2 = f|\partial\Delta^2$, (iii) *mesh* $T < \varepsilon$, (iv) $g(\Delta^2) \cap T^{(2)} = \emptyset$, *and* (v) *the diameter of each component of $g(\Delta^2) \cap \Sigma$ is less than ε.*

Proof. By [5, Cor. 2C.2.1] or [6, Th. 3.2] we can assume that $f^{-1}(f(\Delta^2) \cap \Sigma)$ is 0-dimensional.

Step 1. Determine a positive number δ so small that any δ-subset of Σ is contained in an open $(n-1)$-cell in Σ of diam $< \varepsilon/4$, and cover $f^{-1}(f(\Delta^2) \cap \Sigma)$ by a finite collection of very small, pairwise disjoint, open 2-cells $Y_1, Y_2, \cdots, Y_k$ in Int Δ^2. Use general position techniques to approximate f by a map $s: \Delta^2 \to S^n$ such that

$$(1) \qquad s|\Delta^2 - \bigcup Y_i = f|\Delta^2 - \bigcup Y_i\,,$$

$$(2) \qquad \rho(s, f) < \varepsilon/4\,,$$

$$(3) \qquad \mathrm{diam}\, s(Y_i) < \min\{\delta, \varepsilon/4\} \qquad (i = 1, \cdots, k)\,,$$

$$(4) \qquad s(Y_i) \cap s(Y_j) = \emptyset \qquad (1 \leqq i < j \leqq k)\,.$$

Choose a positive number α such that

$$(5) \quad \alpha < (1/3)\rho(s(Y_i) \cap \Sigma, s(Y_j) \cap \Sigma) \qquad (1 \leqq i < j \leqq k)\,,$$

and choose a triangulation T of Σ such that

$$(6) \qquad \mathrm{mesh}\, T < \min\{\alpha, \varepsilon/4\}\,,$$

$$(7) \qquad W \text{ is 1-ULC in } W \cup (\Sigma - T^{(2)})\,.$$

Step 2. By Condition (3) each set $s(Y_i) \cap \Sigma$ is contained in a small $(n-1)$-cell in Σ. Let X denote the component of $\Delta^2 - s^{-1}(s(\Delta^2) \cap \Sigma)$ containing $\partial\Delta^2$, and observe that $\Delta^2 - X \subset \bigcup Y_i$. Apply Tietze's Extension Theorem to extend $s \mid \mathrm{Bd}(Y_i - X)$ to a map of $\mathrm{Cl}(Y_i - X)$ into an $(n-1)$-cell in Σ of diam $< \varepsilon/4$, thereby defining a map $t: \Delta^2 \to \mathrm{Cl}\, W$ such that

$$(8) \qquad t \mid X = s \mid X ,$$

$$(9) \qquad \operatorname{diam} t(Y_i) < \varepsilon/2 ,$$

$$(10) \qquad \rho(t, s) < \varepsilon/2 ,$$

$$(11) \qquad t^{-1}(t(\Delta^2) \cap \Sigma) = \Delta^2 - X .$$

Step 3. In this step we indicate how to approximate t so that the images of distinct Y_i's are disjoint. For $i = 1, 2, \cdots, k$ let R_i denote $\bigcup_{j \neq i} \mathrm{St}(s(Y_j) \cap \Sigma, T)$, and let Q_i denote $\mathrm{St}(s(Y_i) \cap \Sigma, T)$. The choice of α and T implies

$$(12) \qquad Q_i \cap R_i = \emptyset \qquad (i = 1, 2, \cdots, k) .$$

For $i = 1, 2, \cdots, k$ there exists a compact 2-manifold-with-boundary H_i such that

$$(13) \qquad \mathrm{Bd}(Y_i - X) \subset \mathrm{Int}\, H_i \subset H_i \subset Y_i ,$$

$$(14) \qquad t(H_i) \cap \Sigma \subset \mathrm{Int}\, Q_i .$$

By approximating $t \mid \Delta^2 - \mathrm{Cl}\, X: \Delta^2 - \mathrm{Cl}\, X \to \Sigma$ by a general position map, we can assume in addition that

$$(15) \quad t(Y_i - \mathrm{Cl}\, X) \cap t(Y_j - \mathrm{Cl}\, X) = \emptyset \qquad (1 \leqq i < j \leqq k) ,$$

