PACIFIC JOURNAL OF MATHEMATICS
Vol. 52, No. 1, 1974

ON THE EXCEPTIONAL SETS FOR SPACES OF POTENTIALS

DAVID R. ADAMS

New results on the Bessel and Besov-Lipschitz potentials on R^n are obtained via recent results in nonlinear potential theory. In particular their respective exceptional classes are shown to be identical when $p > 2 - \alpha/n$. By the same techniques, results on thin sets and traces of potentials are obtained.

1. Introduction. In the theory of "perfect functional completion" of a given normed linear space of smooth functions defined on R^n, the idea is to look for a Banach space with respect to the given norm in, say, the class of Lebesgue measurable functions by taking limits in the norm of smooth functions. Associated in a natural way with any such completion is a σ-algebra of exceptional sets of R^n. These exceptional sets give the limits up to which one can pick a canonical equivalence class representative that is defined on the largest possible set. In this note, the exceptional sets for two important perfect functional completions are reexamined in light of recent development in nonlinear potential theory — see e.g., [3], [4], and [7]. The two classes of interest are: $\Lambda_{\alpha,p} = \Lambda_{\alpha\,p}(R^n)$, the Besov-Lipschitz potentials on R^n, and $L_{\alpha.p} = L_{\alpha\,p}(R^n)$, the Bessel potentials on R^n. Their respective exceptional classes are denoted by $\mathfrak{A}^{\alpha,p}$ and $\mathfrak{B}^{\alpha,p}$ in [5], where they are studied extensively — see especially Chapter III page 289 in [5] where a criterion for belonging to $\mathfrak{A}^{\alpha,p}$ or $\mathfrak{B}^{\alpha,p}$ is given. This is utilized in Proposition 1 below.

$L_{\alpha,p}(R^n) = g_\alpha(L_p(R^n))$, i.e., the convolution image of the p-summable functions on R^n, $1 \leqq p \leqq \infty$, under the Bessel kernel $g_\alpha = g_\alpha^{(n)}(x)$, the $L_1(R^n)$ function whose Fourier transform is $(1 + |\xi|^2)^{-\alpha/2}$, $\xi \in R^n$, $\alpha > 0$. The norm on $L_{\alpha.p}$ is $\|u\|_{\alpha\,p} = \|f\|_p$, where $u = g_\alpha * f$ ($\|\cdot\|_p$ the usual norm on L_p). For $\Lambda_{\alpha,p}$, we say $u \in \Lambda_{\alpha\,p}$, $1 \leqq p \leqq \infty$, $0 < \alpha < 1$, if $u \in L_p$ and

$$(1) \qquad |u|_{\alpha,p} \equiv \|u\|_p + \left\{\int_{R^n}\int_{R^n}\left(\frac{|\Delta_y u(x)|}{|y|^\alpha}\right)^p \frac{dxdy}{|y|^n}\right\}^{1/p}$$

is finite, $\Delta_y u(x) = u(x-y) - u(x)$. For $1 \leqq \alpha < 2$, $\Delta_y u(x)$ is replaced by $\Delta_y^2 u(x) = u(x-y) + u(x+y) - 2u(x)$ in (1). And finally for $\alpha \geqq 2$, $u \in \Lambda_{\alpha,p}$ iff $u \in L_p$ and $\partial u/\partial x_k \in \Lambda_{\alpha-1,p}$, $k = 1, \cdots, n$. Other equivalent definitions of $\Lambda_{\alpha,p}$ can be found in [9].

