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HELLY AND RADON-TYPE THEOREMS IN INTERVAL CONVEXITY SPACES

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The notion of interval convexity T on a point set S is defined. If T is an interval convexity defined on S, $\mathscr{C}(T)$ will denote the collection of nonempty T-convex subsets of S. Properties $k, H(k)$ (a Helly property), and $R(k, n)$ (a Radon property) are defined on $\mathscr{C}(T)$, and relationships between these properties are investigated.

A partial order convexity $\leqq$ on a point set S is a special type of interval convexity. Some sufficient conditions are imposed on $\leqq$ and $\mathscr{C}(\leqq)$ to insure the existence of certain Radon-type properties.

1. Introduction. Suppose S is a point set, and $\mathscr{P}(S)$ is the collection of nonempty subsets of S. The statement that T is an *interval convexity* on S means that T is a transformation from $S \times S$ into $\mathscr{P}(S)$. A subset M of S is said to be *T-convex* provided that $T(x, y)$ is a subset of M for every x and y in M. Let $\mathscr{C}(T)$ denote the collection of all nonempty T-convex subsets of S. For each $M \in \mathscr{P}(S)$, the *convex hull* of M relative to T, denoted by $\mathrm{Co}\,(M)$, is the intersection of the elements of $\mathscr{C}(T)$ which contain M. We assume that if each of x and y is in S, then $T(x, y)$ is T-convex, $T(x, y)$ contains x and y, and $T(x, y) = T(y, x)$.

Let *m-set* mean a set of m points of S. A subset M of S is said to be *n-divisible* provided it may be partitioned into n mutually exclusive subsets whose T-convex hulls have a common point of S. In this paper we consider the relationship of the following Helly and Radon-type properties on a set S with an interval convexity T. $\mathscr{C}(T)$ has *property* $R(k)$ if each $(k+1)$-set of S is 2-divisible, and more generally, $\mathscr{C}(T)$ has *property* $R(k, n)$ with respect to some integer valued function f if each $[f(k, n)]$-set is n-divisible. We say that $\mathscr{C}(T)$ has *property* $r(k)$ if k is the smallest integer for which $\mathscr{C}(T)$ has property $R(k)$. $\mathscr{C}(T)$ is said to have *property* $H(k)$ provided that if $\mathscr{G}$ is a finite subcollection of $\mathscr{C}(T)$ containing at least k elements, then the following two statements are equivalent:

(a) Each k elements of $\mathscr{G}$ have a common point.

(b) The elements of $\mathscr{G}$ have a common point.

In (2) we give sufficient conditions for property $R(k)$ to be equivalent to property $H(k)$. We also consider in (2) the existence of sets with property $R(k)$ in partially ordered spaces and more generally, in (3) the existence of sets with property $R(k, n)$.

2. Theorems concerning properties k, $R(k)$, and $H(k)$. From a theorem of Levi [7] we have that property $R(k)$ implies property $H(k)$. In [1] Calder introduces the following property: $\mathscr{C}(T)$ has *property* k provided that if M is a finite point set containing at least $k+1$ points, then there exists a point p such that $p \in \mathrm{Co}\,[M \sim \{m\}]$ for each m in M. He proves that property k is equivalent to property $H(k)$ and then proves that property $R(k)$ is equivalent to property $H(k)$ in a partially ordered space. It should be noted that the partial order does not have to be antisymmetric. Calder also gives an example of an interval convexity T such that property $H(k)$ is not equivalent to property $R(k)$ in $\mathscr{C}(T)$. In the first two theorems of this section we give sufficient conditions on T for properties $H(k)$ and $R(k)$ to be equivalent.

If each of A and B is in $\mathscr{P}(S)$, then $A * B$ denotes the set

$$\bigcup_{a \in A, b \in B} T(a, b)\ .$$

THEOREM 2.1. *Let T be an interval convexity on S such that for each M in $\mathscr{P}(S)$, $\mathrm{Co}\,(M) = M * M$; and if a, b, c, and d are four distinct points such that d is in $T(a, b)$ and $T(a, c)$, then b is in $T(a, c)$, or c is in $T(a, b)$. Then property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathscr{C}(T)$.*

THEOREM 2.2. *Let T be an interval convexity on S such that for each M in $\mathscr{P}(S)$, $\mathrm{Co}\,(M) = \bigcup_{m \in M} T(m, m)$. Then property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathscr{C}(T)$.*

The proofs of Theorems 2.1 and 2.2 are easy modifications of the proof of Theorem 3.2 of Calder [1].

