PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 1, 1974

RINGS WHOSE PROPER HOMOMORPHIC IMAGES ARE RIGHT SUBDIRECTLY IRREDUCIBLE

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The structure of the lattice of ideals in a ring whose every proper homomorphic image is right subdirectly irreducible has been determined in all cases except when the ring is primitive and contains a nonzero primitive ideal. In the commutative case, the rings described in the title have been shown to be noetherian and their proper homomorphic images to be self-injective.

Right subdirectly irreducible rings have been introduced in [1, 2] and their structure determined under chain conditions. In this paper, we consider rings whose proper homomorphic images are right subdirectly irreducible. It is customary in literature to define rings whose proper homomorphic images satisfy a certain property P as restricted P rings and we will use this terminology. The main object of this paper is to obtain characterizations of restricted right subdirectly irreducible rings (denoted by r-RSI) in terms of their ideal structure. As an interesting side result we show that commutative restricted subdirectly irreducible rings are noetherian and that they are closely related to the restricted self-injective rings of Faith [3], and Levy [5].

Preliminaries. All rings considered here contain an identity. A ring R is called r-RSI if each proper homomorphic image of R is RSI, while an RSI ring is one in which the intersection of all nonzero right ideals (which is always two-sided) is nonzero. For properties of such rings, we refer to [1, 2].

If A is a nonzero ideal in a r-RSI ring R, then R/A is RSI. This implies, R has an ideal B which properly contains A and that A is maximal in the family of right ideals of R properly contained in B. The proof of the following proposition depends on this fact.

PROPOSITION 1. *Let R be a r-RSI ring. Then*

(i) *any two nonzero ideals in R are either comparable or their intersection is zero;*

(ii) *if L is a nonzero ideal and K a strictly right (not two-sided) ideal, then either $L \cap K$ is a strictly right ideal or $L \subseteqq K$;*

(iii) *intersection of any family of strictly right ideals is zero or is a strictly right ideal.*

We will prove (ii) and leave (i) and (iii) whose proofs are similar.

Proof of (ii). If $L \cap K$ is a nonzero two-sided ideal, then $R/[L \cap K]$ is a RSI ring in which $L/[L \cap K]$ and $K/[L \cap K]$ are right ideals with intersections zero. Hence $L \cap K$ is either equal to L or K. Being two-sided, it must equal L. Hence $L \subseteq K$.

PROPOSITION 2. *If R is a r-RSI ring and P a nonzero prime (or primitive) ideal, then P is a maximal right ideal of R.*

Proof. Since R/P is a prime ring, the heart of R/P cannot be nilpotent. Being the unique minimal right ideal, it is idempotent and consequently [1, Proposition 1.3] R/P is a division ring.

We can now consider a classification of all r-RSI rings on the basis of the number of prime ideals. Since primitive ideals in a ring are prime, the following lemma implies that rings without nonzero prime ideals are simple.

LEMMA 3. *If R is a ring without a nonzero primitive ideal, then it is a simple ring.*

Proof. Since a primitive ideal is the largest two-sided ideal contained in a maximal right ideal, any nonzero two-sided ideal must be contained in a maximal right ideal and therefore, is contained in a primitive ideal.

Now suppose R is a r-RSI ring with two distinct nonzero prime ideals P and Q. Then each is a maximal right ideal by Proposition 2 and $P \cap Q = 0$ by Proposition 1. Thus, $R = P \oplus Q$ is a direct sum of two division rings. This gives us the preliminary classification of r-RSI rings as stated in the following.

THEOREM 4. *Let R be a r-RSI ring. Then either*

(i) *R has no nonzero prime ideal and it is a simple ring, or*

(ii) *R has exactly two nonzero prime ideals and it is isomorphic to a direct sum of two division rings, or*

(iii) *R has precisely one nonzero prime ideal.*

Since the ideal structure of rings of type (i) and (ii) above is trivial, we consider only case (iii) in what follows.

