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HOMOTOPY TYPES OF SPHERICAL FIBRE SPACES OVER SPHERES

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It is known that the fibre homotopy type of a spherical fibre space over a sphere is determined by its characteristic class. Our purpose is to describe the homotopy type of the total space of a spherical fibre space over a sphere in terms of its characteristic class, and to classify homotopy types of them by defining a kind of equivalence between characteristic classes.

I. M. James and J. H. C. Whitehead classified homotopy types of the total space of sphere bundles over spheres in [2] and [3]. Our results are a generalization of their theorems and also an answer to one of problems proposed by J. D. Stascheff in [7]. Let $\mathscr{G}_k$ be the space of maps of a k-sphere into itself with degree 1 and let $\mathscr{F}_k$ be the subspace of $\mathscr{G}_k$ consisting of maps preserving the base point ${}^t(0, \cdots, 0, 1)$. We denote by $\mathscr{E}_{k,n}(\chi)$ the total space of an orientable k-spherical fibre space over an n-sphere with $\chi \in \pi_{n-1}(\mathscr{G}_k)$ as its characteristic class. First we shall treat with the case where fibrations have cross-sections. Then we may suppose $\chi = i_{k*}(\xi)$ where $i_k\colon \mathscr{F}_k \to \mathscr{G}_k$ denotes the inclusion map.

Now let

$$\lambda\colon \pi_{n-1}(\mathscr{F}_k) \longrightarrow \pi_{k+n-1}(\mathscr{S}^k)$$

be the isomorphism defined by B. Steer in [5]. We are concerned with $\lambda(\xi)$ but not χ.

Then if $i_{k*}(\xi) = i_{k*}(\xi')$ we claim

$$(1) \qquad \lambda(\xi') = \lambda(\xi) + [x, \iota_k]$$

for some $x \in \pi_n(\mathscr{S}^k)$ where $[\,,\,]$ denotes Whitehead product.

For, let i be the inclusion $\mathscr{R}_{k+1} \to \mathscr{G}_k$ where $\mathscr{R}_{k+1}$ is the rotation group of $\mathscr{S}^k$. Clearly i induces a fibre map of the fibration $\mathscr{R}_{k+1} \to \mathscr{S}^k$ into the fibration $\mathscr{G}_k \to \mathscr{S}^k$. Since the restriction of λ on the image of $\pi_{n-1}(\mathscr{R}_k)$ is equal to (up to sign) ([5]), the homomorphism $\mathscr{J}$ which is defined by G. W. Whitehead in [6], λ maps $\partial\pi_n(\mathscr{S}^k)$ onto the group $[\pi_n(\mathscr{S}^k), \iota_k]$ by the formula $\mathscr{J}\partial(x) = -[x, \iota_k]$ where ∂ denotes the boundary homomorphism taken from the homotopy sequences of fibrations. Thus, since $\xi' - \xi$ is contained in the $\partial\pi_n(\mathscr{S}^k)$, we obtain (1).

Let Σ be the natural projection

$$\pi_{k+n-1}(\mathscr{S}^k) \longrightarrow \pi_{k+n-1}(\mathscr{S}^k)/[\pi_n(\mathscr{S}^k), \iota_k]\ .$$

A map of $\mathscr{S}^k$ into itself with degree -1 canonically induces an endmorphism of $\pi_{k+n-1}(\mathscr{S}^k)/[\pi_n(\mathscr{S}^k), \iota_k]$. We denote by $\tilde{\Sigma}$ the composition of Σ and the endmorphism. The set

$$\mathscr{M}(\mathscr{E}_{k,n}(\chi)) = (\pm\Sigma\lambda(\xi), \pm\tilde{\Sigma}\lambda(\xi))$$

is independent from the choice of ξ by (1). Then we shall prove

THEOREM 1. *If the fibration $\chi_i (i = 1, 2)$ has a cross-section $(n, k \geqq 2)$, $\mathscr{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathscr{E}_{k,n}(\chi_2)$ if and only if*

(1) *if $n \neq k$, or $n = k =$ even $\mathscr{M}(\mathscr{E}_{k,n}(\chi_1)) = \mathscr{M}(\mathscr{E}_{k,n}(\chi_2))$*

(2) *if $n = k =$ odd $d\cdot\lambda(\xi_1) \equiv \lambda(\xi_2) \bmod [\pi_n(\mathscr{S}^k), \iota_k]$ for some integer d, $(d, m) = 1$, where m is the order of $\lambda(\xi_2) \bmod [\pi_n(\mathscr{S}^k), \iota_k]$.*

If $\mathscr{E}_{k,n}(\chi)$ has the same homotopy type as $\mathscr{S}^k \times \mathscr{S}^n$ the fibration has a cross-section. Hence we have

COROLLARY 1.1. *$\mathscr{E}_{k,n}(\chi)$ has the same homotopy type as $\mathscr{S}^k \times \mathscr{S}^n$ if and only if the fibration χ is fibre homotopically trivial.*

Secondly we consider fibrations which do not necessarily have cross-sections. Therefore, we are mainly concerned in the case $n > k$. However, the case $n = k + 1$ is different from others, so we suppose $n \geqq k + 2 \geqq 4$.

Let $\bar{\rho}: \mathscr{S}^k \to \mathscr{S}^k$ be the homeomorphism defined by

$$\bar{\rho}(x_1, x_2, \cdots, x_{k+1}) = (-x_1, x_2, \cdots, x_{k+1}) ,$$

and let $\rho: \mathscr{G}_k \to \mathscr{G}_k$ be the homeomorphism induced by $\bar{\rho}(\rho(f) = \bar{\rho} f \bar{\rho})$. For any $\alpha \in \pi_{n-1}(\mathscr{S}^k)$, from the diagram

$$\pi_{k+n-1}(\mathscr{S}^{n-1}) \xrightarrow[\alpha_*]{} \pi_{k+n-1}(\mathscr{S}^k) \xleftarrow[\lambda]{} \pi_{n-1}(\mathscr{F}_k) \xrightarrow[i_{k*}]{} \pi_{n-1}(\mathscr{G}_k) ,$$

we have the subgroup of $\pi_{k+n-1}(\mathscr{G}_k)$ defined by

$$\mathscr{G}(\alpha) = i_{k*}\lambda^{-1}\cdot\alpha_*\pi_{k+n-1}(\mathscr{S}^{n-1}) .$$

