TWO PROPERTIES OF THE SORGENFREY PLANE

William G. Bade

The Sorgenfrey line S is the real line with the topology generated by the half open intervals $[a, b)$. R. H. Sorgenfrey proved that S is paracompact, while $S \times S$ is not paracompact, or even normal. The two main results of this paper are that $S \times S$ is strongly zero-dimensional, and that every real continuous function on $S \times S$ is in the first Baire class for the Euclidean topology of the plane. These results answer questions asked by P. Nyikos.

Interest in $S \times S$ arises from the important problem of whether the product of strongly zero-dimensional spaces is strongly zero-dimensional. The proof given for $S \times S$ shows the difficulties which are involved in settling the general case. On the other hand, the algebra $C_b(S \times S)$ of all bounded continuous functions on $S \times S$ is an interesting example of a function algebra related to the Baire classes. Its structure space $\beta(S \times S)$ is shown to be totally disconnected.

Since this paper was written we have learned that the strong zero-dimensionality of S^2 was proved independently and simultaneously by Mrowka [3] and by two pairs: P. Nyikos—P. Roy and R. Heath—D. Lutzer (their proofs will appear in a paper written by Nyikos). Mrowka [3] proved that each continuous function on S^n (n an arbitrary cardinal) is of the first Baire class for the Euclidean topology. Recently Mrowka [4] has proved the strong zero-dimensionality of S^n for every cardinal n. In presenting this paper we claim no priority for the results, as we find they had been announced prior to submission of our paper. However, we believe that the proofs given here are conceptually very simple and intuitive.

1. **Strong zero-dimesionality.** Following Nyikos [5] we call a completely regular space X *totally disconnected* if no connected set has more than one point, and *zero-dimensional* if it has a base of clopen (= open and closed) sets. We say X is *strongly zero-dimensional* if given any two disjoint zero sets Z_1 and Z_2 in X there exists a clopen set C such that $Z_1 \subseteq C$, $Z_2 \cap C = \phi$. For compact spaces these properties are all equivalent (see [1], page 247, where the term "zero-dimensional" is used for "strongly zero-dimensional"). The following general theorem is known. (See [5] where further equivalent conditions are given.)

THEOREM 1.1. *Let X be completely regular. The following conditions are equivalent:*

(i)   *X is strongly zero-dimensional.*

(ii) $\beta(X)$ *is totally disconnected.*

(iii)   *Every cozero set in X is a countable of clopen sets.*

The Sorgenfrey plane $S \times S$ has as base for its topology the clopen boxes $[a, b) \times [c, d)$. If $p = (x, y) \in S \times S$ and $\sigma > 0$ we write $B(p, \sigma) = [x, x + \sigma) \times [y, y + \sigma)$ and call it the clopen square cornered at p with side σ. In what follows, there will be an interplay between the Sorgenfrey and Euclidean topologies of the plane. We denote these topologies by $\mathscr{S}$ and $\mathscr{E}$ respectively. Thus $\text{int}_{\mathscr{E}}(A)$ means the Euclidean interior of A, $\text{cl}_{\mathscr{S}}(A)$ is the closure of A for $\mathscr{S}$, etc. The term clopen always refers to $\mathscr{S}$.

The Sorgenfrey plane is a zero-dimensional completely regular space. To get a feeling for its asymmetrical character, note that the relative topology on any line with negative slope is discrete, while a line with positive slope is homeomorphic in its relative topology to S. The Sorgenfrey line S is strongly zero-dimensional [5]. We shall prove that $S \times S$ is strongly zero-dimensional by establishing (iii) above. Our first task is to determine a sufficiently large class of clopen sets in $S \times S$. The clopen boxes are insufficient, since there are obvious clopen sets such as the triangle

$$x + y \geqq 1 , \qquad 0 < x < 1 , \qquad 0 < y < 1$$

which are not countable unions of clopen boxes. We give a way of generating clopen sets from $\mathscr{E}$-compact sets.

LEMMA 1.2.   *Let K be $\mathscr{E}$-compact, $\sigma > 0$ and*

$$W = \bigcup_{p \in K} B(p, \sigma) .$$

If $K \cap \text{int}_{\mathscr{E}}(W) = \phi$, then W is clopen.

Proof. Clearly W is $\mathscr{S}$-open. We show it is $\mathscr{S}$-closed. Suppose that q_0 is a point of W^c such that every clopen square $B(q_0, \varepsilon)$ intersects W. Taking $\varepsilon = 1/n$, we can find a sequence of points $q_n \in B(q_0, 1/n) \cap W$. Then $\{q_n\}$ converges to q_0 for both $\mathscr{E}$ and $\mathscr{S}$. Now $q_n \in B(p_n, \sigma)$, where $\{p_n\}$ is a sequence of points in K. Without loss of generality we can suppose $\{p_n\}$ converges for $\mathscr{E}$ to a point $p_0 \in K$, and that the Euclidean distance of each p_n from p_0 is less than $\sigma/2$. Then $q_0 \in \text{cl}_{\mathscr{E}} B(p_0, \sigma)$. Since $q_0 \notin W$, it must lie on one of the two open sides of $B(p_0, q)$.