It is well known that $\Lambda_{\alpha,2} = L_{\alpha,2}$ for all $\alpha > 0$ and hence $\mathfrak{A}^{\alpha,2} = \mathfrak{B}^{\alpha,2}$. But for $1 \leqq p \leqq 2$, $\Lambda_{\alpha,p} \subset L_{\alpha,p}$ and for $2 \leqq p \leqq \infty$, $L_{\alpha,p} \subset \Lambda_{\alpha,p}$. Moreover, these inclusion are *proper* for $p \neq 2$. (See [9].) Thus, although the classes $\Lambda_{\alpha,p}$ and $L_{\alpha,p}$ are quite different in many respects it can be shown (via nonlinear potential theory) that $\mathfrak{A}^{\alpha,p} = \mathfrak{B}^{\alpha,p}$, $\alpha > 0$, $2 - \alpha/n < p < \infty$. When $1 \leqq p \leqq 2 - \alpha/n$, the result remains open. It is this and related results that are discussed here.

2. Main result. Consider the following set functions (capacities) defined initially for compact set $K \subset \boldsymbol{R}^n$:

$$B^{(n)}_{\alpha,p}(K) = \inf \| \varphi \|^p_{\alpha,p}$$

and

$$A^{(n)}_{\alpha,p}(K) = \inf | \varphi |^p_{\alpha,p}$$

where, in each case, the infimum is over $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ for which $\varphi(x) \geqq 1$ on K. $C_0^\infty(\boldsymbol{R}^n)$ denotes the infinitely differentiable functions on $\boldsymbol{R}^n$ with compact support.

REMARK. $B^{(n)}_{\alpha,p}$ and $A^{(n)}_{\alpha,p}$ can be extended to all sets of $\boldsymbol{R}^n$ as "outer capacities" — see e.g., [7].

PROPOSITION 1. *$A^{(n)}_{\alpha,p}(K) = 0$ $(B^{(n)}_{\alpha,p}(K) = 0)$ iff $K \in \mathfrak{A}^{\alpha,p}$ $(K \in \mathfrak{B}^{\alpha,p})$, K a compact set of $\boldsymbol{R}^n$.*

THEOREM 1. *$A^{(n)}_{\alpha,p}(K) = 0$ iff $B^{(n)}_{\alpha,p}(K) = 0$, $\alpha > 0$, $2 - \alpha/n < p < \infty$, K a compact set of $\boldsymbol{R}^n$, and for $\alpha p > 1$, $p > 1$ if K is a compact subset of $\boldsymbol{R}^{n-1}$.*

For the proof, we need to draw from two sources — the key facts are Theorems I and II below.

THEOREM I ([8]). *If for $\varphi \in C_0^\infty(\boldsymbol{R}^n)$, $R_m\varphi = \varphi(x_1, \cdots, x_m, 0, \cdots, 0)$, $1 \leqq m \leqq n$, then there exists a linear extension operator E_m such that for $\psi \in C_0^\infty(\boldsymbol{R}^m)$, $R_m(E_m\psi) = \psi$. Furthermore, there is a constant C independent of φ and ψ such that*

(a) $| R_m\varphi |_{\beta,p} \leqq C \| \varphi \|_{\alpha,p}$

(b) $\| E_m\varphi \|_{\alpha,p} \leqq C | \psi |_{\beta,p}$

(c) $| R_m\varphi |_{\beta,p} \leqq C | \varphi |_{\alpha,p}$

(d) $| E_m\psi |_{\alpha,p} \leqq C | \psi |_{\beta,p}$

whenever $1 < p < \infty$, $\beta = \alpha - (n - m/p) > 0$.

THEOREM II ([2]). *For a Borel measure μ, set*

$$U^{(n)}_{\alpha,p}(x) = g^{(n)}_\alpha * (g^{(n)}_\alpha * \mu)^{1/(p-1)}(x) ,$$

then

$$U_{\alpha,p}^{(n)}(x) \sim \int_0^\infty [r^{\alpha p-n}\mu(S_r(x))]^{1/(p-1)}e^{-br}\frac{dr}{r} \tag{2}$$

for $p > 2 - \alpha/n$, $0 < \alpha p \leqq n$. *The symbol* $\sim$ *means that the ratio is bounded above* (*for some* $b > 0$) *and below* (*for another* $b > 0$), *the bounds being independent of* x *and* μ. $S_r(x) =$ *ball of radius* r *about* $x \in \boldsymbol{R}^n$.