Then we claim

$$\mathscr{G}(\alpha) = \mathscr{G}(-\alpha) \text{ and } \rho_*(\mathscr{G}(\alpha)) = \mathscr{G}((-\iota_k)_*\alpha) . \tag{2}$$

For, the former is clear and the latter follows from the following commutative diagram (see Lemma 2.2)

$$\begin{tikzcd}
\pi_{k+n-1}(\mathscr{S}^k) \arrow[d, "(-\iota_k)_*"'] & \pi_{n-1}(\mathscr{F}_k) \arrow[l, "-\lambda"'] \arrow[r, "i_{k*}"'] \arrow[d, "\rho'_*"] & \pi_{n-1}(\mathscr{G}_k) \arrow[d, "\rho_*"] \\
\pi_{k+n-1}(\mathscr{S}^k) & \pi_{n-1}(\mathscr{F}_k) \arrow[l, "\lambda"] \arrow[r, "i_{k*}"'] & \pi_{n-1}(\mathscr{G}_k)
\end{tikzcd} \tag{3}$$

where $\rho \mid \mathscr{F}_k = i_k \cdot \rho'$ is the natural factorization.

Now let $\mathscr{S}[\chi]$ ($\chi \in \pi_{n-1}(\mathscr{G}_k)$) be the set of elements

$$\{\chi, -\chi, \rho_* \chi, -\rho_* \chi\}$$

and let $\mathscr{P}_k: \mathscr{G}_k \to \mathscr{S}^k$ be the projection of the canonical fibration. We define a relation in $\pi_{n-1}(\mathscr{G}_k)$ as follows $\chi_1 \sim \chi_2$ if and only if $\theta_1 \equiv \theta_2 \bmod \mathscr{G}(\mathscr{P}_k^*(\theta_1))$ for some pair (θ_1, θ_2), $\theta_i \in \mathscr{S}[\chi_i]$.

It can be easily checked by (2) that this is an equivalence relation.

THEOREM 2. *If $n \geqq k+2 \geqq 4$, then $\mathscr{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathscr{E}_{k,n}(\chi_2)$ if and only if $\chi_1 \sim \chi_2$.*

If fibrations have cross-sections this is an alternative version of Theorem 1. For, since $\mathscr{P}_{k*}(\chi_i) = 0$ we have $\chi_i = i_{k*}(\xi_i)$. Then the condition $\chi_1 \sim \chi_2$ means that $\chi_1 = \pm \chi_2$ or $\chi_1 = \pm \rho_* \chi_2$, i.e.,

$$i_k^*(\xi_1) = \pm i_{k*}(\xi_2) \quad \text{or} \quad i_{k*}(\xi_1) = \pm i_{k*}((\rho'_*)(\xi_2)) .$$

These are satisfied if and only if $\xi_1 = \pm \xi_2 + \partial\sigma$ or $\xi_1 = \pm \rho'_* \xi_2 + \partial\sigma$ where $\sigma \in \pi_n(\mathscr{S}^k)$. Now apply λ to the both side, then we have that

$$\lambda(\xi_1) \equiv \pm \lambda(\xi_2) \quad \text{or} \quad \pm(-\iota_k)_* \lambda(\xi_2) \bmod [\pi_n(\mathscr{S}^k), \iota_k] .$$

This is so if and only if $\mathscr{M}(\mathscr{E}_{k,n}(\chi_1)) = \mathscr{M}(\mathscr{E}_{k,n}(\chi_2))$.

From Theorem 2 the following is easily deduced.

COROLLARY 2.1. *Suppose that $\mathscr{J}\pi_{n-1}(\mathscr{R}_k) \supset \mathscr{P}_{k*}(\chi)\pi_{k+n-1}(\mathscr{S}^{n-1})$. If $\mathscr{E}_{k,n}(\chi)(n \geqq k+2 \geqq 4)$ has the same homotopy type as the total space of an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}^n$, then the fibration itself is fibre homotopically equivalent to an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}_n$.*

As special cases we have

COROLLARY 2.2. *Suppose that the fibration χ has a cross-section. If $\mathscr{E}_{k,n}(\chi)(n \geqq k+2 \geqq 4)$ has the homotopy type of the total space of an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}^n$, the fibration is fibre homotopically equivalent to an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}^n$.*

COROLLARY 2.3. *A k-spherical fibring over $\mathscr{S}^n$ is stable fibre homotopically equivalent to an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}^n$ if and only if the total space of the fibring has the same homotopy $\mathscr{S}$-type as the total space of an orthogonal $\mathscr{S}^k$-bundle over $\mathscr{S}^n$.*

2. $\mathscr{E}_{k,n}(\chi)$ as a CW-complex. Let $f: (\mathscr{S}^{n-1}, *) \to (\mathscr{G}_k, 1)$ be a representative of χ and let $\tilde{f}: \mathscr{S}^{n-1} \times \mathscr{S}^k \to \mathscr{S}^k$ be the adjoint map. We denote by $\mathscr{K}(f)$ the complex $\mathscr{S}^k \cup \mathscr{D}^n \times \mathscr{S}^k$ obtained from identifying (x, y) with $\tilde{f}(x, y)$ for $(x, y) \in \mathscr{S}^{n-1} \times \mathscr{S}^k$.

Then it is known that $\mathscr{E}_{k,n}(\chi)$ has the same homotopy type as $\mathscr{K}(f)$ (Prop. 1 of [4]). It may be considered that $\mathscr{K}(f)$ is given the natural CW-decomposition $\mathscr{S}^k \cup e^n \cup e^{k+n}$ in which attaching maps for cells are as follows

$$
\begin{aligned}
&\alpha: \mathscr{S}^{n-1} \longrightarrow \mathscr{S}^k, \ \alpha(x) = f(x, *) \\
(4) \qquad &\beta: \mathscr{S}^{k+n-1} = \mathscr{D}^n \times \mathscr{S}^{k-1} \cup \mathscr{S}^{n-1} \times \mathscr{D}^k \\
&\longrightarrow \mathscr{D}^n \times * \cup \mathscr{S}^{n-1} \times \mathscr{S}^k \xrightarrow[\bar{\alpha} \cup \tilde{f}]{} \mathscr{S}^k \cup e^n
\end{aligned}
$$

where $\bar{\alpha}: (\mathscr{D}^n, \mathscr{S}^{n-1}) \to (\mathscr{S}^k \cup e^n, \mathscr{S}^k)$ denotes the characteristic map for $e^n (\alpha = \partial\bar{\alpha})$.

