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CHEBYSHEV CENTERS IN SPACES OF CONTINUOUS FUNCTIONS

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A bounded set F in a Banach space X has a Chebyshev center if there exists in X a "smallest" ball containing F. A Banach space X is said to admit centers if every bounded subset of X has a center. The purpose of this paper is to show that certain spaces of continuous functions admit centers.

1. Introduction. Let X be a real normed linear space, G a subset of X and f an element of X. Then a best approximant, g_*, to f from G (if it exists) is a solution to

$$\inf \{\|g - f\|, g \in G\} . \tag{1.1}$$

It may happen that f is not defined exactly but is known to lie in a bounded set F. It is reasonable then to approximate simultaneously all $f \in F$ by solving

$$\inf \sup \{\|g - f\|, f \in F\} \equiv R_G(F) \tag{1.2}$$

where the inf is taken over all $g \in G$. Thus we may view problem (1.2) as a natural generalization of the best approximation problem (1.1). If $G = X$ then the solutions of (1.2) are called Chebyshev centers of F, following Garkavi [2]. In [3], Kadets and Zamyatin showed that $C([a, b], R)$, the space of real-valued continuous functions on $[a, b]$, admits centers. This means that (1.2) has a solution in $C([a, b], R)$ for F an arbitrary bounded set in $C([a, b], R)$.

The purpose of this note is to show that the Kadets-Zamyatin result holds under much greater generality. Let

$$\Omega = \text{a paracompact Hausdorff space}$$
$$S = \text{a normal space}$$
$$C(A, B) = \text{space of continuous functions from } A \text{ to } B .$$

Our main theorems are:

THEOREM 1. *$C(\Omega, X)$ admits centers if X is a finite dimensional, rotund space.*

THEOREM 2. *$C(S, H)$ admits centers if H is an arbitrary Hilbert space.*

2. *Proof of Theorem* 1.

DEFINITION 2.1. Let X and E be Banach spaces and $F: X \to 2^E$. F is said to be *upper semi-continuous* (u.s.c) if the set $\{x \mid F(x) \subset G\}$ is open in X for every open $G \subset E$. F is said to be *lower semi-continuous* (l.s.c) if the set $\{x \mid F(x) \cap G\}$ is open in X for every open $G \subset E$.

Proof of Theorem 1. We use the following notation:

$$\begin{aligned} F &= \text{fixed but arbitrary bounded set in } C(\Omega, X) \\ \mathscr{N}(t) &= \text{the directed family of open } t\text{-neighborhoods, } t \in \Omega \\ \Omega(t, N) &= \bigcup \{f(s)\} \text{ where } f \in F,\ s \in N \\ A_N(t) &= \text{convex closure of } \Omega(t, N) \\ B(x, K) &= \text{ball of radius } K \text{ centered at } x \\ R(F) &= \text{Chebyshev radius of } F \text{ with respect to } C(\Omega, X) \text{ as} \\ & \text{defined in (1.2).} \end{aligned}$$

Suppose F is bounded by K. Now for each $t \in \Omega$, consider the net $N \to A_N(t)$ defined on $\mathscr{N}(t)$. The range of this net lies in the metric space $\mathscr{F}(B(0, K))$ whose elements are the compact, convex and nonempty subsets of $B(0, K)$. We put

$$A(t) = \lim A_N(t),\ N \in \mathscr{N}(t)\ . \tag{2.2}$$

This limit exists in $\mathscr{F}(B(0, K))$ by virtue of the compactness of this space and the monotonicity of the net $\{A_N(t)\}$, $N \in \mathscr{N}(t)$. It may be verified that

$$A(t) = \bigcap A_N(t),\ N \in \mathscr{N}(t)\ . \tag{2.3}$$

We show that the map $A: \Omega \to \mathscr{F}(B(0, K))$ is u.s.c.. This requires us to choose any nonempty open set $G \subset X$ and then show that $\{t \in \Omega: A(t) \subset G\}$ is open in Ω. Let t_0 belong to this set. Then by (2.2), there is an $N \in \mathscr{N}(t_0)$ for which $A_N(t_0) \subset G$. Hence, if $t \in N$, we have by (2.3) that

$$A(t) \subset A_N(t) = A_N(t_0) \subset G$$

and so A is upper semi-continuous.