$$(16) \qquad t(\Delta^2 - \mathrm{Cl}\, X) \cap T^{(2)} = \emptyset .$$

With the techniques of [5, Cor. 2C.2.1] or [6, Lemma 3.1] we construct a map $u: \Delta^2 \to \mathrm{Cl}\, W$ very close to t such that

$$(17) \qquad \rho(u, s) < \varepsilon/2 ,$$

$$(18) \qquad \operatorname{diam} u(Y_i) < \varepsilon/2 ,$$

$$(19) \qquad u \mid \Delta^2 - \bigcup H_i = t \mid \Delta^2 - \bigcup H_i ,$$

$$(20) \qquad \mathrm{Int}\, H_i \cap u^{-1}(u(\Delta^2) \cap \Sigma) \text{ is 0-dimensional} ,$$

$$(21) \qquad u(H_i) \cap \Sigma \subset \mathrm{Int}\, Q_i \qquad (1 \leqq i \leqq k) .$$

Since W is 1-ULC in $W \cup (S^n - T^{(2)})$, we can adjust the map a little at points of $\mathrm{Int}\, H_i \cap u^{-1}(u(\Delta^2) \cap \Sigma)$ to make certain that $u(\mathrm{Int}\, H_i) \cap$

$T^{(2)} = \emptyset$, and by Conditions (11), (16), and (19) we have

$$u(\Delta^2) \cap T^{(2)} = \emptyset \, . \tag{22}$$

For $i = 1, 2, \cdots, k$ let $B_i = Y_i - (H_i \cup X)$. By taking a general position approximation to $u \mid \bigcup B_i$, keeping $u(\Delta^2 - \bigcup B_i)$ fixed, we can assume that

$$(\text{dual}\,(n-4)\text{-skeleton of } T) \cap R_i \cap u(B_i) = \emptyset \, . \tag{23}$$

Now we push each $u(B_i) \cap R_i$ very close to $T^{(2)} \cup \operatorname{Bd} R_i$, thereby defining a map $g\colon \Delta^2 \to \operatorname{Cl} W$ such that

$$g \mid \Delta^2 - \bigcup B_i = u \mid \Delta^2 - \bigcup B_i \, , \tag{24}$$

$$g(\bigcup B_i) \subset \Sigma \, , \tag{25}$$

$$\rho(g, u) < \operatorname{mesh} T < \varepsilon/4 \, , \tag{26}$$

$$\operatorname{diam} g(Y_i) < \varepsilon (1 \leqq i \leqq k) \, , \tag{27}$$

$$g(B_i) \cap g(H_j) = \emptyset \qquad (\text{all } j \neq i) \, . \tag{28}$$

By continuing to require general position approximations, we can choose g so that

$$g(B_i) \cap g(B_j) = \emptyset \qquad (1 \leqq i < j \leqq k) \, , \tag{29}$$

$$g(\bigcup B_i) \cap T^{(2)} = \emptyset \, . \tag{30}$$

It follows from Conditions (11), (19), and (24) that $g(\Delta^2) \cap \Sigma$ is contained in $g(\bigcup((Y_i - X) \cup H_i))$; from (22), (24), and (30) that $g(\Delta^2) \cap T^{(2)} = \emptyset$; and from (21), (28), and (29) that $\Sigma \cap g(Y_i) \cap g(Y_j) = \emptyset$ whenever $i \neq j$. Furthermore, Conditions (2), (17), and (26) imply that $\rho(g, f) < \varepsilon$. Thus, g is the required map.

LEMMA 2. *Under the hypotheses of Lemma* 1, *there exists a map g of Δ^2 into* Cl W *such that $\rho(g, f) < \varepsilon$ and $g(\Delta^2) \cap \Sigma$ is a tame* 0-*dimensional subset of Σ.*