Let j be the inclusion: $(\mathscr{S}^k \cup e^n, *) \to (\mathscr{S}^k \cup e^n, \mathscr{S}^k)$. Then we have

LEMMA 2.1. *$\mathscr{P}_{k*}(\chi) = \alpha$, and $j_*(\beta) = \pm[\bar{\alpha}, \iota_k]_r$ if $n > k + 1$ or $\alpha = 0$. Thus we can define the orientation of $\mathscr{K}(f)$ by $j_*(\beta) = [\bar{\alpha}, \iota_k]_r$.*

Proof. The former follows from (4) and the definition of $\mathscr{P}_{k*}$. Since the group $\pi_{k+n-1}(\mathscr{S}^k \cup e^n, \mathscr{S}^k)$ is isomorphic to the direct sum

$$\mathscr{Z}[\bar{\alpha}, \iota_k]_r + \bar{\alpha}\pi_{k+n-1}(\mathscr{D}^n, \mathscr{S}^{n-1})$$

under the assumption, $j_*(\beta)$ is of the form

$$m[\bar{\alpha}, \iota_k]_r + \bar{\alpha}x$$

for some integer m and $x \in \pi_{k+n-1}(\mathscr{D}^n, \mathscr{S}^{n-1})$. Let $\mathscr{X}_i (i = k, n, k+n)$ be generators of $\mathscr{H}^i(\mathscr{K}(f)) = \mathscr{Z}$. Then, by the theorem in [1],

$$\mathscr{X}_k \cup \mathscr{X}_n = \pm m \mathscr{X}_{k+n} .$$

On the other hand, since $\mathscr{K}(f)$ has the homotopy type of $\mathscr{E}_{k,n}(\chi)$ we have

$$\mathscr{X}_k \cup \mathscr{X}_n = \pm \mathscr{X}_{k+n} ,$$

i.e., $m = \pm 1$. And moreover $\bar{\alpha}x = 0$ follows from the existence of the projection of the fibration.

Now we consider the special case where $0 = \alpha = \mathscr{P}_{k*}(\chi)$. Then the map f may be considered as a map: $(\mathscr{S}^{n-1}, *) \to (\mathscr{F}_k, 1)$. Since $\tilde{f} \mid \mathscr{S}^{n-1} \times * = *$, $\mathscr{S}^n$ is naturally imbedded as the image of $\mathscr{D}^n \times *$. In this situation, after identifying $\pi_{k+n-1}(\mathscr{S}^k \vee \mathscr{S}^n)$ with $\pi_{k+n-1}(\mathscr{S}^k) +$

$\pi_{k+n-1}(\mathscr{S}^k \vee \mathscr{S}^n, \mathscr{S}^k)$, it follows from Lemma 2.1 that

$$(5) \qquad \beta = \iota_{k*}(x) + [\iota_k, \iota_n] .$$

And also β may be considered as follows

$$(6) \qquad \begin{aligned} \mathscr{S}^{k+n-1} &= \mathscr{D}^n \times \mathscr{S}^{k-1} \cup \mathscr{S}^{n-1} \times \mathscr{D}^k \xrightarrow[1\times\varphi_k]{} \mathscr{D}^n \\ &\times * \cup \mathscr{S}^{n-1} \times \mathscr{S}^k \xrightarrow[\varphi_n \times * \cup * \times \tilde{f}]{} \mathscr{S}^n \times * \cup * \times \mathscr{S}^k \end{aligned}$$

where φ_k denotes the identification map: $\mathscr{D}^k \to \mathscr{S}^k/\varphi^{k-1}$.

We make use of λ to determine x, so we recall the definition of λ. Let ε be the map: $\mathscr{S}^p \to \mathscr{F}_k$ defined by $\varepsilon(\ \ ) =$ the identity of $\mathscr{S}^k$ and let h be a map: $(\mathscr{S}^p, *) \to (\mathscr{F}_k, 1)$. Since adjoint maps $\tilde{h}, \tilde{\varepsilon}: \mathscr{S}^p \times \mathscr{S}^k \to \mathscr{S}^k$ has the same restriction on $\mathscr{S}^p \vee \mathscr{S}^k$, the separation element $d(\tilde{h}, \tilde{\varepsilon}) \in \pi_{p+k}(\mathscr{S}^k)$ is defined. B. Steer defined $\lambda(h)$ by $d(\tilde{h}, \tilde{\varepsilon})$. For example we have (see the diagram (3))

LEMMA 2.2. $-\lambda\rho'_*(\xi) = (-\iota_k)_*\lambda(\xi)(\xi \in \pi_p(\mathscr{F}_k))$.

Proof. Let g be a representative of ξ. Then we have

$$\begin{aligned} (-\iota_k)_*\lambda(\xi) &= (-\iota_k)_*d(\tilde{g}, \tilde{\varepsilon}) = \bar{\rho}_*d(\tilde{g}, \tilde{\varepsilon}) = d(\bar{\rho}\tilde{g}, \bar{\rho}\tilde{\varepsilon}) \\ &= d(\bar{\rho}\tilde{g}, \tilde{\varepsilon}(id \times \bar{\rho})) = d(\bar{\rho}\tilde{g}(id \times \bar{\rho})(id \times \bar{\rho}), \tilde{\varepsilon}(id \times \bar{\rho})) \\ &= -d(\bar{\rho}\tilde{g}(id \times \bar{\rho}), \tilde{\varepsilon}) . \end{aligned}$$

Since $\widetilde{\rho'\cdot g}(x, y) = \bar{\rho}(\tilde{g}(x, \bar{\rho}(y)) = \bar{\rho}'\tilde{g}(id \times \bar{\rho})(x, y)$ we have $\widetilde{\rho' g} = \bar{\rho}\tilde{g}(id \times \bar{\rho})$. Hence $d(\bar{\rho}\tilde{g}(id \times \bar{\rho}), \tilde{\varepsilon}) = d(\widetilde{\rho' g}, \tilde{\varepsilon}) = \lambda(\rho' g) = \lambda(\rho'_*(\xi))$.

LEMMA 2.3. *In the expression in* (4) *we have* $x = \lambda(\xi)$, *up to sign, where* ξ *denotes the homotopy class of* f.

