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FAITHFUL STATES ON A C^*-ALGEBRA

MASAMICHI TAKESAKI

Let A be a separable C^*-algebra which is σ-weakly dense in a von Neumann algebra $\mathscr{M}$. The equivalence of the following two statements is proven: (i) $\mathscr{M}$ is atomic and A contains all miminal projections; (ii) A normal state φ of $\mathscr{M}$ is faithful if the restriction $\varphi|_A$ of φ to A is also.

The results of this paper have grown out of conversations with Haag, Herman, Hugenholtz, Kadison, Kastler, and Pedersen during the Functional Analysis Special Year Program at UCLA in the spring, 1971. Seeing that any state on a C^*-algebra satisfying the Kubo-Martin-Schwinger condition for any one parameter automorphism group of the algebra gives rise to a faithful normal state on the von Neumann algebra generated by the cyclic representation induced by the state, Hugenholtz raised the question as to when this phenomenon occurs in general. Namely, the original question is as to whether given a von Neumann algebra $\mathscr{M}$, with a σ-weakly dense C^*-subalgebra A, a normal state φ on $\mathscr{M}$ is faithful if the restriction $\varphi|_A$ of φ to A is faithful. Kadison answered immediately in the negative with the following example:

EXAMPLE 1. Let $\{r_n\}$ be an enumeration of all rational numbers in $[0, 1]$. Consider the set

$$E = \bigcup_{n=1}^{\infty}\left\{\left(r_n - \frac{1}{3^{n+1}}, r_n + \frac{1}{3^{n+1}}\right) \cap [0, 1]\right\} .$$

Then, denoting the Lebesque measure on $[0, 1]$ by μ, we have

$$\mu(E) \leqq \sum_{n=1}^{\infty} \frac{2}{3^{n+1}} = \frac{1}{3} .$$

Let A denote the C^*-algebra $C[0, 1]$ of all continuous functions on $[0, 1]$ considered as an operator algebra acting on $L^2[0, 1]$. Then the von Neumann algebra generated by A is given by $L^\infty[0, 1]$. We define a normal state ω of $L^\infty(0, 1)$ by

$$\omega(f) = \frac{1}{\mu(E)} \int_E f(t) d\mu(t) , \quad f \in L^\infty(0, 1) .$$

Then, ω is not faithful since $\mu(E) \leqq 2/3 < 1$, while $\omega|_A$ is faithful because E is dense in $[0, 1]$.

Therefore, a dense C^*-subalgebra of a von Neumann algebra does not characterize faithful normal states. However, if we consider the cyclic representation $\{\pi_\omega, \mathfrak{H}_\omega, \xi_\omega\}$ induced by a given faithful state ω on a C^*-algebra A, then the situation is more complicated. That is, if A is abelian, then the von Neumann algebra $\pi_\omega(A)''$, say $\mathscr{M}_\omega$, is maximal abelian, so that the state $\tilde{\omega}$ on $\mathscr{M}_\omega$ given by $\tilde{\omega}(x) = (x\xi_\omega | \xi_\omega)$, $x \in \mathscr{M}_\omega$, is faithful. In the above example, the situation is that $\mathfrak{H}_\omega = L^2(E)$ and $\mathscr{M}_\omega = L^\infty(E)$.

Pedersen then constructed the following counterexample.

EXAMPLE 2. Let A denote the C^*-algebra of all 2×2 matrix-valued continuous functions on $[0, 1]$. Let $\{r_n\}$ be an enumeration of the rational numbers in $[0, 1]$ and r_0 be an irrational number in $[0, 1]$. We define a state ω on A by

$$\omega(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} (x_{11}(r_n) + x_{22}(r_n)) + \frac{1}{2} x_{11}(r_0) .$$

Then $\mathfrak{H}_\omega$ is identified with the direct sum of $\boldsymbol{C}^2$ and the Hilbert space of all 2×2 matrix-valued functions on $\{r_n\}$ equipped with the inner product:

$$(\xi | \eta) = \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \sum_{i,j=1}^{2} \xi_{ij}(r_n)\overline{\eta_{i,j}(r_n)} + \xi_1(r_0)\overline{\eta_1(r_0)} + \xi_2(r_0)\overline{\eta_2(r_0)} .$$

The von Neumann algebra $\mathscr{M}_\omega$ generated by $\pi_\omega(A)$ is identified with the algebra of all bounded 2×2 matrices-valued functions on $\{r_n\} \cup \{r_0\}$, where the action of $\mathscr{M}_\omega$ on $\mathfrak{H}_\omega$ is given by

$$(x\xi)_{i,j}(r_n) = \sum_{k=1}^{2} x_{i,k}(r_n)\xi_{k,j}(r_n) ;$$
$$(x\xi)_i(r_0) = x_{i,1}(r_0)\xi_1(r_0) + x_{i,2}(r_0)\xi_2(r_0)$$

for every $x \in \mathscr{M}_\omega$ and $\xi \in \mathfrak{H}_\omega$. The cyclic vector ξ_ω is given by

$$(\xi_\omega)_{i,j}(r_n) = \delta_{i,j} , \quad (\xi_\omega)_1(r_0) = 1 , \quad (\xi_\omega)_2(r_0) = 0 .$$

Let e denote the projection defined by

$$e(r_n) = 0 , \quad n = 1, 2, \cdots , \quad e_{11}(r_0) = e_{12}(r_0) = e_{21}(r_0) = 0 ,$$
$$e_{22}(r_0) = 1 .$$

Then we have $(e\xi_\omega | \xi_\omega) = 0$. Hence ω does not give rise to a faithful normal state on $\mathscr{M}_\omega$.