Nontrivial r-RSI rings. Suppose R is a r-RSI ring with precisely one nonzero prime ideal. If R is not a primitive ring, i.e., if 0 is not a primitive ideal, then M is the unique primitive ideal of R. Also, M is a maximal right ideal of R by Proposition 2. Hence, M is the unique maximal right ideal and consequently R is in this case a local ring. The following property of local rings will be needed in the sequel and is a partial converse of [1, Theorem 2.1].

PROPOSITION 5. *Let R be a local ring, the maximal right ideal M being nilpotent. Then R has both* a·c·c *and* d·c·c *on right ideals if and only if M is finitely generated as a right ideal.*

Proof. Let $M^n = 0$ and $M^{n-1} \neq 0$. Since M is finitely generated as a right R-module, each of M/M^2, M^2/M^3, $\cdots$, M^{n-1} are finite dimensional vector spaces over R/M and consequently the composition series for each of them can be lifted and connected into a composition series for R_R. The converse is trivial.

COROLLARY 6. *If R is a local ring with nil maximal ideal, then R has* d·c·c *on right ideals whenever it has* a·c·c *on right ideals.*

Proof. With *a·c·c* on right ideals, nil ideals are nilpotent and every right ideal is finitely generated.

Since, for a ring R, homomorphic image of a homomorphic image is itself a homomorphic image of it, each homomorphic image of a r-RSI ring is RSI and r-RSI. In the following theorem we show that such rings are local artinian principal right ideal rings.

THEOREM 7. *Let R be a* RSI, r-RSI *ring. Then R has a unique maximal right ideal M which is nilpotent and all of the right ideals of R are of the form m^kR, $k = 0, 1, 2, \cdots, n$ and m an element of M.*

Proof. We exclude division rings from discussion for which the theorem holds trivially. Hence, the heart H of R is a proper nonzero ideal of R. Then R has at least one primitive ideal M containing H (and hence nonzero) and since R is RSI, its intersection with any nonzero ideal is nonzero. Then by Proposition 1, M contains each ideal of R and thus it is the unique primitive ideal of R. (Note that 0 cannot be a primitive ideal unless R is a division ring). Now, let A and B be any two nonzero ideals of R. Again $A \cap B \neq 0$ and hence $A \subsetneqq B$ or $B \subsetneqq A$. This together with the fact that to each nonzero ideal in a r-RSI ring corresponds a unique smallest ideal containing it implies that the ideals of R form a well-ordered chain under inclusion. In particular, ideals of R satisfy d·c·c. We will now show that each right ideal is two-sided in R. In fact, suppose there exists a strictly right ideal. Then, since the intersection T of all strictly right ideals is nonzero (it contains H), it must be a strictly right ideal by Proposition 1. Let L denote the sum of all two-sided ideals contained in T and containing H. Clearly, by the choice of T and L, there is no right ideal of R between T and L. This however, implies that T/L must be the heart of the RSI ring R/L; or in particular, that T must be a two-sided ideal which is a contradiction.

Now, R has d·c·c on right ideals and hence the unique primitive ideal M of R coincides with the Wedderburn radical and must be nilpotent. If $M^n = 0$ and $M^{n-1} \neq 0$, then M^{n-1} is contained in the left annihilator of M. Since for a RSI ring with d·c·c, the left annihilator of the maximal ideal is H [1, Theorem 3.1], we have $M^{n-1} = H$. Now, if A is any ideal of R, there exists an integer i, $0 \leqq i \leqq n-1$ such that $M^{i+1} \subsetneqq A \subset M^i$. Then applying the above argument to R/M^{i+1}, we have M^i/M^{i+1} contained in the annihilator of M/M^{i+1} and hence it is the heart of R/M^{i+1}. Thus $A \subsetneqq M^{i+1}$ which implies that $A = M^{i+1}$. This shows that $R, M, M^2, \cdots, M^{n-1}, 0$ exhaust all the ideals of R. It is easy to argue that each of these is a principal right ideal.

It may be remarked that in this ring, left ideals need not be two-sided and infact they need not satisfy either a·c·c or d·c·c. An example of such a ring is given below.