EXAMPLE 2.1. Let M be a subset of a linear space S. A subset K of M is said to be *extremal* provided that if k is an element of K, and there exist elements x and y in M such that $k = tx + (1 - t)y$ for some $t \in (0, 1)$, then x and y are elements of K. Obviously, the union and intersection of any collection of extremal subsets of M are extremal.

We define an interval convexity T on M as follows: If each of x and y is an element of M, $T(x, y)$ is the intersection of the extremal subsets of M which contain $\{x, y\}$.

For each subset K of M, $K \subset \bigcup_{k \in K} T(k, k)$. Since $\bigcup_{k \in K} T(k, k)$ is convex, $\mathrm{Co}\,(K) \subset \bigcup_{k \in K} T(k, k)$. However, $\bigcup_{k \in K} T(k, k) \subset \mathrm{Co}\,(K)$. Thus $\mathrm{Co}\,(K) = \bigcup_{k \in K} T(k, k)$, and hence property $H(k) \Leftrightarrow$ property $R(k)$ in $\mathscr{C}(T)$.

Let $\leqq$ be a partial order on the set S. If each of x and y is a point of S, $[x, y] = \{p \mid p = s$, or $p = y$, or $x < p < y$, or $y < p < x\}$.

A subset M of S is said to be *$\leqq$-convex* if for all elements x and y of M, $[x, y]$ is a subset of M. The collection of all $\leqq$-convex subsets of S is denoted by $\mathscr{C}(\leqq)$. In [5], Franklin shows that $\text{Co}(M) = M * M$ for any M in $\mathscr{P}(S)$.

THEOREM 2.3. *Suppose $\leqq$ is a partial order on S, and S is the union of n linearly ordered sets, $S_1, S_2, \cdots, S_n$. Then $\mathscr{C}(\leqq)$ has property $R(2n)$.*

Proof. Suppose $M = \{x_1, x_2, \cdots, x_{2n+1}\}$ is a $(2n+1)$-set. Then for some i, $1 \leqq i \leqq n$, S_i contains at least three points, z_1, z_2, z_3, of M such that $z_1 < z_2 < z_3$. Thus $\text{Co}\{z_2\}$ and $\text{Co}\{z_1, z_3\}$ have a common point, and therefore $\mathscr{C}(\leqq)$ has property $R(2n)$.

It is easy to show that $\mathscr{C}(\leqq)$ has property $r(2)$ if and only if $\leqq$ linearly orders S. Suppose $\leqq$ is a partial order on S which does not linearly order S. Under these conditions on $\leqq$, does $\mathscr{C}(\leqq)$ have property $r(3)$ if and only if S is union of two mutually exclusive, linearly ordered subsets S_1 and S_2? The following example shows the answer to this question is no.

EXAMPLE 2.2. Let $S = \{(x, y) \in R^2 \mid y = 0 \text{ or } y = 1\}$. Define $\leqq$ on S as follows: $(x_1, y_1) \leqq (x_2, y_2)$ if $y_1 = y_2$ and $x_1 \leqq x_2$. Thus $\leqq$ is a partial order on S which does not linearly order S. However, $\leqq$ does linearly order $S_1 = \{(x, 1) \in R^2\}$ and $S_2 = \{(x, 0) \in R^2\}$, and $S = S_1 \cup S_2$. To show that $\mathscr{C}(\leqq)$ does not have property $r(3)$ we choose $M = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Obviously M is not 2-divisible.

3. Property $R(k, n)$. Tverberg shows in [11] that the collection on convex sets in R^{k-1} has property $R(k, n)$ with respect to $f(k, n) = (n-1)k + 1$ for $n, k \geqq 2$. By putting suitable restrictions on T, we have the following:

THEOREM 3.1. *Suppose T is an interval convexity on S such that if $M \in \mathscr{P}(S)$, then $\text{Co}(M) = \bigcup_{m \in M} T(m, m)$. If $\mathscr{C}(T)$ has property $R(k)$, then $\mathscr{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n-1)k + 1$ for $n \geqq 2$.*

Proof. (We use induction on n.) Suppose $\mathscr{C}(T)$ has property $R(k)$, i.e., $\mathscr{C}(T)$ has property $R(k, 2)$. Suppose further that $\mathscr{C}(T)$ has property $R(k, m)$ for some $m \geqq 2$, and let $M = \{x_1, x_2, \cdots, x_{mk+1}\}$ be an $[mk+1]$-set. Let $M_1 = \{x_1, x_2, \cdots, x_{(m-1)k+1}\}$ be the subset of M containing the first $(m-1)k+1$ points of M. Then there exist m points, $y_{11}, y_{12}, \cdots, y_{1m}$, of M_1 and a point p_1 such that

$$p_1 \in \bigcap_{i=1}^m \operatorname{Co}\{y_{1i}\} .$$

Now choose $M_2 = \{x_1, x_2, \cdots, x_{(m-1)k+1}, x_{(m-1)k+2}\} \sim \{y_{11}\}$. Thus M_2 is an $[(m-1)k+1]$-set, and hence there exist m points, $y_{21}, y_{22}, \cdots, y_{2m}$, of M_2 and a point $p_2 \in \bigcap_{i=1}^m \operatorname{Co}\{y_{2i}\}$.