Proof. Let $\varepsilon_1, \varepsilon_2, \cdots$ be a sequence of positive numbers such that $\Sigma \varepsilon_i < \varepsilon$. Apply Lemma 1 repeatedly to obtain a sequence $\{g_n\}$ of maps of Δ^2 into Cl W and a sequence $\{{}_nT\}$ of triangulations of Σ satisfying

$$\rho(g_n, g_{n-1}) < \varepsilon_n \, , \tag{1}$$

$$g_n \mid \partial \Delta^2 = f \mid \partial \Delta^2 \, , \tag{2}$$

$$\operatorname{mesh} {}_nT < \varepsilon_n \, , \tag{3}$$

(4) $$g_n(\Delta^2) \cap \Sigma \cap {}_nT^{(2)} = \varnothing \ .$$

(5) The diameter of each component of $\Sigma \cap g_n(\Delta^2)$ is less than ε_n. Here g_0 denotes f. In addition, after g_n is defined, the remaining maps g_{n+r} are required to be so close to g_n that

(6) there exists a neighborhood U_n of ${}_nT^{(2)}$ such that $g_{n+r}(\Delta^2) \cap U_n = \varnothing$,

(7) there exist finitely many open (in S^n) sets $E_{n,1}, E_{n,2}, \cdots, E_{n,k(n)}$, each of diameter less than ε_n and having pairwise disjoint closures such that

$$g_{n+r}(\Delta^2) \cap \Sigma \subset \bigcup_{i=1}^{k(n)} E_{n,i} \ ,$$

$$g_{n+r} \mid \Delta^2 - g_{n+r}^{-1}\Big(\bigcup_{i=1}^{k(n)} E_{n,i}\Big) = g_n \mid \Delta^2 - g_n^{-1}\Big(\bigcup_{i=1}^{k(n)} E_{n,i}\Big) \ .$$

Let $g = \lim g_n \circ \cdots \circ g_1$. Clearly g is a map of Δ^2 into Cl W such that $\rho(g, f) < \varepsilon$. That $g(\Delta^2) \cap \Sigma$ is 0-dimensional follows from Condition (7): $g(\Delta^2) \cap \Sigma$ is covered by $\bigcup_{i=1}^{k(n)} \mathrm{Cl}\, E_{n,i}$ for every n, and these sets can be expanded to give an open cover of $g(\Delta^2) \cap \Sigma$ by arbitrarily small, pairwise disjoint, open subsets of S^n. Furthermore, Condition (6) implies that $g(\Delta^2) \cap {}_nT^{(2)} = \varnothing$ for every n, from which it is easy to demonstrate that $\Sigma - (g(\Delta^2) \cap \Sigma)$ is 1-ULC. Consequently, the tameness of $g(\Delta^2) \cap \Sigma$ follows from [9].

THEOREM 3. *If Σ is an $(n-1)$-sphere in $S^n (n \geqq 6)$ and W is a component of $S^n - \Sigma$ such that there exist triangulations R of Σ of arbitrarily small mesh for which W is 1-ULC in $W \cup (\Sigma - R^{(2)})$, then there exists a tame (relative to Σ) 0-dimensional F_σ set F such that $W \cup F$ is 1-ULC.*

Proof. For each $\varepsilon > 0$ there exists a finite collection of open sets $\{V_i\}$ covering S^n such that any map of $\partial\Delta^2$ into $V_i \cap W$ extends to a map f of Δ^2 into an ε-subset of Cl W. By Lemma 2 there exists a map g of Δ^2 into an ε-subset of Cl W such that $g \mid \partial\Delta^2 = f \mid \partial\Delta^2$ and $g(\Delta^2) \cap \Sigma$ is a tame 0-dimensional subset of Σ. Since there are just a countable number of homotopy classes of maps of $\partial\Delta^2$ in $V_i \cap W$, one can define a set F as the countable union of sets of the form $g(\Delta^2) \cap \Sigma$ so as to make it obvious that W is 1-ULC in $W \cup F$. The argument given for Theorem 4.2 of [2] can be used to prove from this that $W \cup F$ is 1-ULC.

The hypothesis regarding $R^{(2)}$ in these first three results may appear unnatural, but it is necessary in order to deal effectively with a sphere Σ which is well-behaved (in terms of Theorm 3) from only one of its complementary domains. In the following theorem, the main result of this section, where one must consider both complementary domains of Σ, one naturally resorts to the condition that $R^{(2)}$ be tame. Seebeck has established a similar theorem [12, Th. 4] for an m-manifold Σ in S^n such that $n - m > 1$. Using different means Bryant has demonstrated the equivalence of Conditions 1 and 3 [3, Th. 3].