EXAMPLE 8. Let F be a field and σ a monomorphism of F into itself which is not onto and α an element of F, transcendental over F^σ. Let R be the ring of ordered pairs of members of F with componentwise addition and with $(a, b)(c, d) = (ac, bc + a^\sigma d)$ as the multiplication. Then there is exactly one nontrivial proper two-sided ideal in R, namely $H = \{(0, b): b \in F\}$ and it is also the only right ideal. However, $L_n = \{(0, p\alpha^n): p \in F^\sigma\}$ are distinct left ideals for $n = 1, 2, \cdots$ whose sum is direct. Clearly R does not satisfy a·c·c on left ideals.

Levy [5] has considered commutative noetherian rings whose proper homomorphic images are self-injective. We first observe that if R is commutative and each of its proper homomorphic images is subdirectly irreducible, then for any nonzero ideal A in R, R/A satisfies the above theorem and has a·c·c on ideals. In particular, this means any chain of ideals $A \subsetneqq A_1 \subsetneqq A_2 \cdots$ is finite and thus R is noetherian. Further, the homomorphic image R/A is local, satisfies d·c·c and also the ideals in R/A satisfy the annihilator condition Ann (Ann I) $= I$. Thus R/A is a commutative quasi-Frobenius local ring. Also, by a well-known theorem of Morita [6], quasi-Frobenius local rings are exactly the same as the completely indecomposable rings which are in particular RSI. We have thus proved the following theorem, which is another of the instances of an observation by K. Morita that complete indecomposability and injectivity are closely related concepts.

THEOREM 9. *If R is commutative, then the following are equivalent:*

(i) *R is restricted subdirectly irreducible;*

(ii) *R has* a·c·c *on ideals and each proper homomorphic image is a local self-injective ring;*

(iii) *each proper homomorphic image of R is a quasi-Frobenius local ring.*

REMARK. It is well-known that a right principal ideal ring with d·c·c on right ideals is the same as a right uniserial ring. Faith [3] has proved that a commutative ring is restricted uniserial if and only if it is (i) itself uniserial or (ii) a local artinian ring with $(\mathrm{Rad}\, R)^2 = 0$ and $\mathrm{Rad}\, R$ is a direct sum of two minimal ideals or (iii) it is a dedekind domain not a field. (He has also remarked in the same paper that commutative restricted uniserial rings are the same as restricted quasi-Frobenius rings.) While it is easy to see that rings satisfying the above Theorem 9 are restricted uniserial, a dedekind domain (e.g. Z) is not restricted subdirectly irreducible.

r-RSI local rings. We now consider the nonprimitive r-RSI rings. That these rings are local has been observed before. We will denote the maximal ideal by M. Clearly M is nilpotent or not according as R is not or is a prime ring. If M is nilpotent, the ideal structure of R is completely determined by the following:

THEOREM 10. *Let R be a r-*RSI *local ring with a nilpotent maximal ideal M. Then either*

(a) *R is* RSI *and hence satisfies Theorem 7 above, or*

(b) *$M^2 = 0$, every nonzero proper ideal of R other than M is a minimal ideal and M is the direct sum of any two of these minimal ideals. Further, there is no right ideal of R strictly between M and any one of these minimal ideals.*

Proof. If $M^2 \neq 0$, so that the index of nilpotency $n \geqq 3$, we will show that M is a principal right ideal. In fact, $M^{n-1} \neq 0$ implies that for some $m \in M$, $(m \notin M^2)$ we have $mM^{n-2} \neq 0$. In particular, we have $mR \cap M^{n-1} \neq 0$. In addition, $mR \cap M^{n-1}$ is a two-sided ideal because if $mx \in M^{n-1}$ is nonzero, the conditions $m \notin M^2$ implies that x cannot be a unit, or equivalently, $x \in M$. Now, R/M^{n-1} is a ring satisfying Theorem 7 above and hence M/M^{n-1} is a right ideal in it generated by each $u + M^{n-1}$ such that $u \in M \backslash M^2$. Thus, if $a \in R$ is arbitrary, $am = mb + v$ for some $b \in R$ and $v \in M^{n-1}$. Consequently, $amx = mbx + vx = mbx \in mR$. Since M^{n-1} is a two-sided ideal, $amx \in M^{n-1}$ is trivial and we have $amx \in mR \cap M^{n-1}$ proving that $mR \cap M^{n-1}$

