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COMPATIBLE TOPOLOGIES AND CONTINUOUS IRREDUCIBLE REPRESENTATIONS

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Suppose there are two locally compact group topologies on a group and that the sets of irreducible unitary representations of the group continuous with respect to each of the topologies coincide. Then the topologies are equal if they are comparable or there is a normal subgroup open and σ-compact in one of the topological groups. This is a result of Klaus Bichteler's, but the work presented here represents a much shorter method than that used by Bichteler, using little representation theory, but using results involving compatible topologies: Topologies containing in common a Hausdorff topology.

Klaus Bichteler in his papers [1] and [2] presented some rather interesting ideas on the relationships between two locally compact group topologies on a group when the irreducible unitary representations continuous with respect to each of the two topologies coincide. I wish to present here an alternative proof of Bichteler's conclusions, a proof which uses only a few elementary results from representation theory and as such is rather more simple and considerably shorter than the original. Some ideas for proofs can be traced to Bichteler's [2] and Rajagopolan's [5]. However, the key to this proof is found in my work in what I have called compatible topologies. This paper forms part of a Ph.D. thesis submitted to La Trobe University in Melbourne, Australia. The research was supervised by Dr. Graham Elton of that University, and to whom I am much indebted.

I mentioned compatible topologies above; if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two topologies on the one space, we say $\mathscr{A}_1$ and $\mathscr{A}_2$ are *compatible* if there is a Hausdorff topology weaker than both $\mathscr{A}_1$ and $\mathscr{A}_2$. This and the following theorems are taken from my paper [6].

THEOREM 1. *Let G be a group on which are defined two compatible group topologies $\mathscr{A}_1$ and $\mathscr{A}_2$, $(G, \mathscr{A}_1)$ being locally compact and $(G, \mathscr{A}_2)$ locally countably compact. If there is a nonemty $\mathscr{A}_2$-open set which is contained in an $\mathscr{A}_1$-Lindelöf set, then $\mathscr{A}_1 \subseteqq \mathscr{A}_2$. In particular if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two compatible locally compact group topologies on a group G, and $(G, \mathscr{A}_1)$ is σ-compact, then $\mathscr{A}_1 \subseteqq \mathscr{A}_2$.*

THEOREM 2. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two compatible group topologies defined on a group G, such that G is $\mathscr{A}_1$-locally compact and $\mathscr{A}_2$-*

locally countably compact. Suppose there is a subgroup U of G which is contained in an $\mathscr{A}_1$-Lindelöf set and is such that if $\hat{\mathscr{A}_1}$ and $\hat{\mathscr{A}_2}$ are the natural topologies for G/U from $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively, we have $\hat{\mathscr{A}_1} \subseteqq \hat{\mathscr{A}_2}$. Then $\mathscr{A}_1 \subseteqq \mathscr{A}_2$.

We also need the following Baire category theory result, which can be found as (5.28) on page 42 of [3].

THEOREM 3. *A locally countably compact regular space is not the union of a countable number of closed sets all having void interior.*

Bichteler introduced a good notation which we will adopt: If $\mathscr{A}_1$ and $\mathscr{A}_2$ are locally compact group topologies for the group G, $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$ will mean that the sets of irreducible unitary representations of G continuous with respect to each of $\mathscr{A}_1$ and $\mathscr{A}_2$ coincide. Theorem 4 is the result of Bichteler's which we will prove in the series of lemmas following.

THEOREM 4. *Let G be a group on which are defined two locally compact group topologies $\mathscr{A}_1$ and $\mathscr{A}_2$, such that $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$; G is not necessarily abelian. Then $\mathscr{A}_1 = \mathscr{A}_2$ if one of the following is true:* (i) *$\mathscr{A}_1$ and $\mathscr{A}_2$ are comparable; or* (ii) *there is a normal subgroup of G open and σ-compact in one of $(G, \mathscr{A}_1)$ or $(G, \mathscr{A}_2)$.*

LEMMA 5. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two locally compact group topologies on a group G such that $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. Then*

(i) *$\mathscr{A}_1$ and $\mathscr{A}_2$ are compatible topologies;*

(ii) *if N is a normal subgroup of G which is $\mathscr{A}_2$-closed, then N is also $\mathscr{A}_1$-closed and $(G/N, \hat{\mathscr{A}_1}) \sim (G/N, \hat{\mathscr{A}_2})$;*

(iii) *if U is an $\mathscr{A}_2$-open subgroup of G, then U is $\mathscr{A}_1$-closed and every continuous unitary irreducible representation of $(U, \mathscr{A}_2)$ is $\mathscr{A}_1$-continuous on U.*

Moreover, if $\mathscr{A}_1 \subseteqq \mathscr{A}_2$, then (iii) *becomes:*

(iii)′ *if U is a subgroup of G either $\mathscr{A}_1$- or $\mathscr{A}_2$-open, then U is $\mathscr{A}_1$- and $\mathscr{A}_2$-closed and $(U, \mathscr{A}_1) \sim (U, \mathscr{A}_2)$.*

Proof. (i)–(iii) are precisely Lemma 3.2 of [2]. For (iii)′ we only need consider U to be $\mathscr{A}_2$-open, and because of (iii) we only need show that every continuous unitary irreducible representation of $(U, \mathscr{A}_1)$ is on U $\mathscr{A}_2$-continuous. But this is immediate as $\mathscr{A}_1 \subseteqq \mathscr{A}_2$.

