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LAPLACE TRANSFORM METHODS IN MULTIVARIATE SPECTRAL THEORY

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The Laplace transform of the semigroup $\exp(tA)$ generated by an operator A gives the resolvent of A. An integral formula is obtained for the Laplace transform of $\exp(tA+B)$, where B is another operator which does not commute with A. The new transform has analytic continuation to the same domain as the resolvent, but the analytic continuation is not single-valued. The integral formula is then applied to the joint spectral theory of noncommutative operators. Explicit computations with matrices of degree two illustrate the results.

1. Introduction. Any bounded linear operator A on a Banach space generates a semigroup $\exp(tA)$, $0 \leqq t < \infty$, and the Laplace transform $\mathscr{L}(s, A)$ of this semigroup converges for $\operatorname{Re} s$ sufficiently large and equals the resolvent $(s-A)^{-1}\cdot\mathscr{L}(s, A)$ therefore has unique analytic continuation to the component containing ∞ of the resolvent set A.

Multivariate problems requiring integration of $\exp(\sum t_i A_i)$ one variable at a time, lead us to consider the Laplace transform $\mathscr{L}(s, A, B)$ of $\exp(tA+B)$, $0 \leqq t < \infty$, where B is a fixed bounded operator.

The main result is:

THEOREM 1. *$\mathscr{L}(s, A, B)$ has the contour integral representation*
(1.1)

$$(s-A)^{-1} + \int_s^\infty (u-A)^{-1}B(u-A)^{-1}\exp[B(u-A)^{-1}(u-s)]du$$

valid for $\operatorname{Re} s$ *sufficiently large. Therefore, $\mathscr{L}(s, A, B)$ can be analytically continued along any arc not intersecting $\sigma(A)$.*

Examples are given in §5 which show that the analytic continuation is not always unique.

In §§3 and 4 our result is applied to problems in spectral theory. According to the Weyl functional calculus [1], [2], [3], two self-adjoint operators have a joint spectral distribution in the plane, which if A and B commute is simply the tensor product of their spectral measures. Two operators A and B which are merely bounded have instead a two-dimensional Laplace transform $\mathscr{L}(s, \sigma, A, B)$ which if A and B commute is simply the product of their resolvents. $\mathscr{L}(s, \sigma, A, B)$ may be regarded as a functional on the space of entire functions on C^2. Its carrier may be regarded as a joint spectrum of A, B. Although there is no unique minimal carrier in general for functionals of this type, Theorem 1 can be exploited to obtain information about the carriers in terms of the spectral properties of A and B. It turns out that the actual spectrum of A can be used to construct a carrier, if accuracy with respect to B is sacrificed.

Suppose now that A is bounded and B is self-adjoint. The Weyl calculus for the three self-adjoint operators $\operatorname{Re} A$, $\operatorname{Im} A$, and B gives a spectrum projecting onto the whole numerical range of A. However, in § 4 we construct a hybrid functional for A and B which is an analytic functional with respect to A. The motivating question is whether the carrier of this functional will still be the whole numerical range of A or whether the actual spectrum of A will reappear. Theorem 6 gives the transition between the two competing theories and offers no help in shrinking the carrier. But the examples of § 5 show that in some cases the actual spectrum of A does suffice as the carrier.

2. Proof of Theorem 1.

Proof of Theorem 1. The Laplace transform of $\exp(tA + B)$ cannot be computed directly unless B commutes with A, in which case the trivial result is:

$$\mathscr{L}(s, A, B) = \exp(B)\mathscr{L}(s, A) .$$

We therefore resort to the following contour integral method.

Let C be a simple closed curve containing $\sigma(A)$ (spectrum of A) in its interior. Then C also encloses $\sigma(A + t^{-1}B)$ for $|t|$ greater than some constant k, and by the Riesz functional calculus [6] applied to the operator $A + t^{-1}B$, for $|t| > k$,

$$\exp(tA + B) = \exp(t(A + t^{-1}B)) = \frac{1}{2\pi i}\oint_C e^{tz}[z - (A + t^{-1}B)]^{-1}dz .$$

