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LOCAL COMPACTNESS OF FAMILIES OF CONTINUOUS POINT-COMPACT RELATIONS

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The purpose of this paper is to prove that the pointwise closure of an equicontinuous family of point-compact relations from a compact Hausdorff space to a locally compact Hausdorff uniform space is locally compact in the topology of uniform convergence. This is a generalization of a recent result of R. V. Fuller.

1. **Introduction.** The purpose of this paper is to study conditions under which a family of continuous point-compact relations is locally compact. Our theorem generalizes a theorem of Fuller [2]. For the most part we use the concepts and results of Smithson ([5], [6], [7]), and Michael [4].

We use the term relation where other authors use multivalued function or multifunction. If F is a relation from X to Y and $B \subset Y$, we write

$$F^{-1}(B) = \{x \in X: F(x) \cap B \neq \emptyset\} .$$

A relation F from a topological space X to a topological space Y is called *continuous* iff

(a) $F^{-1}(A)$ is closed in X whenever A is closed in Y, and

(b) $F^{-1}(B)$ is open in X whenever B is open in Y.

F is *point-closed* (respectively *point-compact*) iff $F(x)$ is closed (respectively compact) for each $x \in X$.

We recall three topologies defined on the collection of all nonempty subsets of a topological space X (see Michael [4]). The collection of all sets of the form $\{B \subset X: B \subset U\}$ where U is open in X, is a base for the *upper semi-finite* (u.s.f.) topology. The collection of all sets of the form $\{B \subset X: B \cap U \neq \emptyset\}$ where U is open in X, is a subbase for the *lower semi-finite* (l.s.f.) topology. The *finite* topology is the supremum of the l.s.f. and u.s.f. topologies. Equivalently, the finite topology has as a basis all sets of the form $\langle U_1, \cdots, U_n \rangle = \{A \subset X: A \cap U_i \neq \emptyset, 1 \leqq i \leqq n \text{ and } A \subset \bigcup_{i=1}^n U_i\}$, where $U_1, \cdots, U_n$ are open in X. A relation $F: X \rightarrow Y$ is continuous if and only if the function $F: X \rightarrow P(Y)$ (the power set of Y with the finite topology) is continuous, c.f. remark following Theorem 2.7 of [7].

Let X and Y be topological spaces and let $\mathscr{F}$ be a set of relations from X to Y. The pointwise topology $\mathscr{P}$ [7] on $\mathscr{F}$ has a subbase consisting of the sets of the form $\{F \in \mathscr{F}: F(x) \cap U \neq \emptyset\}$ or $\{F \in \mathscr{F}: F(x) \subset V\}$ where $x \in X$, and U, V are open in Y. We note that the projections $\{\Pi_x : x \in X\}$ defined by $\Pi_x(F) = F(x)$ are continuous functions into the collection of all nonempty subsets of Y with the finite topology.

Let X be a topological space, $(Y, \mathscr{U})$ a uniform space and let $\mathscr{F}$ be a set of relations from X to Y. For $V \in \mathscr{U}$, let $W(V) = \{(F, G) \in \mathscr{F} \times \mathscr{F}$: for all $x \in X$, $(y, G(x)) \cap V \neq \varnothing$ for all $y \in F(x)$, and $(F(x), y') \cap V \neq \varnothing$ for all $y' \in G(x)\}$. Let $\mathscr{W}$ be the uniformity on $\mathscr{F}$ generated by the collection of all such encourages as $W(V)$. The topology generated by $\mathscr{W}$ is the topology of *uniform convergence* [5] and is denoted by $\mathscr{U}\mathscr{C}$. If $(Y, \mathscr{U})$ is a uniform space, $\mathscr{F}$ is called equicontinuous at $x \in X$ [5] iff for every $V \in \mathscr{U}$ there is a nbhd. U of x such that for all $F \in \mathscr{F}$,

(a) $F(U) \subset V(F(x))$, and

(b) $F(z) \cap V(y) \neq \varnothing$ for all $z \in U$ and for all $y \in F(x)$.

We now state a theorem of Smithson which we use in the final section.

THEOREM 1.1. ([5]). *If $\mathscr{F}$ is an equicontinuous family of point-compact relations from a compact space X to a uniform space Y, then on $\mathscr{F}$, $\mathscr{P} = \mathscr{U}\mathscr{C}$.*

For further details and a survey the reader is referred to Smithson [7].