For the proof of Lemma 2.3 we prepare the following general

LEMMA 2.4. *Let* $\mathscr{L}$ *be a* 1*-connected CW-complex and let* $\mathscr{K}$ *be a complex* $\mathscr{L} \cup e^N(\alpha \sim 0)$. *Let* f, g *be maps*: $\mathscr{K} \to \mathscr{X}$ *such that* $f \mid \mathscr{L} = g \mid \mathscr{L}$ *and let* ζ *be a map*: $\mathscr{S}^N \to \mathscr{K}$ *which induces the isomorphism*: $\mathscr{H}_N(\mathscr{S}^N, *) \to \mathscr{H}_N(\mathscr{K}, \mathscr{L})$. *Then we have* $d(f, g) = f_*(\zeta) - g_*(\zeta)$ *(up to sign).*

Proof. Since $\alpha \sim 0$ there exists a homotopy equivalence $\varphi: (\mathscr{L} \vee \mathscr{S}^N, \mathscr{L}) \to (\mathscr{K}, \mathscr{L})$ relative to $\mathscr{L}$. Let δ be the inclusion $\mathscr{S}^N \to \mathscr{L} \vee \mathscr{S}^N$. Then

$$d(f, g) = \pm d(f\varphi, g\varphi) = \pm((f\varphi)_*\delta - (g\varphi)_*\delta) .$$

From $\varphi^{-1}\zeta \in \pi_N(\mathscr{L} \vee \mathscr{S}^N)$ and the assumption on ζ we have

$$\varphi_*^{-1}\zeta = \pm\delta + \eta(\eta \in \pi^N(\mathscr{L})), \text{ i.e., } \zeta = \pm\varphi_*(\delta) + \varphi_*(\eta) .$$

Hence

$$\begin{aligned} f_*(\zeta) - g_*(\zeta) &= f_*(\pm\varphi_*(\zeta) + \varphi_*(\eta)) - g_*(\pm\varphi_*(\delta) + \varphi_*(\eta)) \\ &= \pm(f_*\varphi_*(\delta) - g_*\varphi_*(\delta)) = \pm d(f, g) . \end{aligned}$$

Proof of Lemma 2.3. Let $\mathscr{Q}$ be the identification map:

$$\mathscr{S}^{n-1} \times \mathscr{S}^k \to \mathscr{S}^{n-1} \times \mathscr{S}^k/\mathscr{S}^{n-1} \times * .$$

The maps

$$\tilde{f}\mathscr{Q}^{-1}, \tilde{\varepsilon}\mathscr{Q}^{-1}\colon \mathscr{S}^{n-1} \times \mathscr{S}^k/\mathscr{S}^{n-1} \times * \to \mathscr{S}^k$$

are well-defined and has the same restriction on $* \times \mathscr{S}^k/\mathscr{S}^{n-1} \times *$. The complex $\mathscr{S}^{n-1} \times \mathscr{S}^k/\mathscr{S}^{n-1} \times *$ has a form $\mathscr{S}^k \cup e^{k+n-1}(\alpha \sim 0)$. Then we apply Lemma 2.4 to the case where

$$\begin{gathered} \mathscr{K} = \mathscr{S}^{n-1} \times \mathscr{S}^k/\mathscr{S}^{n-1} \times * , \quad \mathscr{L} = * \times \mathscr{S}^k/\mathscr{S}^{n-1} \times * , \\ N = n + k - 1 , \quad f = \tilde{f}\mathscr{Q}^{-1} , \quad g = \tilde{\varepsilon}\mathscr{Q}^{-1} \text{ and } \mathscr{X} = \mathscr{S}^k . \end{gathered}$$

Thus we have

$$\lambda(f) = d(\tilde{f}, \tilde{\varepsilon}) = d(\tilde{f}\mathscr{Q}^{-1}, \tilde{g}\mathscr{Q}^{-1}) = \pm((f\mathscr{Q}^{-1})_*(\zeta) - (\tilde{\varepsilon}\mathscr{Q}^{-1})_*(\zeta))$$

for any $\zeta\colon (\mathscr{S}^{k+n-1}, *) \to (\mathscr{K}, \mathscr{L})$ which induces an isomorphism

$$\zeta_*\colon \mathscr{H}_{k+n-1}, (\mathscr{S}^{k+n-1}, *) \to \mathscr{H}_{k+n-1}(\mathscr{K}, \mathscr{L}) .$$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathscr{S}^{k+n-1} = \mathscr{D}^n \times \mathscr{S}^{k-1} \cup \mathscr{S}^{n-1} \times \mathscr{D}^n \to \mathscr{D}^n & & \\ \times * \cup \mathscr{S}^{n-1} \times \mathscr{S}^k & \xrightarrow[* \cup \mathscr{Q}]{} & \mathscr{S}^{n-1} \times \mathscr{S}^k/\mathscr{S}^{n-1} \times * \\ \Big\downarrow{\scriptstyle \varphi_n \cup \tilde{f}} & & \Big\downarrow{\scriptstyle \tilde{f}\mathscr{Q}^{-1}} \\ \mathscr{S}^n \vee \mathscr{S}^k & \xrightarrow{\qquad\qquad} & \mathscr{S}^k . \end{array}$$

Since we can take ζ with the composition of two maps in the upper row it follows from $(\tilde{\varepsilon}\mathscr{Q}^{-1})_*(\zeta) = 0$ that $\lambda(f) = \pm(\tilde{f}\mathscr{Q}^{-1})_*(\zeta)$. From the diagram (6) the proof is completed.

3. Proof of Theorem 1. Let $\mathscr{K}$ be a complex of the form

$$\mathscr{S}^k \vee \mathscr{S}^n \cup e^{k+n}$$

where $\beta = \iota_k\alpha + [\iota_k, \iota_n]$ under the decomposition

$$\pi_{k+n-1}(\mathscr{S}^k \vee \mathscr{S}^n) = \pi_{k+n-1}(\mathscr{S}^k) + \pi_{k+n-1}(\mathscr{S}^n) + \mathscr{Z}[\iota_k, \iota_n] .$$

By the cellular homotopy theorem $\mathscr{K}_1$ has the same homotopy type as $\mathscr{K}_2$ if and only if there exists a homotopy equivalence $(n, k \geqq 2)$

$$\Phi: \mathscr{S}^k \vee \mathscr{S}^n \to \mathscr{S}^k \vee \mathscr{S}^n$$

such that $\Phi_*(\beta_1) = \pm\beta_2$. Now consider the case $n \neq k$. It is obvious that a map Φ is homotopy equivalence if and only if $\Phi \mid \mathscr{S}^k = \pm\iota_k + \iota_n\circ\tau(\tau \in \pi_k(\mathscr{S}^n))$, and $\Phi \mid \mathscr{S}^n = \pm\iota_n$ if $n < k = \pm\iota_k$, and $\Phi \mid \mathscr{S}^n = \iota_k\circ\sigma + \pm\iota_n(\sigma \in \pi_n(\mathscr{S}^k))$ if $n > k$. From easy computation of $\Phi_*(\beta_1)$ we can obtain