We note that all through the forthcoming analysis we store in the back of our minds the fact that two topologies are compatible if they satisfy the conditions of Lemma 5.

LEMMA 6. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be two locally compact group topologies on a group* G *such that* $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ *and* $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. *If* $(G, \mathscr{A}_1)$ *is also compact and* $(G, \mathscr{A}_2)$ *is totally disconnected, then* $\mathscr{A}_1 = \mathscr{A}_2$ *and in particular* $(G, \mathscr{A}_2)$ *is also compact.*

Proof. We can choose a basis at the identity of $(G, \mathscr{A}_2)$ of open compact subgroups. We shall show these subgroups to be also open in $(G, \mathscr{A}_1)$.

Let K be an open compact subgroup of $(G, \mathscr{A}_2)$. If $K = G$, then Theorem 1 gives $\mathscr{A}_1 = \mathscr{A}_2$.

Otherwise suppose x_1 is in K'. Then the group generated by $\{x_1\} \cup K$, say K_1, is $\mathscr{A}_2$-σ-compact (as it is the countable union of compact sets) and $\mathscr{A}_2$-open (as it contains K). Now K is a proper subgroup of K_1 and so at least two left translates of it are required to cover K_1. If $K_1 = G$ then Theorem 1 gives $\mathscr{A}_1 = \mathscr{A}_2$.

We continue defining K_n's inducively unless we have a $K_n = G$, when we cease operations: Suppose $K_{n-1} \neq G$ and let $x_n \in K'_{n-1}$. Let K_n be the group generated by $\{x_n\} \cup K_{n-1}$; K_n is $\mathscr{A}_2$-σ-compact and $\mathscr{A}_2$-open. At least 2 left translates of K_{n-1} are needed to cover K_n, and as left translates of a group by different points are disjoint, there will be needed at least 2^n left translates of K to cover K_n.

If, for any n, $K_n = G$, then $\mathscr{A}_1 = \mathscr{A}_2$. Otherwise, let L be the subgroup $\bigcup_{n=1}^{\infty} K_n$, an open σ-compact subgroup of $(G, \mathscr{A}_2)$. If $L = G$ then $\mathscr{A}_1 = \mathscr{A}_2$; suppose $L \neq G$. We will need at least a countably infinite number of left translates of K to cover L; we will show that if $L \neq K$ we need only a finite number, and hence $\mathscr{A}_1 = \mathscr{A}_2$.

Now $(L/K, \hat{\mathscr{A}_2})$, the quotient space with the natural topology derived from $\mathscr{A}_2$, is σ-compact and discrete, and hence must be countable. Also L and K are closed subgroups of $(G, \mathscr{A}_1)$ from Lemma 5, and therefore $(L/K, \hat{\mathscr{A}_1})$ is a compact regular space (see [3] page 38, (5.21) and (5.22)). Theorem 3 again leaps to the rescue and makes $(L/K, \hat{\mathscr{A}_1})$ a discrete space and K an open subgroup of $(L, \mathscr{A}_1)$. But $(L, \mathscr{A}_1)$ is a compact group and hence only a finite number of left translates of K are needed to cover it.

LEMMA 7. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be two locally compact group topologies on a group* G, *such that* $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ *and* $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. *If* $(G, \mathscr{A}_1)$ *is totally disconnected then* $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. Let S be any compact open subgroup of $(G, \mathscr{A}_1)$. Now

$\mathscr{A}_1 \subseteqq \mathscr{A}_2$ causes $(G, \mathscr{A}_2)$ to be totally disconnected and S to be an $\mathscr{A}_2$-open totally disconnected locally compact subgroup of $(G, \mathscr{A}_2)$. Lemma 5 (iii)′ also gives $(S, \mathscr{A}_1) \sim (S, \mathscr{A}_2)$, allowing us to apply Lemma 6 to S. Then S is an $\mathscr{A}_2$-compact, $\mathscr{A}_1$-open subgroup of G; Theorem 1 swings into action to give $\mathscr{A}_2 \subseteqq \mathscr{A}_1$ and hence $\mathscr{A}_1 = \mathscr{A}_2$.

LEMMA 8. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two locally compact group topologies on a group G such that $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ and $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. If $(G, \mathscr{A}_1)$ is a Lie group and $(G, \mathscr{A}_2)$ is totally disconnected then $\mathscr{A}_1$ and $\mathscr{A}_2$ are both discrete.*

Proof. Let C be the component of the identity in $(G, \mathscr{A}_1)$, then all the relevant conditions carry down to C as it is $\mathscr{A}_1$-open (see Lemma 5).