2. Local compactness of a space of relations.

We begin by proving two lemmas.

LEMMA 2.1. *Let F be a point-compact relation from X to Y, and let $A = \{(\{x\}, F(x)) : x \in X\}$ be compact in $P(X) \times P(Y)$, where each of $P(X)$, $P(Y)$ has the finite topology. Then F is a compact subset of $X \times Y$.*

Proof. Let $\mathscr{O}$ be an open cover of F in $X \times Y$. For each $x \in X$, $\{x\} \times F(x)$ is compact; so there is a finite subcollection $V_i^x \times U_i^x$, $1 \leqq i \leqq n$ of $\mathscr{O}$ which covers the set. We can assume that $x \in V_i^x$ for each i and that $F(x) \cap U_i^x \neq \varnothing$. $\langle V_1^x, \cdots, V_n^x \rangle \times \langle U_1^x, \cdots, U_n^x \rangle$ is an open set in $P(X) \times P(Y)$. For each $x \in X$, we obtain such a set, and this leads to an open cover of A. Since A is compact, there is a finite subcover $\langle V_1^{x_i}, \cdots, V_{n_i}^{x_i} \rangle \times \langle U_1^{x_i}, \cdots, U_{n_i}^{x_i} \rangle$, $1 \leqq i \leqq k$. Finally, $\{V_j^{x_i} \times U_j^{x_i} : 1 \leqq j \leqq n_i, 1 \leqq i \leqq k\}$ is a cover of F and so F is compact.

LEMMA 2.2. *If F is a continuous relation from X to Y, then the function $g: X \to P(X) \times P(Y)$ defined by $g(x) = (\{x\}, F(x))$ is continuous. ($P(X)$ and $P(Y)$ both have finite topology.)*

Proof. Let $\langle V\rangle \times \langle U_1, \cdots, U_n\rangle$ be a basic open nbhd. of $(\{x\}, F(x))$. The function $F: X \to P(Y)$ is continuous, hence there exists a nbhd. $N \subset V$ of x such that $F(N) \subset \langle U_1, \cdots, U_n\rangle$. Clearly, $g(N) \subset \langle V\rangle \times \langle U_1, \cdots, U_n\rangle$.

From the above lemmas it follows that if X is compact, Y is T_2 and F is a continuous point-compact relation, then F is a compact subset of $X \times Y$.

The proof of the following lemma is straightforward.

LEMMA 2.3. *Let $\mathscr{F}$ be a family of relations from a topological space X to a topological space Y. Then the* l.s.f. *topology on $\mathscr{F}$ is contained in $\mathscr{P}$.*

LEMMA 2.4. *Let X be compact Hausdorff, $(Y, \mathscr{V})$ a uniform space, and $\mathscr{F}$ a family of continuous point-compact relations from X to Y. Then on $\mathscr{F}$ the* u.s.f. *topology is smaller than $\mathscr{U}\mathscr{C}$.*

Proof. If $F \in \mathscr{F}$, then F is a compact subset of $X \times Y$. Suppose $F \subset N$ is an open subset of $X \times Y$. Let $\mathscr{U}$ be the (unique) uniformity on X. Then from [3] page 199, it follows that there exist $U \in \mathscr{U}$, $V \in \mathscr{V}$ such that $F \subset \cup \{U(x) \times V(y): x \in X, y \in F(x)\} \subset N$. Then $W(V)[F] \subset N$, thus completing the proof.

LEMMA 2.5. *Let $\mathscr{F}$ be an equicontinuous family of relations from a T_1-space X to a uniform space $(Y, \mathscr{V})$. Then on $\mathscr{F}$, $\mathscr{P} \subset$ the finite topology.*