Continuing this process we get $M_j = [M_{j-1} \cup \{x_{(m-1)k+j}\}] \sim \{y_{j-11}\}$ for $3 \leqq j \leqq k+1$, and each of the sets is an $[(m-1)k+1]$-set. Thus there exist m points, $y_{j1}, y_{j2}, \cdots, y_{jm}$, of M_j and a point $p_j \in \bigcap_{i=1}^m \operatorname{Co}\{y_{ji}\}$.

Let $K = \{p_1, p_2, \cdots, p_{k+1}\}$. If $p_i = p_j$ for some $i \neq j$, the theorem is proved. Suppose $p_i \neq p_j$ if $i \neq j$. Since $\mathscr{C}(T)$ has property $R(k)$, there exist points, p_i and p_j, $i < j$, in K and a point

$$p_0 \in \operatorname{Co}\{p_i\} \cap \operatorname{Co}\{p_j\} .$$

Since for each $x \in S$, $T(x, x)$ is convex, we have $p_0 \in \operatorname{Co}\{y_{i1}\} \cap \operatorname{Co}\{y_{j1}\} \cap \cdots \cap \operatorname{Co}\{y_{jm}\}$. Thus M is $(m+1)$-divisible and $\mathscr{C}(T)$ has property $R(k, m+1)$. Therefore, $\mathscr{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n-1)k+1$ for all $n \geqq 2$.

EXAMPLE 3.1. In R^2 let l/P and $\overline{l/P}$ denote, respectively, the open and the closed half planes determined by the line l and containing the point P. PQ denotes the line determined by the points P and Q, and $P[m]$ denotes the line through P with slope m. Let $P_0 = (0, 0)$, $P_1 = (1, 0)$, $P_2 = (-1/2, \sqrt{3}/2)$, $P_3 = (-1/2, -\sqrt{3}/2)$, $P_4 = (1, 1)$, $P_5 = (-1, 0)$, $P_6 = (1, -1)$. Choose $S = S_1 \cup S_2 \cup S_3$ where $S_1 = P_0P_1/P_4 \cap P_0P_2/P_4$, $S_2 = P_0P_2/P_5 \cap P_0P_3/P_5$, and $S_3 = P_0P_1/P_6 \cap P_0P_3/P_6$ We define an interval convexity T on S as follows:

$$\text{(a)} \quad T(P, P) = \begin{cases} S_1 \cap \overline{P[-\sqrt{3}]/P_0} & \text{if} \quad P \in S_1 ; \\ S_2 \cap \overline{P[\sqrt{3}]/P_0} & \text{if} \quad P \in S_2 ; \\ S_3 \cap \overline{P[0]/P_0} & \text{if} \quad P \in S_3 . \end{cases}$$

$$\text{(b)} \quad T(P, Q) = T(P, P) \cup T(Q, Q) .$$

Thus if $M \in \mathscr{P}(S)$, $\operatorname{Co}(M) = \bigcup_{m \in M} T(m, m)$. It is easily seen that $\mathscr{C}(T)$ has property $r(3)$. Thus if $k \geqq 3$, $\mathscr{C}(T)$ has property $R(k, n)$ with respect to $f(k, n) = (n-1)k+1$ for $n \geqq 2$.

THEOREM 3.2. *Suppose $\leqq$ is a partial order on S such that $\mathscr{C}(\leqq)$ has property $R(k)$. Then $\mathscr{C}(\leqq)$ has property $R(k, n)$ with respect to $f(k, n) = (2n-3)k+1$ for all $n \geqq 2$.*

Proof. (The proof is a slight modification of the proof of Theorem 3.1.) The statement is true for $n = 2$ since property $R(k)$ is the same as property $R(k, 2)$. Now suppose the statement is true for