THEOREM 4. *Let Σ denote an $(n-1)$-sphere in $S^n (n \geqq 6)$. The following statements are equivalent*:

1. *There exist triangulations R of Σ of arbitrarily small mesh for which $R^{(2)}$ is tame relative to S^n.*
2. *There exists a tame (relative to Σ) 0-dimensional F_σ set F in Σ such that, for each component W of $S^n - \Sigma$, $W \cup F$ is* 1-ULC.
3. *For each k-dimensional polyhedron $P (k \leqq n-3)$ topologically embedded in Σ and $\varepsilon > 0$, there exists an ε-push h of (Σ, P) such that $h(P)$ is tame relative to S^n.*

Proof. Clearly, (3) implies (1). We shall prove that (1) implies (2) and (2) implies (3).

Assume (1). Let W_1 and W_2 denote the components of $S^n - \Sigma$, and T a triangulation of Σ for which $T^{(2)}$ is tame in S^n. Any very small loop in W_i is contractible in a very small subset of $(S^n - T^{(2)})$. The technique of Step 2 in Lemma 1 can be used to cut off this contraction on a small subset of Σ, and a general position approximation (in Σ), as in Step 3 of Lemma 1, can be used to force the contraction to operate in a small subset of $\mathrm{Cl}(W_i) - T^{(2)}$.

Consequently, Theorem 3 implies the existence of a tame (relative to Σ) 0-dimensional F_σ set F_i such that $W_i \cup F_i$ is 1-ULC $(i = 1, 2)$. Let $F = F_1 \cup F_2$. Clearly, W_i is 1-ULC in $W_i \cup F (i = 1, 2)$ and the argument of [2, Th. 4.2] can be applied again to prove that each $W_i \cup F$ is 1-ULC.

Assume (2). Construct an ε-push h of (Σ, P) such that $h(P) \cap F = \varnothing$. It is relatively easy to prove that $S^n - h(P)$ is 1-ULC. Hence, by [4, Th. 3] and [10, Th. 1], $h(P)$ is tame.

Furthermore, the argument in the preceding paragraph, omitting the last sentence, produces the following result.

COROLLARY 5. *Suppose Σ is an $(n-1)$-sphere in $S^n (n \geqq 6)$ satisfying any of the statements of Theorem 4. For each k-dimensional compactum $K (k \leqq n-3)$ in Σ and positive number ε there exists an ε-push h of (Σ, K) such that $S^n - h(K)$ is* 1-ULC.

REMARK 1. The examples constructed in [7] indicate that the hypothesis of Theorem 3 is necessary. There exists an n-cell C in $S^n (n \geqq 4)$ that is locally tame modulo a Cantor set, but some 2-cell in $\Sigma = \partial C$ cannot be pushed to a tame 2-cell by a small push of Σ. Thus, no tame (relative to Σ) 0-dimensional F_σ set F will cause $(S^n - C) \cup F$ to be 1-ULC (at least in the case $n \geqq 5$; a somewhat more complicated contradiction can be found for the case $n = 4$).

REMARK 2. In case $n = 4$ the equivalence of Statements (1) and (2) in Theorem 4 can be demonstrated with methods more elementary than those developed here; however, in case $n = 5$, the equivalence of these statements is an open question.

In case Σ is only partially wild, there is another condition implying the existence of the F_σ set, perhaps simpler than that of Theorem 4. Quite obviously it cannot stand as a necessary condition; any Σ locally tame modulo a Cantor set tame relative to Σ but wild relative to S^n would serve as a counterexample.

THEOREM 6. *If the $(n - 1)$-sphere Σ in $S^n (n \geqq 6)$ is locally tame modulo an $(n - 3)$-dimensional set X and each tame Cantor set in Σ is tame relative to S^n, then Σ contains a tame (relative to Σ) 0-dimensional F_σ set F such that, for each component W of $S^n - \Sigma$, $W \cup F$ is* 1-ULC.

Proof. The idea here is elementary: for each tame 2-complex P in Σ and $\varepsilon > 0$, we build an ε-push of (Σ, P) such that $h(P) \cap X$ is 0-dimensional. This is accomplished by pushing the 1-skeletons of increasingly fine triangulations of P off X. The hypothesis $\dim X \leqq n - 3$ makes this possible by guaranteeing that near each arc A in Σ is an arc A' in $\Sigma - X$.