LEMMA 3.1. *If $n \neq k$, $\mathscr{K}_1$ has the same homotopy type as $\mathscr{K}_2$ if and only if the set $\{\pm\alpha_1, \pm(-\iota_k)_*\alpha_1\}$ is equal to the set*

$$\{\pm\alpha_2, \pm(\iota_k)_*\alpha_2\} \bmod [\pi_n(\mathscr{S}^k), \iota_k] .$$

Next we consider the case $n = k$. By the same way as in [2] we have

LEMMA 3.2. (*James and Whitehead*). *If $n = k =$ even, $\mathscr{K}_1$ and $\mathscr{K}_2$ have the same homotopy type if and only if*

$$\{\pm\alpha_1\} \equiv \{\alpha_2\} \bmod [\pi_n(\mathscr{S}^k), \iota_k] .$$

LEMMA 3.3. (*James and Whitehead*). *If $n = k =$ odd, $\mathscr{K}_1$ and $\mathscr{K}_2$ have the same homotopy type if and only if there exists an integer d which is prime to m_2 and $d\alpha_1 \equiv \alpha_2 \bmod [\pi_n(\mathscr{S}^k), \iota_k]$ where m_2 is the order of $\alpha_2 \bmod [\pi_n(\mathscr{S}^k), \iota_k]$.*

Thus Theorem 1 follows from Lemmas 3.1, 3.2, 3.3, and 2.3.

4. Some Lemmas. Let $\mathscr{L}$ be a complex of the form $\mathscr{S}^k \cup e^n$ with the characteristic map $\bar{\alpha}: (\mathscr{D}^n, \mathscr{S}^{n-1}) \to (\mathscr{L}, \mathscr{S}^k)$ for the n-cell. Let $\bar{\mathscr{L}}$ be the complex obtained from identifying $\mathscr{S}^k$ of two copies of $\mathscr{L}$, i.e., $\bar{\mathscr{L}} = e^n \cup \mathscr{S}^k \cup e^n$. It may be considered that two maps $\mu_i(i = 1, 2): \mathscr{L} \to \bar{\mathscr{L}}$ and a map $\nu: \bar{\mathscr{L}} \to \mathscr{L}$ are naturally defined and satisfy $\nu\mu_i =$ the identity. Since $\mu_1 \mid \mathscr{S}^k = \mu_2 \mid \mathscr{S}^k$ the separation element $d(\mu_1, \mu_2)$ is defined. Then we have

LEMMA 4.1. *If $\beta \in \pi_{k+n-1}(\mathscr{L})$ and $j_*(\beta) = m[\bar{\alpha}, \iota_k]_r$, then $\mu_{1*}(\beta) - \mu_{2*}(\beta) = m[d(\mu_1, \mu_2), \iota_k]$.*

Proof. Consider the following commutative diagram

$$
\begin{tikzcd}
\pi_*(\mathscr{S}^k) \arrow[r, "i_+"'] \arrow[d, leftrightarrow, "id"] & \pi_*(\overline{\mathscr{L}}) \arrow[r, "j_+"'] \arrow[d, shift left=3, leftarrow] \arrow[d] \arrow[d, shift right=3, leftarrow] & \pi_*(\overline{\mathscr{L}}, \mathscr{S}^k) \arrow[d, shift left=3, "\nu_*"'] \\
\pi_*(\mathscr{S}^k) \arrow[r, "i_*"'] & \pi_*(\mathscr{L}) \arrow[r, "j_*"'] & \pi_*(\mathscr{L}, \mathscr{S}^k) \arrow[u, shift right=3, "\mu_{1*}"'] \arrow[u, shift left=9, "\mu_{z*}"']
\end{tikzcd}
$$

which is taken from the homotopy sequence of the pair and $* = k + n - 1$.

From the commutativity it follows that

$$j_+(\mu_{1*}(\beta) - \mu_{2*}(\beta)) = m[\mu_{1*}\bar{\alpha} - \mu_{2*}\bar{\alpha}, \iota_k]_r .$$

On the orther hand, we have

$$j_+[d(\mu_1, \mu_2), \iota_k] = [j_+d(\mu_1, \mu_2), \iota_k]_r = [\mu_{1*}\bar{\alpha} - \mu_{2*}\bar{\alpha}, \iota_k]_r .$$

Thus, for some element $\gamma \in \pi_*(\mathscr{S}^k)$, it holds

$$m[d(\mu_1, \mu_2), \iota_k] = \mu_{1*}(\beta) - \mu_{2*}(\beta) + i_+(\gamma) .$$

Applying ν_* to the both side, then, from

$$\nu_* d(\mu_1, \mu_2) = d(\nu\mu_1, \nu\mu_2) = d(id, id) = 0 \text{ and } \nu\mu_i(\beta) = \beta ,$$

we have $\nu_* i_+(\gamma) = 0$. Hence $i_+(\gamma) = 0$ from the commutativity of the diagram.

As an application of Lemma 4.1 we have

LEMMA 4.2. *Let f, g be maps: $\mathscr{L} \to \mathscr{X}$ such that $f \mid \mathscr{L} = g \mid \mathscr{L}$. For any β, $j_*(\beta) = m[\bar{\alpha}, \iota_k]_r$, we have*

$$f_*(\beta) - g_*(\beta) = m[d(f, g), f \mid \mathscr{S}^k] .$$

Proof. Define a map $f \cup g: \overline{\mathscr{L}} \to \mathscr{X}$ by

$$(f \cup g)\mu_1 = f , \quad \text{and} \quad (f \cup g)\mu_2 = g .$$

Since $d(f, g) = d((f \cup g)\mu_1, (f \cup g)\mu_2) = (f \cup g)_* d(\mu_1, \mu_2)$ the proof is completed by applying $(f \cup g)_*$ to the both side of the equality in Lemma 4.1.