If in addition

$$|t| > k_1 = \sup_{z \notin \operatorname{int} C} \| B(z - A)^{-1} \|$$

then

$$\begin{aligned} [z - A - t^{-1}B]^{-1} &= [(I - t^{-1}B(z - A)^{-1})(z - A)]^{-1} \\ &= (z - A)^{-1} \sum_{n=0}^{\infty} [B(z - A)^{-1}]^n t^{-n} \end{aligned}$$

and

$$\exp(tA + B) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} e^{zt}t^{-n}(z - A)^{-1}[B(z - A)^{-1}]^n dz$$
$$= \frac{1}{2\pi i}\oint_C \sum_{n,j=0}^{\infty} \frac{z^j t^{j-n}}{j!}(z - A)^{-1}[B(z - A)^{-1}]^n dz \, .$$

The double series in the integrand is absolutely and uniformly convergent on the domain $z \in C$, $\max(k, k_1) < |t| < k_2$ where k_2 is any constant. The contour integral can therefore be evaluated term-by-term. All terms having $j < n$ are $O(|z|^{-2})$ for large z, and so by enlargement of the contour, they vanish. The remaining terms can therefore be rewritten as the sum

$$\sum_{q=0}^{\infty} t^q \frac{1}{2\pi i}\int_C (z - A)^{-1} \sum_{p=0}^{\infty} \frac{z^{p+q}}{(p + q)!}[B(z - A)^{-1}]^p dz \, .$$

This power series expansion of the entire function $\exp(tA + B)$ is valid in the annular region given above, and consequently holds for all t.

Since $\exp(tA + B)$ has exponential growth rate, the Laplace transform of its power series expansion may be taken term-by-term. This fact is discussed fully in Widder's book on the Laplace transform [8]. Therefore

$$\mathscr{L}(s, A, B) = \sum_{q=0}^{\infty} \frac{q!}{s^{q+1}} \frac{1}{2\pi i}\oint_C (z - A)^{-1} \sum_{p=0}^{\infty} \frac{z^{p+q}}{(p + q)!}[B(z - A)^{-1}]^p dz \, .$$

Next we note that

$$\sum_{q,p=0}^{\infty} \frac{q!}{s^{q+1}} \frac{z^{p+q}}{(p + q)!}[B(z - A)^{-1}]^p$$

is absolutely and uniformly convergent when $z \in C$, $|z/s| < l < 1$, l being any constant less than one.

To reduce the double series to closed form, consider

$$F(a, b) = \sum_{q,p=0}^{\infty} a^q b^p \frac{q!}{(p + q)!} \, .$$

If the series $\sum_{q=0}^{\infty} a^{q+p} q!/(p + q)!$ is differentiated p times, we obtain a geometric series which converges to $(1 - a)^{-1}$. By elementary means, therefore,

$$\sum_{q=0}^{\infty} a^q \frac{q!}{(p + q)!} = \frac{1}{a^p}\int_0^a \frac{(a - t)^{p-1}}{(p - 1)!} \frac{dt}{1 - t} \, , \qquad p \geqq 1$$

and

$$F(a, b) = \frac{1}{1-a} + \sum_{p=1}^{\infty}\left(\frac{b}{a}\right)^p \int_0^a \frac{(a-t)^{p-1}}{(p-1)!}\frac{dt}{1-t}$$
$$= \frac{1}{1-a} + b\sum_{p=1}^{\infty} b^{p-1}\int_0^1 \frac{(1-u)^{p-1}}{(p-1)!}\frac{du}{1-au}$$
$$= \frac{1}{1-a} + b\int_0^1 e^{b(1-u)}\frac{du}{1-au} .$$

We now substitute $a = z/s$ and $b = zB(z-A)^{-1}$.

$$\mathscr{L}(s, A, B) = \frac{1}{2\pi i s}\oint_C (z-A)^{-1}\left\{\left(1-\frac{z}{s}\right)^{-1}\right.$$
$$\left. + zB(z-A)^{-1}\int_0^1 \exp\,[zB(z-A)^{-1}(1-u)]\left(1-\frac{z}{s}u\right)^{-1}du\right\}dz$$
$$= (s-A)^{-1} + \int_0^1 \frac{1}{2\pi i}\oint_C z(z-A)^{-1}B(z-A)^{-1}$$
$$\cdot \exp\,[zB(z-A)^{-1}(1-u)]\left(\frac{s}{u}-z\right)^{-1}dz\frac{du}{u} .$$