$n = m$, and let $M = \{x_1, x_2, \cdots, x_{(2m-3)k+1}, \cdots, x_{(2m-1)k+1}\}$ be a $[(2m-1)k + 1]$-set. (Note that $[(2m-1)k+1] - [(2m-3)k+1] = 2k$.) Let $K_0 = \{x_1, x_2, \cdots, x_{(2m-3)k+1}\}$ be the subset of M containing the first $(2m-3)k+1$ points of M. Thus there exist m mutually exclusive subsets, $K_{01}, K_{02}, \cdots, K_{0m}$, of K_0 and a point $y_0 \in \bigcap_{i=1}^m \mathrm{Co}(K_{0i})$. It follows then that there exist points s_0 and t_0 in K_0 such that $s_0 < y_0 < t_0$. Now let K_1 be the $[(2m-3)k+1]$-set $[K_0 \sim \{s_0, t_0\}] \cup \{x_{(2m-3)k+2}, x_{(2m-3)k+3}\}$. Again there exist m mutually exclusive subsets, $K_{11}, K_{12}, \cdots, K_{1m}$, of K_1 such that $\bigcap_{i=1}^m \mathrm{Co}(K_{1i}) \neq \varnothing$. If $y_0 \in \bigcap_{i=1}^m \mathrm{Co}(K_{1i})$, the theorem is proved.

Suppose $y_0 \notin \bigcap_{i=1}^m \mathrm{Co}(K_{1i})$. Let $y_1 \in \bigcap_{i=1}^m \mathrm{Co}(K_{1i})$. Then there exist points s_1 and t_1 in K_1 such that $s_1 < y_1 < t_1$.

Continuing this process for $2 \leqq i \leqq k$, we obtain $K_i = [K_{i-1} \sim \{s_{i-1}, t_{i-1}\}] \cup \{x_{(2m-3)k+2i}, x_{(2m-3)k+(2i+1)}\}$ and correspondingly m mutually exclusive subsets, $K_{i1}, K_{i2}, \cdots, K_{im}$, of K_i such that $\bigcap_{p=1}^m \mathrm{Co}(K_{ip})$ contains a point y_i. Now if for some j and i, $0 \leqq j < i \leqq k$, $y_j \in \bigcap_{p=1}^m \mathrm{Co}(K_{ip})$, the theorem is proved.

Suppose that if $0 \leqq j < i \leqq k$, $y_j \notin \bigcap_{p=1}^m \mathrm{Co}(K_{ip})$. Then the $(k+1)$-set $C = \{y_0, y_1, \cdots, y_k\}$ is 2-divisible. Let C_1 and C_2 be mutually exclusive subsets of C such that $\mathrm{Co}(C_1) \cap \mathrm{Co}(C_2) \neq \varnothing$. It can be shown that if $w \in \mathrm{Co}(C_1) \cap \mathrm{Co}(C_2)$ then there are $m+1$ mutually exclusive subsets, $M_1, M_2, \cdots, M_{m+1}$, of M such that $w \in \bigcap_{i=1}^{m+1} \mathrm{Co}(M_i)$. Hence M is $m+1$ divisible. Therefore, $\mathscr{C}(\leqq)$ has property $R(k, n)$ with respect to $f(k, n) = (2n-3)k+1$ for all $n \geqq 2$.

REFERENCES

1. J. R. Calder, *Some elementary properties of interval convexities*, J. London Math. Soc., **3** (1971), 422-428.

2. L. Danzer, N. Grümbaum, and V. Klee, *Helly's theorem and its relatives*, Proceedings of Symposia in Pure Mathematics, **7** (1963), Convexity, 101-177.

3. J. Eckhoff, *Der Satz von Radon in honvexen Productstrukturen I — II*, Monatsh. Math., **72** (1968), 303-314, and **73** (1969), 7-30.

4. J. W. Ellis, *A general set-separation theorem*, Duke Math. J., **19** (1952), 417-421.

5. S. P. Franklin, *Some results on order convexity*, Amer. Math. Monthly, **69** (1962), 357-359.

6. David Kay and E. W. Womble, *Axiomatic convexity theory and relationships between the Caratheordory, Helly and Radon numbers*, Pacific J. Math., **38** (1971), 471-485.

7. F. W. Levi, *On Helly's theorem and the axioms of convexity*, J. Indian Mathematical Society, (N. S.) Part A, **15** (1951), 65-76.

8. T. S. Motzkin, *Linear Inequalities*, Mimeographed lecture notes, University of California, Los Angeles, 1951.

9. J. R. Reay, *Caratheordory theorems in convex product structure*, Pacific J. Math., **35** (1970), 227-230.

10. ———, *An extension of Radon's theorem*, Illinois J. Math., **12** (1968), 184-189.

11. H. Tverberg, *A generalization of Radon's theorem*, J. London Math. Soc., **41** (1966), 123-128.

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