Let id be the identity map of $\mathscr{L}(n \geqq k + 2 \geqq 4)$ and let $w: \mathscr{L} \to \mathscr{L}$ be a map with $w \mid \mathscr{S}^k = id \mid \mathscr{S}^k$. In general, $d(id, w)$ is belonging to $\pi_n(\mathscr{L})$. However, we have

LEMMA 4.3. *w is a homotopy equivalence preserving the orientation of the n-cell if and only if $d(id, w)$ is contained in $i_*\pi_n(\mathscr{S}^k)$.*

Proof. Let x_n, y_n be the orientation generators of $\mathscr{H}_n(\mathscr{L})$, and $\mathscr{H}_n(\mathscr{S}^n)$ respectively, and let δ be $d(id, w)$. Since $x_n - w_*(x_n) = \delta_*(y_n)$, $x_n = w_*(x_n)$ holds if and only if $\delta_*(y_n) = 0$. On the other hand, the diagram

$$\pi_n(\mathscr{S}^k) \xrightarrow[i_*]{} \pi_n(\mathscr{L}) \longrightarrow \pi_n(\mathscr{L}, \mathscr{S}^k) = \mathscr{H}_n(\mathscr{L}, \mathscr{S}^k) = \mathscr{H}_n(\mathscr{L})$$

shows that $\delta_*(x_n) = 0$ is equivalent to $\delta \in i_*\pi_n(\mathscr{S}^k)$.

Now we prepare lemmas for the proof of Theorem 2. In what follows, we use the notations in § 2 and suppose $n \geqq k + 2 \geqq 4$.

LEMMA 4.4. *Let i be the inclusion: $\mathscr{S}^k \to \mathscr{S}^k \cup e^n \subset \mathscr{K}(f)$. Then we have*

$$i_*^{-1}(0) = \alpha_*\pi_{k+n-1}(\mathscr{S}^{n-1}) .$$

Proof. Since the pair $(\mathscr{K}(f), \mathscr{S}^k)$ is homotopy equivalent to $(\mathscr{E}_{k,n}(\chi), \mathscr{S}^k)$

$$\pi_{k+n}(\mathscr{K}(f), \mathscr{S}^k) = \pi_{k+n}(\mathscr{S}^n) = E\pi_{k+n-1}(\mathscr{S}^{n-1}) .$$

Hence from the homotopy sequence of the triple $(\mathscr{K}(f), \mathscr{S}^k \cup e^n, \mathscr{S}^k)$ we obtain

$$\pi_{k+n}(\mathscr{S}^k \cup e^n, \mathscr{S}^k) = \partial\pi_{k+n+1}(\mathscr{K}(f), \mathscr{S}^k \cup e^n) \cup \bar{\alpha}_*\pi_{k+n}(\mathscr{D}^n, \mathscr{S}^{n-1}) .$$

Thus we have that

$$i_*^{-1}(0) = \partial\pi_{k+n}(\mathscr{S}^k \cup e^n, \mathscr{S}^k) = \alpha_*\pi_{k+n-1}(\mathscr{S}^{n-1}) .$$

Let $\chi_i (i = 1, 2)$ be elements such that $\mathscr{P}_{k*}(\chi_1) = \mathscr{P}_{k*}(\chi_2) = \alpha$. Then $\beta_i \in \pi_{k+n-1}(\mathscr{S}^k \cup e^n)$ and there exists an element $\xi \in \pi_{n-1}(\mathscr{F}_k)$ which satisfies $i_{k*}(\xi) = \chi_1 - \chi_2$.

LEMMA 4.5. *There exists a homotopy equivalence $\varphi: \mathscr{S}^k \cup e^n \to \mathscr{S}^k \cup e^n$ which satisfies*

(1) $\varphi_*(e^k) = e^k$, $\varphi_*(e^n) = e^n$

(2) $\beta_1 - \varphi_*(\beta_2) = i_*\lambda(\xi)$ *(up to sign).*

Proof. Let $\kappa: \mathscr{S}^n \to \mathscr{S}_1^n \vee \mathscr{S}_2^n$ be a map of type (1, −1) and let χ be the fibration induced from $\chi_1 \vee \chi_2$ by κ, i.e., $\chi = \chi_1 - \chi_2$. Since $i_{k*}(\xi) = \chi\mathscr{K}(f)$ has the form $\mathscr{S}^k \vee \mathscr{S}^n \cup e^{k+n}$ by (5). It may be considered that κ induces a map $\bar{\kappa}$:

$$\begin{aligned}\mathscr{K}(f) = \mathscr{S}^k \vee \mathscr{S}^n \cup e^{k+n} &\longrightarrow \mathscr{K}(f_1) \cup \mathscr{K}(f_2)\\ &= \mathscr{D}_1^n \times \mathscr{S}^k \cup \mathscr{S}^k \cup \mathscr{D}_2^n \times \mathscr{S}^k\end{aligned}$$

which satisfies

$$\bar{\kappa}_*(e^{k+n}) = e_1^{k+n} - e_2^{k+n}\ , \quad \bar{\kappa}_*(e^n) = e_1^n - e_2^n \text{ and } \bar{\kappa}_*(e^k) = e^k\ .$$

Let $\bar{\bar{\kappa}}$: $\mathcal{S}^k \vee \mathcal{S}^n \to e_1^n \cup \mathcal{S}^k \cup e_2^n$ be the map obtained from the restriction of $\bar{\kappa}$ on $\mathcal{S}^k \vee \mathcal{S}^n$ and let i_j be the inclusion: $e_j^n \vee \mathcal{S}^k \to e_1^n \cup \mathcal{S}^k \cup e_2^n$. Then we have

$$(*) \qquad \bar{\bar{\kappa}}_*(\beta) = i_{1*}(\beta_1) - i_{2*}(\beta_2)\ .$$

Define the map r: $e_1^n \cup \mathcal{S}^k \cup e_2^n \to \mathcal{S}^k \cup e^n$ by

$$r \mid e_1^n \cup \mathcal{S}^k = \text{identity} = r \mid \mathcal{S}^k \cup e_2^n\ .$$

We claim that

(**) $r_*(\omega)$ is contained in $\tilde{i}_*$-image where $\omega = \bar{\bar{\kappa}} \mid \mathcal{S}^n$ and $\tilde{i}$ denotes the inclusion: $\mathcal{S}^k \to e_1^n \cup \mathcal{S}^k \cup e_2^n$.