Since the integrand of the contour integral is holomorphic in the neighborhood of $z = \infty$ ontside C, the Cauchy integral formula yields

$$\mathscr{L}(s, A, B) = (s-A)^{-1}$$
$$+ \int_0^1 \frac{s}{u}\left(\frac{s}{u}-A\right)^{-1}B\left(\frac{s}{u}-A\right)^{-1}\exp\left[\frac{s}{u}B\left(\frac{s}{u}-A\right)^{-1}(1-u)\right]\frac{du}{u} .$$

Replacing u by s/u, we get Theorem 1:

$$\mathscr{L}(s, A, B) = (s-A)^{-1}$$
$$+ \int_s^{\infty}(u-A)^{-1}B(u-A)^{-1}\exp\,[B(u-A)^{-1}(u-s)]\,du .$$

COROLLARY 2. *$\mathscr{L}(s, A, B)$ has unique analytic continuation to R_∞, the component of the resolvent set of A containing ∞, iff for every component σ_i of $\sigma(A)$ meeting $\bar{R}_\infty$, any contour C_i enclosing component σ_i only, and for all $j \geqq 1$,*

$$\oint_{C_i}(u-A)^{-1}[B(u-A)^{-1}]^j \exp\,[B(u-A)^{-1}u]du = 0 . \tag{2.1}$$

Proof. Suppose the contour integral of Theorem 1 is continued along two different arcs terminating at s. The difference between the values of s so obtained is, by homotopy arguments, an integral combination of the closed contour integrals

$$\oint_{C_i}(u-A)^{-1}B(u-A)^{-1}\exp\,[B(u-A)^{-1}(u-s)]du$$

or

$$\frac{d}{ds}\oint_{C_i}(u-A)^{-1}\exp[B(u-A)^{-1}(u-s)]du\ .$$

The result follows by power series expansion in s.

COROLLARY 3. *The Laplace transform* $\mathscr{L}(s, \sigma, A, B)$ *of* $\exp(tA+\xi B)$ *is given by*

$$\sigma^{-1}(s-A)^{-1}+\int_s^\infty(u-A)^{-1}B(u-A)^{-1}[\sigma-B(u-A)^{-1}(u-s)]^{-2}du \tag{2.2}$$

for $\sigma>0$, $s>0$ *sufficiently large.*

Proof. Replace B by ξB in the formula for $\mathscr{L}(s, A, B)$. The integration with respect to ξ is elementary.

3. Analytic functionals in spectral theory. Suppose the bounded operators $A_1, \cdots, A_n$ are all self-adjoint, so that for $\xi\in R^n$, $\exp(i\xi\cdot A)$ is a unitary operator. Then by Fourier inversion a tempered distribution $\mathscr{F}^{-1}\exp(i\xi\cdot A)$ is determined. In previous papers by the present author, [1], [2], [3], this distribution was called the "joint spectral distribution" of $A_1, \cdots, A_n$ and denoted $T(A)$.

In order to gain further insight into this type of spectral distribution, we consider the slightly different case when $iA_1, \cdots, iA_n$ are assumed only to be the generators of contraction semigroups. This is equivalent to the condition that $A_1, \cdots, A_n$ have numerical range in the upper half plane. In this case, so does $\xi\cdot A$ if $\xi_1, \cdots, \xi_n\geqq 0$ (abbreviation $\xi\geqq 0$), so $\|\exp(i\xi\cdot A)\|\leqq 1$ when $\xi\geqq 0$.

DEFINITION. When $iA_1, \cdots, iA_n$ generate contraction semigroups, $S(A)$ denotes the tempered distribution defined for $f\in\mathscr{S}(R^n)$ by

$$S(A)f=(2\pi)^{-n/2}\int_{\xi\geqq 0}(\mathscr{F}f)(\xi)\exp(i\xi\cdot A)d\xi\ . \tag{3.1}$$

In one dimension, simple computation shows that

$$S(A)f=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(x)\mathscr{L}(x, A)dx \tag{3.2}$$

where the Laplace transform $\mathscr{L}(x, A)=(x-A)^{-1}$, provided the spectrum of A does not intersect the real line.

