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PERTURBATION THEORY FOR GENERALIZED FREDHOLM OPERATORS

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Successful development of the theory of Fredholm and semi-Fredholm operators and a general recognition of the importance of the subject provides an impetus for the study of various generalizations. In this paper, a study is made of operators whose ranges and null spaces are closed complemented subspaces. In particular, if T is such an operator, a rather general sufficient condition is obtained to ensure that $T-U$ is of the same kind. This perturbation theorem includes, as special cases, previous results due to Dieudonné, Saphar, and Crownover.

1. Definitions and basic properties. If X is a Banach space, write $B(X)$ to denote the space of continuous linear operators defined on X. We shall study the class of operators T in $B(X)$ which have the property that both the null space $N(T)$ and the range $R(T)$ are closed complemented subspaces of X. This class of operators contains all the Fredholm operators and will therefore be called the *generalized Fredholm* operators, denoted $GF(X)$. We observe immediately that when X is a Hilbert space, then $GF(X)$ consists merely of those operators which have closed range and that in all cases, $GF(X)$ contains all projections. Such an observation would discourage one's optimism about paralleling the theory of Fredholm operators to any reasonable extent. However, the fact that $GF(X)$ has already been studied under another guise encourages further scrutiny; more precisely, we introduce the notion of *generalized inverse* as follows: an operator T in $B(X)$ is said to have a generalized inverse S in $B(X)$ if the following equations are valid:

$$TST = T \tag{1}$$

$$STS = S\,. \tag{2}$$

The above notion has attracted a great deal of interest in the finite dimensional case and has been the subject of at least three recent monographs ([3], [8], [9]). It is perhaps appropriate to remark that conditions (1) and (2) are usually augmented by others leading to a variety of notions with a rather variable terminology. Some extensions to infinite dimensional situations have been considered by F. J. Beutler [2], and W. T. Reid [10] and others. Particular attention should be paid to the early paper of Atkinson [1] in which he calls operators satisfying (1) and (2) *relatively regular* and derives many of their properties. The fact that generalized Fredholm operators are exactly those which have a generalized inverse has been known for a long time but since its proof is usually given in a finite dimensional setting, it is included here in the general case.

LEMMA. *T is a generalized Fredholm operator if and only if T has a generalized inverse.*

Proof. Let T belong to $GF(X)$ so that there exist closed subspaces X_1 and X_2 such that X can be decomposed as $N(T) \oplus X_1$ and $R(T) \oplus X_2$. Let P denote the projection onto $R(T)$ parallel to X_2. Now write T_1 for the restriction of T to X_1 so that T_1 has a continuous inverse defined on $R(T)$. Therefore, $T_1^{-1}P$ is well defined in $B(X)$; it is easy to check that $T_1^{-1}P$ is a generalized inverse of T.

Conversely, if S is a generalized inverse of T, then equations (1) and (2) imply that TS is a projection of X onto $R(T)$ parallel to $N(S)$ and $I - ST$ is a projection of X onto $N(T)$ parallel to $R(S)$. Hence T belongs to $GF(X)$.

2. Perturbation theory. In general, $GF(X)$ is not an open set nor is it stable under compact perturbations. To see this, we recall a construction due to R. J. Whitley ([7], V.2.6): if T is an operator with closed range of infinite codimension and $N(T)$ is infinite dimensional, then it is possible to construct a compact operator B such that $T + \lambda B$ does not have closed range for any $\lambda \neq 0$. The perturbation problem which we study therefore, gives a sufficient condition on the operators concerned and our result will include various interesting special cases including those of Saphar [11] and Crownover [4].

THEOREM. *Let T be a generalized Fredholm operator defined on Banach space X and suppose that S is a generalized inverse of T. Let U be an operator in $B(X)$ such that $\| U \| < \| S \|^{-1}$ and $(I - US)^{-1}U$ maps $N(T)$ into $R(T)$. Then $S(I - US)^{-1}$ and $(I - SU)^{-1}S$ are equal and their common value V is a generalized inverse of $T - U$. Moreover, $N(T - U)$ is a complementary subspace to $R(S)$ and $R(T - U)$ is a complementary subspace to $N(S)$ so that $N(T - U)$ and $R(T - U)$ are linearly homeomorphic, respectively to $N(T)$ and $R(T)$.*