As a result $h(P)$ is locally tame modulo a Cantor set in $h(P) \cap X$. By hypothesis such a Cantor set is tame in S^n. From this one can prove quite easily that $h(P)$ is tame by showing that $S^n - h(P)$ is 1-ULC.

This means that Σ contains triangulations of arbitrarily small mesh having tame (in S^n) 2-skeletons, and the desired conclusion follows from Theorem 4.

3. Side approximations to the boundary of a cell.

Bing's Side Approximation Theorem [1, Th. 16] has been so essential to the study of embeddings of surfaces in S^3 that there may be value in making some observations about generalizations to it in higher dimensions. In the definitions that follow Σ denotes an $(n - 1)$-sphere in S^n and W a component of $S^n - \Sigma$. We say that Σ can be *almost approximated from* W if for each $\varepsilon > 0$ there exists an ε-homeomorphism h of Σ into S^n and there exists a finite collection $E_1, E_2, \cdots, E_k$ of open, pairwise disjoint, ε-sets in Σ such that $h(\Sigma) \cap \Sigma$ is contained in $\bigcup E_i$ and the diameter of each component of $h(\Sigma) - W$ is less than ε. We say that Σ can be *strongly almost approximated from* W if the preceding defining properties are satisfied with the additional hypothesis that the E_i's be open $(n-1)$-cells on Σ. If the reembeddings h can be obtained so that, in addition, $h(\Sigma)$ is locally flat, then we say that Σ can be (strongly) *almost approximated from* W *by locally flat spheres.*

The examples of [7] indicate that some spheres Σ can be almost approximated from a complementary domain W but cannot be strongly almost approximated from W. This is clarified by the remark following Corollary 5 and by Theorem 8, which relates, for the case where $S^n - W$ is an n-cell, the existence of a tame (relative to Σ) 0-dimensional F_σ set F such that $W \cup F$ is 1-ULC and the property that Σ can be strongly almost approximated from W.

LEMMA 7. *Suppose Σ is an $(n-1)$-sphere in S^n and W is a component of $S^n - \Sigma$ such that Σ can be strongly almost approximated from W. Then Σ contains a tame (relative to Σ) 0-dimensional F_σ set F such that $W \cup F$ is* 1-ULC.

Since the details of this argument would read like a too-familiar story, we sketch a brief outline. For each map $f\colon \Delta^2 \to \mathrm{Cl}\, W$ such that $f(\partial\Delta^2) \subset W$ and each $\varepsilon > 0$, we cut off f, first on a very close approximation Σ' to Σ and then on some small cells containing $\Sigma' \cap \Sigma$, in order to define a map $g\colon \Delta^2 \to \mathrm{Cl}\, W$ such that $\rho(g, f) < \varepsilon$, $g \mid \partial\Delta^2 = f \mid \partial\Delta^2$, and $g(\Delta^2) \cap \Sigma$ is contained in the union of finitely many pairwise disjoint, open $(n-1)$-cells in Σ, each of diam $< \varepsilon$. We then follow the procedures in the proof of Lemma 2 to obtain a sequence $\{g_n\}$ of maps of Δ^2 into $\mathrm{Cl}\, W$ that converges to a map $g\colon \Delta^2 \to \mathrm{Cl}\, W$ such that $\rho(g, f) < \varepsilon$, $g \mid \partial\Delta^2 = f \mid \partial\Delta^2$, and $g(\Delta) \cap \Sigma$ is covered by collections of finitely many pairwise disjoint, arbitrarily small, open $(n-1)$-cells in Σ. Lemma 2 of [9] implies then that $g(\Delta^2) \cap \Sigma$ is a tame 0-dimensional subset of Σ.

THEOREM 8. *Suppose the $(n-1)$-sphere Σ in $S^n (n \geqq 5)$ bounds an n-cell C. Then Σ contains a tame (relative to Σ) 0-dimensional F_σ set F such that $(S^n - C) \cup F$ is* 1-ULC *if and anly if Σ can be strongly almost approximated from $S^n - C$ by locally flat spheres.*

Proof. The sufficiency half of the theorem is an immediate consequence of Lemma 7.