However, $f\in L^2(R^1)$ may be written as $f=f_++f_-$, where $\mathscr{F}f_+=\mathscr{F}f$ for $x\geqq 0$, $\mathscr{F}f_-=\mathscr{F}f$ for $x\leqq 0$. f_+ is the boundary value of a function $f_+(z)$ holomorphic in the upper half plane, and $|f_+(z)|=$

$0((\operatorname{Im} z)^{-1})$. If C is any contour in the upper half plane enclosing spectrum A, we obtain

$$S(A)f = \frac{1}{2\pi i} \oint_C f_+(z) \mathscr{L}(z, A) dz \, . \tag{3.3}$$

This is just the Riesz calculus (see Ch. XI of [6]), but in two dimensions we can similarly obtain the formula

$$S(A, B)f = \left(\frac{1}{2\pi i}\right)^2 \oint_{C_1} \oint_{C_2} f_+(s, \sigma) \mathscr{L}(s, \sigma, A, B) ds \, d\sigma \tag{3.4}$$

where $\mathscr{L}(s, \sigma)$ is holomorphic for s, σ outside C_1, C_2 respectively, and $\mathscr{F} f_+ = \mathscr{F} f$ for $\xi \geqq 0$, $\mathscr{F} f_+ = 0$ otherwise.

Formula (3.4) defines a continuous linear functional on the space of entire functions in two complex variables, and the numerical range of A, B need not be restricted. Such functionals are discussed, for example, in Hormanders' book [5]. Such functionals are in one-to-one correspondence with entire functions of exponential growth, in our case $\exp(i\xi \cdot A)$. In one dimension, there is a canonical representation of a functional similar to (3.2), but not in higher dimensions. If K_i denotes the compact set bounded by C_i, and $K = K_1 \times K_2$, then $\| S(A, B)f \| \leqq c \sup_{s,\sigma \in K} | f(s, \sigma) |$, so K is an example of a "carrier" of $S(A, B)$. In general, there is no unique minimal carrier of a functional in dimension greater than 1.

LEMMA 4. *If K_1, K_2 contain neighborhoods of the numerical ranges of A, B resp., then $K = K_1 \times K_2$ is a carrier of $S(A, B)$. If K_1 is simply connected and contains the spectrum of A in its interior, then there exists K_2 such that $K = K_1 \times K_2$ is a carrier of $S(A, B)$.*

Proof. $\mathscr{L}(s, \sigma, A, B)$ is holomorphic when $s, \sigma > 0$ if A and B have numerical range in the left half-plane. By translating and rotating A and B independently, the general result is obtained. The second result follows by inspection of formula (2.2) in Corollary 3.

4. A hybrid functional.

DEFINITION. Let iA generate a contraction semigroup and let B be self-adjoint. Then the tempered distribution ST (A, B) is defined for $f \in \mathscr{S}(R^2)$ by

$$\mathrm{ST}\,(A, B)f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_0^{\infty} (\mathscr{F} f)(t, \xi) \exp\left(i(tA + \xi B)\right) dt \, d\xi \, . \tag{4.1}$$

NOTATION. Given $f \in \mathscr{S}(R^2)$, let $f_+(z, x)$ denote the analytic continuation to $\operatorname{Im} z \geqq 0$ of the function $f_+ \in L^2(R^2)$ satisfying

$$(\mathscr{F} f_+)(t, \xi) = \begin{cases} (\mathscr{F} f)(t, \xi) , & t \geqq 0 \\ 0 & t < 0 . \end{cases}$$

Note that for each fixed $z_0, f_1(z_0, x) \in \mathscr{S}(R^1)$.

LEMMA 5. *For each z outside the closure of the numerical range of A, there is a tempered distribution $\Phi(z) \in \mathscr{S}'(R^1)$ acting on $\varphi(x) \in \mathscr{S}(R^1)$, such that $\Phi(z)$ is (weakly) holomorphic in z, and such that for a contour C enclosing the numerical range of A,*

$$\operatorname{ST}(A, B)f = \frac{1}{2\pi i} \oint_C \Phi(z) f_+ dz . \tag{4.2}$$

Proof. It is easily checked that

$$\| \exp(tA + i\xi B) \| \leqq \exp(|t| \, \|A\|) .$$

Therefore, $\exp(tA + i\xi B) = \sum_{j=0}^{\infty} t^j G_j(\xi)$ where for all j, $G_j(\xi) \in C^\infty(R^1)$ and

$$\| G_j(\xi) \| \leqq \left(\frac{e\|A\|}{j} \right)^j \text{ uniformly in } \xi .$$