Proof. Since $\| U \| < \| S \|^{-1}$, the Neumann series of $(I - US)^{-1}$ and $(I - SU)^{-1}$ converge and it is easy to see, by writing them out, that $S(I - US)^{-1}$ and $(I - SU)^{-1}S$ are equal. Then

$$\begin{aligned} V(T-U)V &= (I-SU)^{-1}S(T-U)S(I-US)^{-1} \\ &= (I-SU)^{-1}(STS-SUS)(I-US)^{-1} \\ &= (I-SU)^{-1}S(I-US)(I-US)^{-1} \\ &= V\,. \end{aligned}$$

Now consider

$$\begin{aligned} T-U-(T-U)V(T-U) &= [I-(T-U)S(I-US)^{-1}](T-U) \\ &= [I-US-(T-U)S](I-US)^{-1}(T-U) \\ &= (I-TS)(I-US)^{-1}(T-UST+UST-U) \\ &= (I-TS)(I-US)^{-1}[(I-US)T+U(ST-I)] \\ &= (I-TS)(I-US)^{-1}U(ST-I) \\ &= 0 \end{aligned}$$

if $(I-US)^{-1}U$ maps $R(ST-I)$ into $N(I-TS)$. But, as noted in the proof of the Lemma, $R(ST-I)$ is $N(T)$ and $N(I-TS)$ is $R(T)$. Hence the first part of the proof is complete.

Now

$$R(V) = R[S(I-US)^{-1}] = R(S)$$

so that

$$X = N(T-U) \oplus R(V) = N(T-U) \oplus R(S)\,.$$

Similarly

$$N(V) = N[(I-SU)^{-1}S] = N(S)$$

so that

$$X = R(T-U) \oplus N(V) = R(T-U) \oplus N(S)\,.$$

COROLLARIES. 1. *If $N(U) \supseteqq N(T)$ or $R(U) \subseteqq R(T)$ with $\|U\| \leqq \|S\|^{-1}$, then $T-U$ is generalized Fredholm.*

Proof. If $N(U) \supseteqq N(T)$, then $(I-US)^{-1}U$ maps $N(T)$ onto $\{0\}$. If $R(U) \subseteqq R(T)$, then we observe that $(I-US)^{-1}U$ can be written as $U(I-SU)^{-1}$ so that its range lies inside $R(U)$ and hence it maps $N(T)$ into $R(T)$.

2. *The class of left invertible operators is given by $\{T \in GF(X)\colon N(T) = \{0\}\}$; the class of right invertible operators is given by $\{T \in GF(X)\colon R(T) = X\}$. Hence from Corollary* 1, *we obtain the well-known result* (Dieudonné [5]) *that these classes are open sets in $B(X)$.*

3. *Suppose T is a generalized Fredholm operator with $R(T) \supseteqq N(T)$. Let M be closed subspace of X such that $R(T) \supseteqq M \supseteqq N(T)$*

and $TM = M$. *Then if* $UM \subseteqq M$ *and* $\| U \| < \| S \|^{-1}$, *the conclusions of the Theorem are valid.*

Proof. $(TS)M = (TST)M = TM = M$ so that $SM \subseteqq T^{-1}M$. Now if $m \in T^{-1}M$, then $Tm \in M = TM$ so that there exists $m' \in M$ such that $Tm = Tm'$. But $m - m' \in N(T) \subseteqq M$ so that $m \in M$. Hence $T^{-1}M \subseteqq M$. Therefore, $SM \subseteqq M$ and so $(I - US)^{-1}UM \subseteqq M$. Hence we can write

$$\begin{aligned} R[(I - TS)(I - US)^{-1}U(ST - I)] &= (I - TS)(I - US)^{-1}UN(T) \\ &\subseteqq (I - TS)(I - US)^{-1}UM \\ &\subseteqq (I - TS)M \\ &\subseteqq (I - TS)R(T) = \{0\} . \end{aligned}$$

Hence the result.

4. *If* $R(T^n) \supseteqq N(T)$ *for each* n *and all* $R(T^n)$ *are closed, then* $M = \bigcap_1^\infty R(T^n)$ *satisfies the conditions of Corollary* 3. *This gives Saphar's main result in* [11] (Théorèmes 1 and 2).

5. *If* $N(T) = \{0\}$ *and* $R(T)$ *has codimension* 1 (*and is therefore closed*) *and* $\bigcap_1^\infty R(T^n) = \{0\}$, *then Crownover* [3] *calls such an operator a shift on* X. *Clearly, for any such shift, all* $R(T^n)$ *are closed since* T *is Fredholm. Therefore, shifts satisfy the condition of Corollary* 4. *Crownover's Theorem* 2 ([4], p. 236) *considers* $T - \lambda I$ *when* T *is a shift and derives a result essentially the same as ours in this very special case.*

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