is a two-sided ideal. Now Proposition 1 applies and we must have $M^{n-1} \subseteqq mR$. Since, by considering right ideals in R/M^{n-1}, it is clear that there are no right ideals in R containing M^{n-1} other than the powers of M, we have proved $M = mR$. By a theorem of Feller [4, Corollary 3.6] it follows that all of the right ideals in R may be listed as $R, mR, m^2R, \cdots, m^{n-1}R, 0$ proving part (a) of our theorem.

Now suppose $M^2 = 0$. Any other nonzero ideal A in R must be contained in M and by applying Theorem 9 to R/A we see that there is no right ideal of R between A and M. Thus, if B is any other nonzero ideal, we have $M = A + B$ and Proposition 1 requires that $A \cap B = 0$. Thus, all ideals of R other than M are minimal and M is a direct sum of any two of them. This completes the proof.

Again we compare this theorem with that of Faith and Levy in the commutative case. If R is commutative r-RSI local ring with maximal ideal nil, by Theorem 9 it satisfies a·c·c and then by Corollary 6 it is artinian. Thus we have,

COROLLARY 11. *If R is commutative, the following are equivalent:*

(i) *R is a restricted subdirectly irreducible local ring with maximal ideal nil,*

(ii) *R is either a uniserial ring or a local artinian ring with* $(\mathrm{Rad}\, R)^2 = 0$ *and* $\mathrm{Rad}\, R$ *is a direct sum of two minimal ideals.*

The observation of Levy [5, page 152] about the structure of rings of type (ii) above holds in noncommutative case as well. A typical example of such a noncommutative ring is $D[x, y]/(x^2, y^2, xy)$ where D is a division ring. In this case, M is generated by x and y. The minimal ideals are generated by $x + \alpha y$ for distinct α in the center of D; while if α is not in the center, $x + \alpha y$ generates the ideal M. M is a direct sum of any two right ideals $(x + \alpha_1 y)R + (x + \alpha_2 y)R$ for α_1, α_2 not in the center of D.

r-RSI rings with nonnil maximal ideal. If M is not nil in a r-RSI local ring R, then M cannot be the prime radical of R, and since the only other prime ideal that R may have must be zero, R is a prime ring. The ring of formal power series in one variable over a field or over a division ring is an example of a local r-RSI prime ring. In general, if R is an arbitrary r-RSI local ring which is prime, then as in Theorem 7 above, we can conclude that there is no right ideal of R containing a power of M unless it is itself a power of M. Also, there is no ideal in R other than one of the powers of M. Note that $\bigcap_{i=1}^{\infty} M^i$ must be 0 or it must equal M^n for some n because otherwise $R/\bigcap_{i=1}^{\infty} M^i$ cannot be RSI. This determines the ideals of R, and gives us the following characterization of r-RSI prime local rings:

THEOREM 12. *If R is a r-RSI local ring whose maximal ideal M is not nil, then R is a prime ring, the only ideals of R are powers of M and there is no right ideal of R between any two powers of R. In addition, we either have $\bigcap_{i=1}^{\infty} M^i = 0$ or, for some integer n, $M^n = M^{n+1}$ and this is the unique minimal two-sided ideal of R.*

Questions. It appears that in all cases of a r-RSI ring, we can conclude that it is local except when it is primitive. The authors have been unable to find an example of a nonsimple primitive r-RSI ring with a nonzero primitive ideal M. It is clear that in such a case, each proper homomorphic image of R must satisfy Theorem 7 and hence must be local artinian. It is conjectured that a primitive r-RSI ring exists which is not simple. Also, we have not been able to settle the question: Whether in a r-RSI local ring, the condition M is nil implies M is nilpotent. The answer is yes in the commutative case.

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Received January 29, 1973.

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