Now by $I(a)$, $A_{\beta,p}^{(m)}(K) \leqq CB_{\alpha,p}^{(n)}(K)$ for some $C > 0$ independent of $K \subset \boldsymbol{R}^m$ since the restriction R_m of each test function for B is a test function for A. Similarly, $B_{\alpha,p}^{(n)}(K) \leqq CA_{\beta,p}^{(m)}(K)$ using $I(b)$. Hence, $I(a-d)$ implies that $B_{\alpha,p}^{(n)} \sim A_{\beta,p}^{(m)} \sim A_{\alpha,p}^{(n)}$, *on compact subsets of* $\boldsymbol{R}^m$, $m \leqq n-1$. To remove this restriction on K, we use II. By [7] we know that for any compact $K \subset \boldsymbol{R}^m$, $B_{\alpha,p}^{(n)}(K) > 0$ iff there is a non-zero Borel measure concentrated on K such that $U_{\alpha,p}^{(n)}$ is bounded. But since $\alpha p - n = \beta p - m$, II gives $B_{\beta,p}^{(m)}(K) > 0$ iff $B_{\alpha,p}^{(n)}(K) > 0$, $p > 2 - \beta/m$, $0 < \alpha p \leqq n$. (This relation trivially holds when $\alpha p > n$.) It might be noted that "$B_{\alpha,p}^{(n)}(K) = 0$ implies $B_{\beta,p}^{(m)}(K) = 0$ for $1 < p < \infty$" is an immediate consequence of the definition of Bessel capacity — a fact we have improved for $p > 2 - \beta/m$.

Note, if we change the notation slightly in the above arguments, we have: $B_{\alpha,p}^{(n)}(K) = 0$ iff $B_{\alpha+k/p,p}^{(n+k)}(K) = 0$, $p > 2 - \alpha/n$, K compact in $\boldsymbol{R}^n$, k a positive integer.

COROLLARY 1. $B_{\alpha,p}^{(n)}(K) = 0$ *iff* $B_{\beta,p}^{(m)}(K) = 0$, $\beta = \alpha - (n-m)/p > 0$, $p > 2 - \beta/m$, K *compact in* $\boldsymbol{R}^m$.

3. Thin sets. A set $E \subset \boldsymbol{R}^n$ is called $\mathscr{T}_{\alpha,p}^{(n)}$ — thin at $x_0 \in \bar{E}$ iff there exists a Borel measure μ such that $U_{\alpha,p}^{(n)}(x)$ is bounded and

$$U_{\alpha,p}^{(n)}(x) < \liminf_{\substack{x \to x_0 \\ x \in E}} U_{\alpha,p}^{(n)}(x) .$$

Recently in [4], necessary and sufficient conditions of the Wiener type have been given for a set to be $\mathscr{T}_{\alpha,p}^{(n)}$ — thin at x_0, provided $p > 2 - \alpha/n$. The condition, which depends strongly on II, is by [4] and Theorem 1

$$\int_0^1 [r^{\alpha p-n}A_{\alpha,p}^{(n)}(E \cap S_r(x_0))]^{1/(p-1)}\frac{dr}{r} < \infty . \tag{3}$$

It was also shown in [4] that (3) is *not* equivalent to $\mathscr{T}_{\alpha,p}^{(n)}$ — thinness for $1 < p < 2 - \alpha/n$. So, although it remains unknown as to what the appropriate replacement for (3) is when $p < 2 - \alpha/n$, it is of interest to know just what (3) means when $p \leqq 2 - \alpha/n$. In this vain, Theorem 1 gives