For, consider the commutative diagram

$$\begin{array}{ccccc}
\pi_n(\mathcal{S}^k \vee \mathcal{S}^n, \mathcal{S}^k) & \xrightarrow[\bar{\bar{\kappa}}_*]{} & \pi_n(e_1^n \cup \mathcal{S}^k \cup e_2^n, \mathcal{S}^k) & \xrightarrow[r_*]{} & \pi_n(\mathcal{S}^k \vee e^n, \mathcal{S}^k) \\
\uparrow{\scriptstyle j_{1*}} & & \uparrow{\scriptstyle j_{2*}} & & \uparrow{\scriptstyle j_{3*}} \\
\pi_n(\mathcal{S}^k \vee \mathcal{S}^n) & \xrightarrow[\bar{\bar{\kappa}}_*]{} & \pi_n(e_1^n \cup \mathcal{S}^k \cup e_2^n) & \xrightarrow[r_*]{} & \pi_n(\mathcal{S}^k \cup e^n)\ .
\end{array}$$

Let z_n be the element of $\pi_n(\mathcal{S}^k \vee \mathcal{S}^n)$ which is represented by $\mathcal{S}^n$. Then we have

$$\begin{aligned}
j_{3*}r_*(\omega) &= j_{3*}r_*\bar{\bar{\kappa}}(z_n) = r_*j_{2*}(\bar{\bar{\kappa}}(z_n)) = r_*\bar{\bar{\kappa}}_*j_{1*}(z_n) \\
&= r_*(i_{1*}(\bar{\alpha}_1) - i_{2*}(\bar{\alpha}_2)) = \bar{\alpha} - \bar{\alpha} = 0\ .
\end{aligned}$$

Thus (**) is proved.

Now, by applying r_* to the both side of (*) we have

$$r_*\bar{\bar{\kappa}}_*(\beta) = \beta_1 - \beta_2\ .$$

On the other hand, by using (5), we have

$$\begin{aligned}
r_*\bar{\bar{\kappa}}_*(\beta) &= r_*\bar{\bar{\kappa}}_*(\iota_k(\pm\lambda(\xi)) + [\iota_k, z_n]),\ (\iota_n = z_n) \\
&= i_*(\pm\lambda(\xi)) + [\iota_k, r_*(\omega)] \\
&= i_*(\pm\lambda(\xi) + [\iota_k, \omega']),\ (\omega' \in \pi_n(\mathcal{S}^k),\ \tilde{i}_*\omega' = r_*(\omega) \text{ by } (**)) \\
&= i_*(\pm\lambda(\xi) \pm [\omega', \iota_k])
\end{aligned}$$

$$\text{i.e.,} \quad \beta_1 - \beta_2 = i_*(\pm\lambda(\xi) \pm [\omega', \iota_k])\ .$$

If we take a map φ: $\mathcal{S}^k \cup e^n \to \mathcal{S}^k \cup e^n$ such that $d(id, \varphi) = \mp\omega'$, it follows from Lemma 4.2 and Lemma 2.1 that

$$\beta_2 - \varphi_*(\beta_2) = i_*(\mp[\omega', \iota_k]) \quad \text{i.e.,} \quad \beta_1 - \varphi_*(\beta_2) = i_*(\pm\lambda(\xi))\ .$$

Since $d(id, \varphi) \in i_*\pi_n(\mathcal{S}^k)$ φ satisfies (1) by Lemma 4.3.

LEMMA 4.6. *There exist homotopy equivalences* $u': \mathscr{K}(-f) \to \mathscr{K}(f)$ *and* $u'': \mathscr{K}(\rho\rho) \to \mathscr{K}(f)$ *which satisfy*

(1) $u'_*(e^k) = e^k$ *and* $u'_*(e^n) = -e^n$,

(2) $u''_*(e^k) = -e^k$ *and* $u''_*(e^n) = e^n$.

Proof. Let $\mathscr{U}$ be the identification map: $\mathscr{S}^k + \mathscr{D}^n \times \mathscr{S}^k \to \mathscr{K}(f)$ and define u', u'' as follows

$$\rho_n(x_1, x_2, \cdots, x_n) = (-x_1, x_2, \cdots, x_n), \ ((x_1, x_2, \cdots, x_n) \in \mathscr{D}^n)$$
$$u'(x) = x\,, \quad u''(x) = \bar{\rho}x \text{ if } x \in \mathscr{S}^k \text{ and}$$
$$u'(y, z) = \mathscr{U}(\rho_n y, z)\,, \quad u''(y, z) = \mathscr{U}(y, \bar{\rho}z) \text{ if } (y, z) \in \mathscr{D}^n \times \mathscr{S}^k\,.$$

u' and u'' are well-defined by the formulas

$$-\bar{f} = \bar{f}((\rho_n \mid \mathscr{S}^{n-1}) \times id) \text{ and } \widetilde{\rho f} = \bar{\rho}\bar{f}(id \times \bar{\rho})\,.$$

5. **Proof of Theorem 2.** First of all we prove

LEMMA 5.1. *If* $\mathscr{E}_{k,n}(\chi_1)$ *has the same homotopy type as* $\mathscr{E}_{k,n}(\chi_2)$ *there exists a pair* (θ_1, θ_2), $\theta_i \in \mathscr{S}[\chi_i]$ *which satisfies*

($\mathscr{A}$) $\mathscr{P}_{k*}(\theta_1) = \mathscr{P}_{k*}(\theta_2)$

($\mathscr{B}$) *there exists a homotopy equivalence* $\psi: \mathscr{K}(g_1) \to \mathscr{K}(g_2)$ *with* $\psi_*(e_1^i) = e_2^i$ $(i = k, n)$.

Proof. Let $h: \mathscr{K}(f_1) \to \mathscr{K}(f_2)$ be a homotopy equivalence which may be considered as a cellular map. Then we have

$$\mathscr{P}_{k*}(\chi_1) = \pm(\mathscr{P}_{k*}(\chi_2)) \text{ or } \pm(-\iota_k)_*\mathscr{P}_{k*}(\chi_2)\,.$$

Since it is clear that each element on the right hand side can be obtained as $\mathscr{P}_{k*}(\theta_2)$ of a suitable $\theta_2 \in \mathscr{S}[\chi_2]$, there exists a pair (χ_1, θ_2) which satisfies ($\mathscr{A}$), and a homotopy equivalence $u: \mathscr{K}(f_1) \to \mathscr{K}(g_2)$ by Lemma 4.6.

We suppse that $u_*(e_1^k) = \varepsilon_k e_2^k$ and $u_*(e_1^n) = \varepsilon_n e_2^n \cdot (\varepsilon_k, \varepsilon_n = \pm 1)$. Then we have the equation

($\mathscr{C}$) $(\varepsilon_k \iota_k)(\mathscr{P}_{k*}(\chi_1) = \varepsilon_n \mathscr{P}_{k*}(\theta_2)$.