Let $R_X(A) = \sup\{R_X(A(t)): t \in \Omega\}$. Following Olech [4], we introduce the map $G: \Omega \to 2^X$ defined by

$$G(t) = \{\beta \in X: A(t) \subset B(\beta, R_X(A))\}\ .$$

Olech proved (under the assumption that X is uniformly rotund which is the same as rotund in finite dimensions) that the values $G(t)$ are compact, convex and nonempty subsets of X and G is lower semi-continuous in t. Thus by appealing to the Michael selection theorem, there is a continuous selection f for G.

It is clear that $\|f - g\| \leqq R_X(A)$ for all $g \in F$. It remains to show that $R_X(A) \leqq R(F)$. Let ε be arbitrary and choose $t \in \Omega$ so that $R_X(A(t)) > R_X(A) - \varepsilon$. Since $f \in C(\Omega, X)$, we may choose $N \in \mathscr{N}(t)$ for which $\operatorname{osc}(f: N) < \varepsilon$. Due to (2.2) and (2.3) we may assume that N has been chosen so "small" that there is a $\gamma \in N$ and $g \in F$ for which $R_X(A) - 2\varepsilon < |g(\gamma) - f(\gamma)| \leqq R(F)$.

3. *Proof of Theorem* 2. The problem with X being infinite dimensional is that we have no right to expect $\lim A_N(t)$, $N \in \mathscr{N}(t)$, to exist as in Theorem 1. Thus the method of proof of Theorem 1 must be abandoned. Nevertheless, Theorem 2 may still be proved.

Proof of Theorem 2. Let $F \subset C(S, H)$ be bounded by K. There exist "approximate centers", call then f_n, such that f_n is within $R(F) + 1/n$ of each element of F. We clearly have for any approximate center f_i and f_j the relationship $\|f_i - f_j\| \leqq 4K$.

Step 1. We show that for arbitrary $\delta > 0$, there exists an $\varepsilon_\delta > 0$ such that for any $(R(F) + \varepsilon_\delta)$-approximate center f_1, we may construct an $(R(F) + \varepsilon_{\delta/2})$-center f_2 such that $f_2 \in B(f_1, \delta)$.

Proof of Step 1. Pick $\varepsilon_\delta > 0$ so that $\delta = (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2}$. Pick g where g is an $(R(F) + \varepsilon_{\delta/2})$-approximate center for F. It is clear that $\|g - f_1\| \leqq 4K$. Let $F(t) = \{f(t): f \in F\}$. By definition of approximate center, for all $t \in S$,

$$B(f_1(t), R(F) + \varepsilon_\delta) \cap B(g(t), R(F) + \varepsilon_{\delta/2}) \supset F(t) .$$

For convenience sake set

$$\begin{aligned} r_1 &= R(F) + \varepsilon_\delta;\ r_2 = R(F) + \varepsilon_{\delta/2} \\ d(t) &= \|f_1(t) - g(t)\| . \end{aligned}$$

Define

$$f_2(t) = f_1(t) + \beta(t)(g(t) - f_1(t))$$

where

$$\beta(t) = \begin{cases} 1 & \text{if } (r_1^2 - r_2^2)/d^2 \geqq 1 \\ ((r_1^2 - r_2^2)/d^2)^{1/2} & \text{if } (r_1^2 - r_2^2)/d^2 < 1 . \end{cases}$$

Note that $0 \leqq \beta(t) \leqq 1$ for all $t \in S$. We now make three claims about f_2.

(1) f_2 is a continuous function, i.e., $f_2 \in C(S, H)$

(2) $\|f_2 - f_1\| \leqq (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2} \leqq \delta$

(3) f_2 is an $(R(F) + \varepsilon_{\delta/2})$-approximate center of F

Proof of (1). Since g and f_1 are continuous functions and $d(t) = \|f_1(t) - f_2(t)\|$ is continuous, $\beta(t)$ is also continuous. This clearly implies the continuity of f_2.