Suppose without loss of generality that q_0 lies on the upper open side; i.e., $q = (x, y_0 + \sigma)$ where $p_0 = (x_0, y_0)$ and $0 \leqq x \leqq x_0 + \sigma$. We investigate where the points p_n must lie.

(1) No $p_n = (x_n, y_n)$ can have $y_n \leqq y_0$. For no neighborhood $B(q_0, \varepsilon)$ could contain a point q_n.

(2) No p_n lies in $\text{int}_{\mathscr{E}} B(p_0, \sigma)$, since then $K \cap \text{int}_{\mathscr{E}}(W) \neq \phi$.

(3) It must be true then that for all n

$$x_0 - \sigma < x_n \leqq x_0 ,$$
$$y_0 < y_n < y_0 + \sigma , \qquad n = 1, 2, \cdots .$$

However, since $p_n \to p_0$ for $\mathscr{E}$, the horizontal line segment

$$(t, y_0 + \sigma) , \qquad x_0 \leqq t < x_0 + \sigma ,$$

lies in $\bigcup_{n=1}^{\infty} B(p_n, \sigma)$. Thus unless $q_0 = (x_0 + \sigma, y_0 + \sigma)$, we would have $q_0 \in W$.

(4) If $q_0 = (x_0 + \sigma, y_0 + \sigma)$, then by (3) for each $\varepsilon > 0$, $B(q_0, \varepsilon) \cap B(p_n, \sigma) = \phi$. Thus no such square $B(q_0, \varepsilon)$ can contain a point q_n.

These cases show that it is impossible that $p_n \to p_0$, and we have the required contradiction.

The condition on K cannot be dropped. For example take K to be the segment $x = y$, $0 \leqq x \leqq 1$, and $\sigma = 1$. The point $(1, 0)$ is an $\mathscr{S}$-limit point of $\bigcup_{k \in K} B(k, 1)$ but is not a member of this set.

THEOREM 1.3. *Let V be any $\mathscr{S}$-open set in $S \times S$. There exists a sequence $\{Q_n\}$ of clopen sets such that*

$$V \subseteqq \bigcup_{n=1}^{\infty} Q_n \subseteqq \text{cl}_{\mathscr{S}} V .$$

Proof. It is clearly sufficient to consider the case where V is bounded for $\mathscr{E}$. Let $\sigma > 0$ and define

$$A_\sigma = \{p \in \text{cl}_{\mathscr{S}} V \mid B(p, \sigma) \subseteqq \text{cl}_{\mathscr{S}} V\} .$$

Then A_σ is $\mathscr{E}$-compact. For let $a_n \in A$, $a_n \to a_0$ for $\mathscr{E}$. We have

$$\text{int}_{\mathscr{E}} B(a_0, \sigma) \subseteqq \bigcup_{n=1}^{\infty} B(a_n, \sigma) \subseteqq \text{cl}_{\mathscr{S}} V .$$

Thus $B(a_0, \sigma) \subseteqq \text{cl}_{\mathscr{S}} V$, so $a_0 \in A_\sigma$. Now define

$$K_\sigma = A_\sigma \sim \text{int}_{\mathscr{E}}(\text{cl}_{\mathscr{S}} V)$$

and

$$W_\sigma = \bigcup_{k \in K_\sigma} B(k, \sigma) .$$

Then K_σ is $\mathscr{E}$-compact, and

$$K_\sigma \cap \text{int}_{\mathscr{E}} W_\sigma = \phi .$$

It follows from Lemma 1.2 that W_σ is clopen.

We prove now that V can be covered by a sequence of clopen sets contained in $\mathrm{cl}_{\mathscr{S}} V$. First suppose that $p \in V \sim \mathrm{int}_{\mathscr{E}}(\mathrm{cl}_{\mathscr{S}} V)$. Since V is $\mathscr{S}$-open, there exists $\delta > 0$ such that $B(p, \delta) \subseteq V$. Thus $p \in K_{\delta} \subseteq W_{\delta}$. If $\delta_1 < \delta_2$ we have $K_{\delta_1} \subseteq K_{\delta_2}$. Thus

$$V \sim \mathrm{int}_{\mathscr{E}}(\mathrm{cl}_{\mathscr{S}} V) \subseteq \bigcup_{n=1}^{\infty} K_{1/n} \subseteq \bigcup_{n=1}^{\infty} W_{1/n} \subseteq \mathrm{cl}_{\mathscr{S}} V \;.$$

Further $\mathrm{int}_{\mathscr{E}}(\mathrm{cl}_{\mathscr{S}} V)$ is open for $\mathscr{E}$, and hence is the union of a sequence $\{Z_n\}$ of clopen boxes. Letting $Q_n = Z_n \cup W_{1/n}$, we have

$$V \subseteq \bigcup_{n=1}^{\infty} Q_n \subseteq \mathrm{cl}_{\mathscr{S}} V \;.$$

An open set U for $\mathscr{S}$ is called *regular* if $U = \mathrm{int}_{\mathscr{S}}(\mathrm{cl}_{\mathscr{S}} U)$.