Assume that Σ contains a tame 0-dimensional F_σ set F such that $(S^n - C) \cup F$ is 1-ULC, and let ε denote a positive number. There exists a countable collection $\{D_i\}$ of tame $(n-1)$-cells in Σ such that $\bigcup D_i \supset F$, diam $D_i < \varepsilon/2$ for all i, $D_i \cap D_j = \emptyset$ whenever $i \neq j$, and lim diam $D_i \to 0$ as $i \to \infty$.

Choose a null sequence $\{C_i\}$ of n-cells in C such that diam $C_i < \varepsilon/2$, $C_i \cap \Sigma = D_i$, and $C_i \cap C_j = \emptyset$ whenever $i \neq j$, in such a way that there exists an $(\varepsilon/2)$-homeomorphism f of C onto $\mathrm{Cl}\,(C - \bigcup C_i)$ satisfying

(1) $f \mid \Sigma - \bigcup D_i = 1$

(2) $f(\bigcup \operatorname{Int} D_i) \subset \operatorname{Int} C$

(3) $f(\Sigma)$ is locally flat in C at each point of $f(\cup \operatorname{Int} D_i)$.

Clearly $S^n - f(C)$ is 1-LC at each point of $f(\cup D_i)$. To see that $S^n - f(C)$ is 1-LC at the other points of $f(\Sigma)$, observe that the C_i's can be used to prove that each very small loop in $S^n - f(C)$ is homotopic, in a small subset of $S^n - f(C)$, to a loop in $S^n - C$, which is contractible in a small subset of $(S^n - C) \cup F \subset S^n - f(C)$. By [11, Th. 9], $f(\Sigma)$ is flat.

As a result, there exists an $(\varepsilon/2)$-homeomorphism g of S^n onto itself such that $gf(\Sigma) \cap f(C) = \emptyset$. Let $h = gf$. Observe that $h(\Sigma) \cap \Sigma$ is contained in $\bigcup \operatorname{Int} D_i$. Since the D_i's form a null sequence of open subsets of Σ, $h(\Sigma)$ can intersect only a finite number of the D_i's. Consequently, the locally flat sphere $h(\Sigma)$ strongly almost approximates (for this choice of ε) Σ from $S^n - C$.

As indicated in the comments following Theorem 9 of [11], we could require the approximating spheres to be PL rather than locally flat, which would better reflect the spirit of Bing's work. However, Theorem 8 as stated has an immediate generalization to closed PL $(n-1)$-manifolds Σ in S^n that are collared from one side, and such a generalization could not be obtained so easily were the locally flat condition replaced by a PL one.

COROLLARY 9. *Suppose Σ is an $(n-1)$-sphere in $S^n (n \geqq 6)$ that bounds an n-cell C. Then the equivalent statements of Theorem 4 all hold if and only if Σ can be strongly almost approximated from $S^n - C$ by locally flat spheres.*

COROLLARY 10. *Let C^* denote an $(n-1)$-cell in $S^{n-1} (n \geqq 5)$ and C the natural suspension of C^* in S^n, the suspension of S^{n-1}. Then the boundary Σ of C can be strongly almost approximated from $S^n - C$ by locally flat spheres.*

Proof. By [8, Cor. 7] there exists a tame (relative to Σ) 0-

dimensional F_σ set F in Σ such that $(S^n - C)$ is 1-ULC in $(S^n - C) \cup F$ (equivalently: $(S^n - C) \cup F$ is 1-ULC).

That the D_i's in the proof of Theorem 8 constitute a null sequence of tame $(n - 1)$-cells enabled us to prove that $h(\Sigma)$ intersects Σ in the union of finitely many $(n - 1)$-cells. If we were to use an arbitrary null sequence of open sets $\{D_i\}$ on Σ covering F, we could construct an associated null seqnence C_i of open sets in C, with C_i homeomorphic to $D_i \times [0, 1)$, in such a way that the argument there will establish the following result.

THEOREM 11. *Suppose the $(n - 1)$-sphere Σ in $S^n (n \geqq 5)$ bounds an n-cell C and contains a 0-dimensional F_σ set F such that $(S^n - C) \cup F$ is 1-ULC. Then Σ can be almost approximated from $S^n - C$ by locally flat spheres.*

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Received January 5, 1973. Partially supported by NSF Grant GP-19966.

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