Proof. Let $[x, U] = \{G \in \mathscr{F}: G(x) \subset U\}$, where $x \in X$ and U is open in Y. If $F \in [x, U]$, then $N = \langle X \times U \cup (X - \{x\}) \times Y\rangle$ is a nbhd. of F in the finite topology, and $F \in N \subset [x, U]$. Suppose $F \in M = \{G \in \mathscr{F}: G(x) \cap W \neq \varnothing\}$ where $x \in X$ and W is open in Y. If $F(x) \subset W$, then the above method works, and so we assume that $F(x) \not\subset W$. Let $p \in F(x) \cap W$ and let $V \in \mathscr{V}$ such that $\overline{V^2(p)} \subset W$. Since $\mathscr{F}$ is equicontinuous at x, there is a nbhd. U of x such that for all $T \in \mathscr{F}$, $T(U) \subset V(T(x))$. Now $F \in \langle U \times [V(p)]^0, U \times (Y - \overline{V^2(p)}), U \times W, (X - \{x\}) \times Y\rangle \subset M$, which completes the proof.

LEMMA 2.6. *Let X be compact Hausdorff, and let $(Y, \mathscr{V})$ be a uniform space. Let $\mathscr{F}$ be an equicontinuous family of point-compact relations from X to Y. If $\hat{\mathscr{F}}$ is the $\mathscr{P}$-closure of $\mathscr{F}$ in the space of all point-compact relations from X to Y, then $\hat{\mathscr{F}}$ is closed in $\mathscr{C}(X \times Y)$, the space of all nonempty compact subsets of $X \times Y$ with the finite topology.*

Proof. Let (F_α) be a net in $\hat{\mathscr{F}}$ converging to $F \in \mathscr{C}(X \times Y)$. Note that the domain of F is X; for if $x \in X - \operatorname{dom} F$, then $\{G \in \mathscr{C}(X \times Y) \colon G \subset X \times Y - \{x\} \times Y\}$ is an u.s.f. nbhd. of F in $\mathscr{C}(X \times Y)$, and F_α is eventually in this nbhd., a contradiction. Clearly F is point-compact. We now show that $(F_\alpha) \to F$ in $\mathscr{P}$. Let U be open in Y, and suppose $F(x) \subset U$. Then the set $N = \langle X \times U \cup (X - \{x\}) \times Y \rangle$ is a nbhd. of F in the finite topology on $\mathscr{C}(X \times Y)$. Since F_α is eventually in N, it follows that F_α is eventually in $[x, U]$. If we are given a nbhd. $M = \{G \colon G(x) \cap W = \emptyset\}$, ($W$ open in Y) of F and $F(x) \not\subset W$, then we employ the technique used in the last part of the proof of Lemma 2.5, and use the fact that $\hat{\mathscr{F}}$ is equicontinuous ([5], Lemma 6).

We now prove the main result.

THEOREM 2.7. *Let $\mathscr{F}$ be an equicontinuous family of point-compact relations from a compact Hausdorff space X to a locally compact Hausdorff uniform space Y. Let $\hat{\mathscr{F}}$ be the $\mathscr{P}$-closure of $\mathscr{F}$ in the space of all point-compact relations from X to Y. Then $\hat{\mathscr{F}}$ is locally compact in $\mathscr{U}\mathscr{C}$.*

Proof. We first note that on $\hat{\mathscr{F}}$, $\mathscr{P} \subset \mathscr{U}\mathscr{C}$ ([5], Lemma 1). From Lemmas 2.3 and 2.4 it follows that the finite topology is contained in $\mathscr{U}\mathscr{C}$, and since $\hat{\mathscr{F}}$ is equicontinuous, we have by Theorem 1.1, $\mathscr{P} = \mathscr{U}\mathscr{C}$ on $\hat{\mathscr{F}}$. From Lemma 2.5, it follows that the finite topology equals $\mathscr{U}\mathscr{C}$ on $\hat{\mathscr{F}}$. Each member of $\hat{\mathscr{F}}$ is compact, and so $\hat{\mathscr{F}} \subset C(X \times Y)$. By Lemma 2.6, $\hat{\mathscr{F}}$ is closed in the finite topology on $C(X \times Y)$. Since $C(X \times Y)$ is locally compact ([4], Prop. 4.4.1), $\hat{\mathscr{F}}$ is locally compact.

If in the above theorem, each $F \in \mathscr{F}$ is a (single valued) function, then it is easy to verify that each member of $\hat{\mathscr{F}}$ is also a function. Hence a recent result of R. V. Fuller [2] on the local compactness of $\hat{\mathscr{F}}$ is a special case of the above theorem.

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