Let U be an open compact subgroup of $(C, \mathscr{A}_2)$. Then U is a closed subgroup of $(C, \mathscr{A}_1)$ which is σ-compact. Hence Theorem 1 gives $(U, \mathscr{A}_1) = (U, \mathscr{A}_2)$ and $(U, \mathscr{A}_1)$ is a totally disconnected Lie group (see [4] page 186), and so is discrete. Then $(U, \mathscr{A}_2)$ is discrete and open in $(C, \mathscr{A}_2)$ and $(C, \mathscr{A}_2)$ is discrete. Any subgroup of C is now $\mathscr{A}_2$-open and consequently $\mathscr{A}_1$-closed. But the only connected Lie group to have this property is a one element group ([5] in Lemma 4); $C = \{e\}$, and $(G, \mathscr{A}_1)$ is discrete.

LEMMA 9. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two locally compact group topologies on a group G such that $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ and $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. If $(G, \mathscr{A}_2)$ is totally disconnected then $\mathscr{A}_1 = \mathscr{A}_2$.*

Proof. There is an open subgroup U of $(G, \mathscr{A}_1)$ such that, if C is the component of the identity in $(G, \mathscr{A}_1)$, $(U/C, \hat{\mathscr{A}}_1)$ is compact (see [4] page 54, 2.3.1). Again we use [4], this time 4.6 on page 175: Let A be any $\mathscr{A}_1$-neighborhood of the identity; $A \cap U$ is an $\mathscr{A}_1$-neighborhood of the identity in U and let N be the corresponding compact normal subgroup of $(U, \mathscr{A}_1)$ which is in A and is such that $(U/N, \hat{\mathscr{A}}_1)$ is a Lie group.

In fact $(U/N, \hat{\mathscr{A}}_1)$ is a locally compact Lie group, $(U/N, \hat{\mathscr{A}}_2)$ a locally compact totally disconnected group (see [3] page 63, (7.11)), $\hat{\mathscr{A}}_1 \subseteqq \hat{\mathscr{A}}_2$, and $(U/N, \hat{\mathscr{A}}_1) \sim (U/N, \hat{\mathscr{A}}_2)$ by Lemma 5. Applying Lemma 8 we have $(U/N, \hat{\mathscr{A}}_1)$ is discrete. Then N is open in $(U, \mathscr{A}_1)$, also in $(G, \mathscr{A}_1)$, and as every $\mathscr{A}_1$-neighborhood of the identity contains such an $\mathscr{A}_1$-open subgroup, the identity itself must be the intersection of all $\mathscr{A}_1$-open subgroups; $(G, \mathscr{A}_1)$ must be totally disconnected. Applying Lemma 7 we obtain: $\mathscr{A}_1 = \mathscr{A}_2$.

THEOREM 10. *Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two locally compact group* *topologies on a group G such that* $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ *and* $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. *Then* $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. The component of the identity in $(G, \mathscr{A}_2)$ is a σ-compact closed normal subgroup of $(G, \mathscr{A}_2)$. By Theorem 2 we would have $\mathscr{A}_1 = \mathscr{A}_2$ if the two natural topologies on the factor group by this component are equal. Noting that the necessary properties descend to the factor group, the previous lemma provides our solutions.

COROLLARY 11. *If every unitary irreducible representation of a locally compact group* $(G, \mathscr{A})$ *is continuous, then* $(G, \mathscr{A})$ *is discrete.*

Proof. Let $\mathscr{A}_d$ be the discrete topology for G. Then $\mathscr{A} \subseteqq \mathscr{A}_d$, and $(G, \mathscr{A}) \sim (G, \mathscr{A}_d)$. The above Theorem 10 then gives $\mathscr{A} = \mathscr{A}_d$.

COROLLARY 12. *Let* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *be two locally compact group topologies on a group G such that* $(G, \mathscr{A}_1) \sim (G, \mathscr{A}_2)$. *Then if G contains a normal subgroup (say N) which is open and* σ*-compact in one of the topologies (say* $\mathscr{A}_1$*), we have* $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. We need only show N to be $\mathscr{A}_2$-open, and then Theorem 1 gives $\mathscr{A}_1 \subseteqq \mathscr{A}_2$ ready for Theorem 10 to be applied.

The group $(G/N, \hat{\mathscr{A}_1})$ is discrete, and since N is $\mathscr{A}_1$-closed and $(G/N, \hat{\mathscr{A}_1}) \sim (G/N, \hat{\mathscr{A}_2})$ (Lemma 5), the previous Corollary 11 gives that $(G/N, \hat{\mathscr{A}_2})$ is also discrete, that is N is open in $(G, \mathscr{A}_2)$.

Theorem 10 and Corollary 12 constitute Theorem 4. Corollary 11 is the result in Bichteler's earlier paper [1].

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