Therefore, for $\varphi(x) \in \mathscr{S}(R^1)$,

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathscr{F}\varphi)(\xi) \exp(tA + i\xi B) d\xi = \sum_{j=0}^{\infty} t^j \Phi_j(\varphi)$$

where $\Phi_j(\varphi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathscr{F}\varphi)(\xi) G_j(\xi) d\xi$ satisfies

$$\| \Phi_j(\varphi) \| \leqq (2\pi)^{-1/2} \left(\frac{e\|A\|}{j} \right)^j \| (\mathscr{F}\varphi) \| .$$

In particular, Φ_j is a tempered distribution on $\mathscr{S}(R^1)$. Define

$$\Phi(z) = \mathscr{L}\left(\sum_{j=0}^{\infty} t^j \Phi_j \right) = \sum_{j=0}^{\infty} \frac{j!}{z^{j+1}} \Phi_j$$

which converges when $|z| > \|A\|$.

By trivial arguments, $\Phi(z)$ has analytic continuation to all z not in the closure of the numerical range of A. The lemma follows immediately for f of the form $\psi(z)\varphi(x)$, which suffices.

THEOREM 6. *Suppose A, B act on Hilbert space, and let $A = \operatorname{Re} A + i \operatorname{Im} A$, where $\operatorname{Re} A$, $\operatorname{Im} A$ are self-adjoint. Then for $\varphi \in \mathscr{S}(R^1)$ and $|z|$ large,*

$$\Phi(z)\varphi = T(\mathrm{Re}\, A, \mathrm{Im}\, A, B)\frac{\varphi(x_3)}{z - (x_i + ix_2)} \tag{4.3}$$

where T is defined for $g(x_1, x_2, x_3) \in \mathscr{S}(R^3)$ as stated at the beginning of § 3 and in [1].

Note. The support of the distribution T contains only (x_1, x_2, x_3) such that $x_1 + ix_2$ is in the closure of the numerical range of A. See [1]. Therefore, (4.3) extends at least to all z outside the closed numerical range of A.

Proof. Both sides expand in Laurent series in z, with coefficient of z^{-j-1} on the left

$$= (j!)\Phi_j(\varphi) = (j!)(2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathscr{F}\varphi)(\xi)G_j(\xi)d\xi$$

and on the right

$$T(\mathrm{Re}\, A, \mathrm{Im}\, A, B)[\varphi(x_3)(x_i + ix_2)^j]$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{+\infty}(\mathscr{F}\varphi)(\xi)T(\mathrm{Re}\, A, \mathrm{Im}\, A, B)[e^{ix_3\xi}(x_i + ix_2)^j]d\xi \ .$$

Now $G_j(\xi)$ is the coefficient of t^j in $\exp(tA + i\xi B)$ or $\sum_{n=0}^{\infty} 1/n!\,(tA + i\xi B)^n$ or, by the monomial substitution rule for T in [1],

$$\sum_{n=0}^{\infty}\frac{1}{n!}T(\mathrm{Re}\, A, \mathrm{Im}\, A, B)[e^{t(x_1+ix_2)}e^{i\xi x_3}] \ .$$

That is,

$$G_j(\xi) = \frac{1}{j!}T(\mathrm{Re}\, A, \mathrm{Im}\, A, B)[e^{ix_3\xi}(x_i + ix_2)^j] \ .$$

Therefore, the two Laurent expansions coincide.

5. Examples. We first examine the hybrid functional in the case when A and B act on the two-dimensional complex Hilbert space.

Let M_1, M_2, M_3 be the three hermitian matrices with eigenvalues ± 1, satisfying $M_iM_j + M_jM_i = 0$, $i \neq j$. (E.g. the Pauli matrices.) Then every 2×2 complex matrix is a unique linear combination of M_1, M_2, M_3, I. Since I commutes with everything, we may as well assume that A, B are linear combinations of M_1, M_2, M_3, and up to unitary equivalence and scale changes we can assume $B = M_1$, $A = i\omega \cdot M$ where ω is a triple of complex numbers.

By simple calculations,

$$\exp(A + i\xi B) = I\cos\sqrt{(\xi + \omega_1)^2 + r^2}$$
$$+ (\xi M_1 + \omega\cdot M)\frac{i\sin\sqrt{(\xi + \omega_1)^2 + r^2}}{\sqrt{(\xi + \omega_1)^2 + r^2}}$$

where $r^2 = \omega_2^2 + \omega_3^2$. Let $\chi(x)$ denote the characteristic function of the interval $-1 \leqq x \leqq 1$, and let δ_1, δ_{-1} denote the unit measures concentrated at the points 1, -1 respectively.