THEOREM 2. *If $E \subset \boldsymbol{R}^n$, then* (3) *is equivalent to E being $\mathscr{F}^{(n+k)}_{\alpha+k/p,p}$ — thin at x_0. Here k is a positive integer chosen large enough so that $p > 1 + 0(1/\sqrt{k})$. Furthermore, k can be chosen to be zero provided $p > 2 - \alpha/n$.*

4. **Traces of $\Lambda_{\alpha,p}$ — potentials.** The techniques of Theorem 1 can also be used to obtain trace inequalities in the spirit of [1] and [2]. If $\Phi(\cdot, \nu)$ is a semi-norm on $C_0^\infty(\boldsymbol{R}^n)$ for each Borel measure ν, such that $\Phi(\phi, \nu) = 0$ when ϕ is zero on the support of ν, then

THEOREM 3. *If for any $\alpha > 0$ and $1 < p < \infty$, there is a constant C independent of $u \in C_0^\infty(\boldsymbol{R}^n)$ such that*

$$\Phi(|u|, \nu) \leqq C \| u \|_{\alpha,p}$$

then

$$\Phi(|u|, \nu) \leqq C' | u |_{\alpha,p}$$

for some constant C' independent of u. The converse holds for $p \geqq 2$.

For various choices of Φ we can obtain trace inequalities for the Λ-spaces analogous to those given in [1] and [2] for the $L_{\alpha,p}$-spaces. In particular, when Φ is a Lorentz norm, we get Sobolev type inequalities for the Λ-spaces from the known inequalities for the Bessel potentials (cf. [6]). Thus from [2] we have the following rather interesting

COROLLARY 2. *Suppose μ is a Borel measure on $\boldsymbol{R}^n$ with compact support such that for all $x \in \boldsymbol{R}^n$ and $r > 0$, and any $d\colon 0 < d \leqq n$, $\mu(S_r(x)) \leqq Cr^d$, then for $\alpha p = n$, $1 < p < \infty$, and some $b > 0$,*

$$\sup_{|u|_{\alpha,p} \leq 1} \int \exp(b \, | u(x) |^{p'}) d\mu(x) < \infty ,$$

$p' = p/(p-1)$.

A further application of the techniques of Theorem 1, is to the results of [10], where Fubini type theorems with respect to $A_{\alpha,p}$ and $B_{\alpha,p}$ null sets are discussed. Theorem 1 improves the apparent assymetry in these results.

REFERENCES

1. D. R. Adams, *Traces of potentials of arising from translation invariant operators*, Ann. Sc. Norm. Sup. Pisa., **25** (1971), 203-217.
2. ———, *Traces of potentials. II*, Indian Univ. Math. J., **22** (1973), 907-918.
3. D. R. Adams and N. G. Meyers, *Bessel potentials Inclusion relations among classes of exceptional sets*, Indiana Univ. Math. J., **22** (1973), 873-905.
4. ———, *Thinness and Wiener criteria for nonlinear potentials*, Indiana Univ. Math. J., **22** (1972), 169-197.
5. N. Aronszajn, F. Mulla, and P. Szeptycki, *On spaces of potentials connected with L^p classes*, Ann. Inst. Fourier, **13** (1963), 211-306.
6. C. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*, J. Math. Mech., **18** (1968), 283-323.
7. N. G. Meyers, *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand., **26** (1970), 255-292.
8. E. M. Stein, *The characterization of functions arising as potentials. II*, Bull. Amer. Math. Soc., **68** (1962), 577-582.
9. M. H. Taibleson, *On the theory of Lipschitz spaces of distributions on Euclidean n-space. I*, J. Math. Mech., **13** (1964), 407-480.
10. P. Szeptycki, *On restrictions of functions in the spaces $P^{\alpha,p}$ and $B^{\alpha,p}$*, Proc. Amer. Math. Soc., **16** (1965), 341-347.

Received August 17, 1973. The research was partially supported by NSF GP-33749.

UNIVERSITY OF CALIFORNIA, SAN DIEGO
AND
UNIVERSITY OF KENTUCKY