Hence, by ($\mathscr{A}$), we have

($\mathscr{D}$) $\varepsilon_n(\varepsilon_k \iota_k)\mathscr{P}_{k*}(\chi_1) = \mathscr{P}_{k*}(\theta_2) = \mathscr{P}_{k*}(\chi_1)$.

The case of $\varepsilon_k = 1$. Since, by Lemma 4.6, there exists a homotopy equivalence $u': \mathscr{K}(\varepsilon_n f_1) \to \mathscr{K}(f_1)$ with $u'_*(e^k) = e^k$ and $u'_*(e^n) = \varepsilon_n e^n$, the set

$$\{\theta_1 = \varepsilon_n \chi_1, \theta_2, \psi = u \cdot u'\}$$

satisfies ($\mathscr{A}$) and ($\mathscr{B}$).

The case of $\varepsilon_k = -1$. Similarly, by Lemma 4.6, there exists a homotopy equivalence $u''\colon \mathscr{K}(\varepsilon_n\rho f_1) \to \mathscr{K}(f_1)$ with $u''_*(e^k) = -e^k$ and $u''_*(e^n) = e^n$. The set

$$\{\theta_1 = \varepsilon_n\rho_*\chi_1,\ \theta_2,\ \psi = u\cdot u''\}$$

satisfies $(\mathscr{A})$ and $(\mathscr{B})$ by $\mathscr{P}_{k*}(\varepsilon_n\rho_*\chi_1) = \varepsilon_n(-\iota_k)\mathscr{P}_{k*}(\chi_1)$.

Proof of Theorem 2. First we suppose that $\mathscr{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathscr{E}_{k,n}(\chi_2)$. We choose $(\theta_1, \theta_2, \psi)$ as stated in Lemma 5.1. Let g_i be a representative of θ_i and let γ_i be the attaching class for the $(k+n)$-cell of $\mathscr{K}(g_i)$. Let $\varphi\colon \mathscr{S}^k \cup e^n \to \mathscr{S}^k \cup e^n$ be a map as stated in Lemma 4.5 ($\chi_i = \theta_i$) and let $\bar\psi$ be the map obtained from the restriction of ψ on $\mathscr{S}^k \cup e^n$. Since $\psi_*(\gamma) = \gamma_2$ we have

$$\begin{aligned}
0 &= \gamma_2 - \bar\psi_*(\gamma_1) \\
&= \gamma_2 - \varphi_*(\gamma_1) + \varphi_*(\gamma_1) - \bar\psi_*(\gamma_1) \\
&= (\gamma_2 - \varphi_*(\gamma_1)) + [d(\varphi, \bar\psi), \iota_k] && \text{by Lemma 4.2 and Lemma 2.1} \\
&= i_*(\pm\lambda(\eta)) + [d(\varphi, \bar\psi), \iota_k] && \text{by Lemma 4.5 and} \\
& && \theta_2 - \theta_1 = i_{k*}(\eta)\,.
\end{aligned}$$

On the other hand, since $d(\varphi, \psi) = d(\varphi, id) + d(id, \bar\psi)$, $d(\varphi\bar\psi)$ is contained in $i_*\pi_n(\mathscr{S}^k)$ by Lemma 4.3. Hence we obtain that

$$\begin{gathered}
\lambda(\eta) = [\delta, \iota_k] + i_*^{-1}(0) \text{ for some } \delta \in \pi_n(\mathscr{S}^k) \text{ i.e.,} \\
\eta \equiv \lambda^{-1}[\delta, \iota_k] \bmod \lambda^{-1}i_*^{-1}(0) = \lambda^{-1}\mathscr{P}_{k*}(\theta_1)\pi_{k+n-1}(\mathscr{S}^{n-1})
\end{gathered}$$

by Lemma 4.4. By applying i_{k*} to the both side we have

$$\theta_2 - \theta_1 \equiv 0 \bmod \mathscr{G}(\mathscr{P}_{k*}(\theta_1))\,, \quad \text{i.e., } \chi_1 \sim \chi_2\,.$$

Secondly we assume that $\chi_1 \sim \chi_2$. Hence there exists a pair (θ_1, θ_2) such that $\theta_1 \equiv \theta_2 \bmod \mathscr{G}(\mathscr{P}_{k*}(\theta_1))$ which means

$$\theta_1 - \theta_2 = i_{k*}(\eta),\ \eta \in \pi_{n-1}(\mathscr{F}_k),\ \lambda(\eta) \in \mathscr{P}_{k*}(\theta_1)\pi_{k+n-1}(\mathscr{S}^{n-1})\,.$$

Since $\mathscr{P}_{k*}(\theta_1) = \mathscr{P}_{k*}(\theta_2)$ there exists a homotopy equivalence $\varphi\colon \mathscr{S}^k \cup e^n \to \mathscr{S}^k \cup e^n$ which satisfies (see Lemma 4.5)

$$\gamma_1 - \varphi_*(\gamma_2) = i_*(\pm\lambda(\eta))\,.$$

Since $i_*(\pm\lambda(\eta)) \in i_*\mathscr{P}_{k*}(\theta_1)\cdot\pi_{k+n-1}(\mathscr{S}^{n-1}) = 0$ by Lemma 4.4, we have $\gamma_1 = \varphi_*(\gamma_2)$, i.e., φ is extendable over $\mathscr{K}(g_1)$ to $\mathscr{K}(g_2)$. Then, by Lemma 4.6, $\mathscr{E}_{k,n}(\chi_1)$ has the same homotopy type as $\mathscr{E}_{k,n}(\chi_2)$.

References

1. I. M. James, *Note on cup products*, Proc. Amer. Math. Soc., **8** (1957), 374-383.
2. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres I*, Proc. London Math. Soc., (3) **4** (1954), 196-218.

3. I. M. James and J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres II*, Proc. London Math. Soc., (3) **5** (1955), 148–166.
4. J. D. Stasheff, *A classification theorem for fibre spaces*, Topology, **2** (1963), 239–246.
5. B. Steer, *Extensions of mappings into H-spaces*, Proc. London Math. Soc., (3) **13** (1963), 219–272.
6. G. W. Whitehead, *On products in homotopy groups*, Ann. of Math., **47** (1946), 460–475.
7. *H*-spaces, Neuchâtel (Swisse) Août 1970. Lecture notes in Math., Vol. 196, Springer-Verlag.

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