Proof of (2). It suffices to show that $\|f_2(t) - f_1(t)\| \leqq (2\varepsilon_\delta R(F) + \varepsilon_\delta^2)^{1/2} \leqq \delta$ for all $t \in S$. Thus for fixed t_0, $\|f_2(t_0) - f_1(t_0)\| = \|\beta(t_0)(g(t_0) - f_1(t_0))\|$. If $\beta(t_0) = 1$, $r_1^2 - r_2^2 \geqq d^2$ so

$$\|\beta(t_0)(g(t_0) - f_1(t_0))\| = \|g(t_0) - f_1(t_0)\| = d(t_0)$$
$$\leqq (r_1^2 - r_2^2)^{1/2} \leqq \delta \ .$$

If $\beta(t_0) < 1$,

$$\|f_2(t_0) - f_1(t_0)\|^2 = \|\beta(t_0)(f_2(t_0) - f_1(t_0))\|^2$$
$$= ((r_1^2 - r_2^2)/d^2)d^2 = r_1^2 - r_2^2$$

so $\|f_2(t_0) - f_1(t_0)\| \leqq (r_1^2 - r_2^2)^{1/2} \leqq \delta$. This proves (2).

Proof of (3). Since by (1) $f_2 \in C(S, H)$, it suffices to show that for each $t_0 \in S$,

$$B(f_2(t_0), R(F) + \varepsilon_{\delta/2})$$
$$\supset B(f_1(t_0), R(F) + \varepsilon_\delta) \cap B(g(t_0), R(F) + \varepsilon_{\delta/2}) \supset F(t_0) \ .$$

The above is equivalent to showing that for all x such that $\|x - f_1\| \leqq r_1$ and $\|x - g\| \leqq r_2$, then $\|x - f_2\| \leqq r_2$.

Without loss of generality assume f_1 is 0. The above problem then simplifies to showing that the implication $\|x\| \leqq r_1$ and $\|x - g\| \leqq r_2$, then $\|x - f_2\| \leqq r_2$ holds for all $x \in V$ and for all $V \subset H$ where V is a two dimensional subspace containing g. Hence we are reduced to a problem in two dimensional Hilbert space and a few simple applications of the Pythagorean theorem prove the assertion.

Step 2. Let f_1 be any $(R(F) + \varepsilon_{\delta_1})$-approximate center of F. Having defined f_n, take f_{n+1} to be an $(R(F) + \varepsilon_{\delta_{n+1}})$-approximate center such that $f_{n+1} \in B(f_n, \delta_n)$ and $\delta_{n+1} = \delta_n/2$, which we may do by Step 1. Evidently $\varepsilon_{\delta_n} \to 0$ as $n \to \infty$.

Now consider $\{f_n\}_{n=1}^\infty$. For all $i, j \geqq K$, $\|f_i - f_j\| \leqq \|f_i - f_K\| + \|f_K - f_j\| \leqq 2\sum_{n=K}^\infty \delta_1/2^n = \delta_1/2^{K-1}$. So $\{f_n\}_{n=1}^\infty$ is a uniformly convergent sequence with limit point f', $f' \in C(S, H)$. Also for each $g \in F$,

$$\sup\{\|g - f'\|, g \in F\} \leqq \sup\{\|g - f_n\| + \|f_n - f'\|), g \in F\}$$
$$\leqq R(F) + \varepsilon_{\delta_n} + \gamma_n$$

where γ_n is a null sequence. Hence $\sup\{\|g - f'\|, g \in F\} = R(F)$ and f' is a Chebyshev center of F.

REMARK 1. Since paracompact spaces are normal [1], Theorem 2 generalizes Theorem 1 in the case that the range space of the space of continuous functions is a finite dimensional Hilbert space.

REMARK 2. This author was unable to resolve the question whether Theorem 2 holds when the range space of $C(S, H)$ is an arbitrary uniformly convex space.

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