Essentially, the Fourier transforms we need are

$$\mathscr{F}^{-1}\frac{\sin\sqrt{\xi^2 + r^2}}{\sqrt{\xi^2 + r^2}} = \sqrt{\frac{\pi}{2}}J_0(r\sqrt{1 - x^2})\chi(x)\ .$$

$$\mathscr{F}^{-1}(\cos\sqrt{\xi^2 + r^2}) = \sqrt{\frac{\pi}{2}}\left[\delta_1 + \delta_{-1} + \frac{rJ_1(r\sqrt{1 - x^2})}{\sqrt{1 - x^2}}\chi(x)\right].$$

Since these formulae are rather hard exercises in contour integration, it is worth mentioning that the first is a corollary of the standard formula (see [4]) in dimension 3:

$$\mathscr{F}^{-1}\frac{\sin|\xi|}{|\xi|} = (2\pi)^{3/2}\mu\ ,$$

where μ is the uniformly distributed measure on the unit sphere. Our formula follows from this equation by taking the partial Fourier transform with respect to two variables.

In order to obtain $\Phi(z)$, we replace A by tA (i.e., replace ω by $t\omega$) and compute the Laplace transform. For the Laplace transforms of the Bessel functions see Watson [7], p. 386. The result is:

THEOREM 7. *If* $B = M_1$, $A = i\omega\cdot M$, *then*

$$\Phi(z) = I\sqrt{\frac{\pi}{2}}\left[\frac{\delta_1}{z - i\omega_1} + \frac{\delta_{-1}}{z + i\omega_1} + \frac{\chi(x)}{((z - i\omega_1 x)^2 + r^2(1 - x^2))^{3/2}}\right]$$
$$+\left(-M_1\frac{\partial}{\partial x} + i\omega\cdot M\frac{\partial}{\partial z}\right)\sqrt{\frac{\pi}{2}}\frac{\chi(x)}{((z - i\omega_1 x)^2 + r^2(1 - x^2))^{1/2}}\ .$$

In particular, $\Phi(z)$ can be analytically continued to the complement of the set $\{z \mid z = i\omega_1 x \pm ir\sqrt{1 - x^2}, -1 \leqq x \leqq 1\}$ which is the ellipsoid parameterized by $0 \leqq \theta \leqq 2\pi$,

$$z = i\omega_1\cos\theta + ir\sin\theta\ .$$

In the case when ω_2/ω_3 is real, this ellipsoid is the boundary of the numerical range of iA, and Lemma 5 gives the actual domain of $\Phi(z)$. The other extreme is the case when $\omega_1 = 0$ and ω_2/ω_3 is imaginary. Then $\omega_1 = r = 0$ and $\Phi(z)$ is singular only at $z = 0$, although the numerical range of iA is the nontrivial ellipsoid $\{z \mid z = i\omega_2\cos\theta + i\omega_3\sin\theta, 0 \leqq \theta \leqq 2\pi\}$. In the latter case, the singularities of $\Phi(z)$ coincide with the spectrum of iA (i.e., $z = 0$), but the former case shows that the analytic continuation established for $\mathscr{L}(s, A, B)$ does not carry over to the hybrid functional ST (A, B).

To obtain examples of the Laplace transform, we utilize the elementary fact that when $g \in \mathscr{S}(R^1)$, the Laplace transform is obtained from the Fourier transform by the formula

$$(\mathscr{L}g)(is) = \sqrt{2\pi}\int_{-\infty}^{\infty} \frac{(\mathscr{F}^{-1}g)(x)}{s-x}\,dx\ .$$

One of the coefficients in $\mathscr{L}(s, iB, A)$, with A, B the 2×2 complex matrices described above, is therefore

$$\pi\int_{-1}^{1} \frac{J_0(r\sqrt{1-x^2})}{s-x}\,dx\ .$$

This coefficient, like the others, has nonunique analytic continuation to all $s \neq \pm 1$. The difference between two values is an integer times $\pi J_0(r\sqrt{1-s^2})$.

This result is easily extended by analytic continuation to any 2×2 matrices A, B.

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