COROLLARY 1.4. *Every regular open set in $S \times S$ is a countable union of clopen sets, and hence a cozero set.*

Theorem 1.3 also solves the problem of strong zero-dimensionality for $S \times S$.

THEOREM 1.5. *The Sorgenfrey plane is strongly zero-dimensional.*

Proof. We verify (iii) of Theorem 1.1. Let U be a cozero set in $S \times S$. Then $U = \bigcup_{n=1}^{\infty} U_k$ where each U_k is a cozero set with

$$U_k \subseteq \mathrm{cl}_{\mathscr{S}} U_k \subseteq U_{k+1} \;.$$

By Theorem 1.3 each U_k has a countable cover by clopen sets contained in U, so U is a countable union of clopen sets.

In the proof of Theorem 1.5 the Euclidean topology seems to play an essential role in several ways—particularly in identifying clopen sets in the product other than the obvious clopen boxes. Any attack on the general problem of whether the class of strongly zero-dimensional spaces is closed under products must surmount this difficulty of providing a rich source of clopen sets.

Finally we give an example of an open set in $S \times S$ which is not a countable union of clopen sets. Let U be the "open" triangle $x + y > 1$, $0 < x < 1$, $0 < y < 1$ together with those points on the hypotenuse whose coordinates are irrational. Then U is $\mathscr{S}$-open. If U were a countable union of clopen sets it would be a cozero set. It follows from Theorem 2.1 below that U would be an F_σ set for $\mathscr{E}$, i.e., $U = \bigcup_{n=1}^{\infty} K_n$ where each K_n is $\mathscr{E}$-compact. This is impossible, since the set of irrationals on the diagonal is not itself an F_σ set.

2. Baire measurability. The following result answers a question asked by P. Nyikos[1].

THEOREM 2.1. *Every $\mathscr{S}$-continuous real function f on $S \times S$ is in the first Baire class for the Euclidean topology of the plane.*

Proof. For a point $p = (x, y) \in S \times S$, let

$$C(p, \sigma) = [x, x + \sigma] \times [y, y + \sigma]$$

be the closed square of side σ cornered at p. The continuity of f on $S \times S$ can be expressed by the condition that for each $p \in S \times S$ and $\varepsilon > 0$, there exists a closed square $C(p, \delta)$ such that

(♯) $\qquad |f(q) - f(p)| < \varepsilon$, if $q \in C(p, \delta)$.

We first show that f is Lebesgue measurable. Let $\varepsilon > 0$ be fixed and let $\mathscr{F}$ be the collection of all closed squares $C(p, \delta)$, for points $p \in S \times S$ such that relation (♯) holds. The collection $\mathscr{F}$ covers the plane in the sense of Vitali. Thus by the Vitali Covering Theorem (as in [2], page 366) there exists a disjoint sequence $\{C_n\} \subseteqq \mathscr{F}$ such that

$$m\!\left((S \times S) \sim \bigcup_{n=1}^{\infty} C_n\right) = 0 \,,$$

where m is Lebesgue measure. Define the function g on $\bigcup_{n=1}^{\infty} C_n$ as follows: if $C_n = C(p, \delta)$, let

$$g(q) = f(p) \,, \qquad q \in C_n \,.$$

At other points let $g(p) = 0$. Then g is Lebesgue (in fact Borel) measurable and

$$|g(p) - f(p)| < \varepsilon$$

almost everywhere for m. Taking successively $\varepsilon = 1/n$, we obtain that f is almost everywhere the uniform limit of Borel measurable functions, and hence Lebesgue measurable.

Now for each n let $G_n = \{q \mid |f(q)| \leqq n\}$, and define the averaging functions

$$h_n(p) = n^2 \int_{C(p, 1/n)} f(q) k_{G_n}(q) \, dm \,, \qquad n = 1, 2, \cdots.$$

Then h_n is $\mathscr{E}$-continuous. Because f is $\mathscr{S}$-continuous

$$\lim_{n \to \infty} h_n(p) = f(p) \,, \qquad p \in S \times S \,,$$

so f is of the first Baire class.

---

[1] In Problem 40 from the Mathematics Problem Book of Carnegie-Mellon University he asks whether such f are Borel measurable.

References

1. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Princeton, 1960.
2. E. J. McShane, *Integration*, Princeton, 1947.
3. S. Mrowka, *Some problems releted to N-compact spaces*, to appear.
4.————, *All powers of the Sorgenfrey space are strongly 0-dimensional*, to appear.
5. P. Nyikos, *Not every 0-dimensional real compact space is N-compact*, Bull. Amer. Math. Soc., **77** (1971), 392–396.
6. R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc., **53** (1947), 631–632.

UNIVERSITY OF CALIFORNIA, BERKELEY