From these examples, we see, in the first place, a remarkable feature of the *KMS*-condition which guarantees the faithfulness of the extended normal state on the generated von Neumann algebra; secondly, it is in general not easy to get any information on the faithfulness of a state extended from a given C^*-algebra. However, we have the following nice example in the noncommutative situation.

EXAMPLE 3. Suppose A is the C^*-algebra $\mathscr{L}\mathscr{C}(\mathfrak{H})$ of all compact oserators on a Hilbert space $\mathfrak{H}$. Then the weak closure $\mathscr{M}$ of A is the von Neumann algebra $\mathscr{L}(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$, and a normal state φ of $\mathscr{M}$ is faithful if and only if its restriction $\varphi|_A$ to A is faithful.

Thus, there does exist a C^*-algebra on a Hilbert space on whose weak closure a normal state is faithful if its restriction to the C^*-algebra is faithful. More generally, the following theorem shows the structure of such a C^*-algebra. To avoid a trivial situation, we assume that the C^*-algebra in question is always separable. The reason for this assumption is, for example, that if the C^*-algebra in question happens to be a von Neumann algebra (a von Neumann algebra is a perfectly good C^*-algebra), then there is nothing to argue.

THEOREM 1. *Let A be a separable C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and $\mathscr{M}$ the weak closure of A. Then the following two statements are equivalent*:

(i) *$\mathscr{M}$ is an atomic von Neumann algebra and A contains all minimal projections of $\mathscr{M}$*;

(ii) *A normal state φ on $\mathscr{M}$ is faithful if the restriction $\varphi|_A$ of φ to A is also*[1].

Proof. (i) $\Rightarrow$ (ii): Suppose a normal state φ on $\mathscr{M}$ is not faithful. Let e be the support projection of φ. Then $e \neq 1$ by assumption. Hence $1 - e$ majorizes a nonzero minimal projection p in $\mathscr{M}$. The assumption in (i) says that p is in A. Since $\varphi(p) = 0$, $\varphi|_A$ is not faithful.

(ii) $\Rightarrow$ (i): Suppose A enjoys the property described in (ii). Since A is separable, there is a normal state φ of $\mathscr{M}$ which is faithful on A. By assumption, φ is then faithful on $\mathscr{M}$. If p is a nonzero projection in $\mathscr{M}$, then the normal positive functional ω on $\mathscr{M}$, given by $\omega(x) = \varphi((1 - p)x(1 - p))$, $x \in \mathscr{M}$, is not faithful on $\mathscr{M}$. Hence ω is not faithful on A either by assumption, so that there exists a nonzero positive element $h \in A$ with $\omega(h) = 0$; hence $(1 - p)h(1 - p) = 0$. Choosing $||h|| \leqq 1$, we have $0 \leqq h \leqq p$. Thus we conclude here that any nonzero projection p in $\mathscr{M}$ majorizes a nonzero positive element in A.

Now we denote by $\tilde{A}$ the universal enveloping von Neumann

[1] After the first draft of this article was distributed to specialists, the author was informed of alternate proofs of the implication (ii) $\Rightarrow$ (i) independently by C. Akemann, G. Elliott, and G. K. Pedersen which were shorter than the original one. The proof presented here is due to Pedersen. The author would like to express his thanks here to Professor C. Akemann, Professor G. Elliott, and Professor G. K. Pedersen.

algebra of A and by z the supremum of all minimal projections in $\tilde{A}$ which is a central projection in $\tilde{A}$. Since A is separable, there exists a countable set $\{\omega_j\}$ of pure states which is weak* dense in the set of all pure states. Let $\omega = \sum_{j=1}^{\infty} 1/(2^j)\omega_j$. Let $\tilde{\omega}$ denote a Hahn-Banach extension of ω to $\mathscr{M}$ and decompose $\tilde{\omega}$ in its normal part and singular part: $\tilde{\omega} = \tilde{\omega}_n + \tilde{\omega}_s$. Then $\tilde{\omega}_n$ is faithful on $\mathscr{M}$. Otherwise, $\tilde{\omega}_n(p) = 0$ for some nonzero projection p in $\mathscr{M}$. Since $\tilde{\omega}_s$ is singular, there exists a nonzero projection $q \leqq p$ in $\mathscr{M}$ with $\tilde{\omega}_s(q) = 0$. From the first part of the proof, it follows that there exists a nonzero h in A with $0 \leqq h \leqq q$. But then we have

$$\begin{aligned} 0 \leqq \omega(h) &= \tilde{\omega}(h) = \tilde{\omega}_n(h) + \tilde{\omega}_s(h) \\ &\leqq \tilde{\omega}_n(q) + \tilde{\omega}_s(q) = 0 , \end{aligned}$$

a contradiction. Therefore, $\tilde{\omega}_n$ is faithful. We consider ω as a normal state of $\tilde{A}$. Let π denote the normal homomorphism of $\tilde{A}$ onto $\mathscr{M}$ induced by the injection of A into $\mathscr{M}$. Let $\psi = {}^t\pi(\tilde{\omega}_n)$. Then ψ is a normal positive functional on $\tilde{A}$ such that

$$\psi|_A = \tilde{\omega}_n|_A ,$$

so that we have $\psi \leqq \omega$. Hence the support e of ψ in $\tilde{A}$ is majorized by the support of ω in $\tilde{A}$; hence $e \leqq z$ because ω is the convex sum of pure states. Hence $\tilde{A}e$ is atomic. For any $x \in \tilde{A}$, $\pi(x) = 0$ if and only if $0 = \tilde{\omega}_n(\pi(x)^*\pi(x)) = \tilde{\omega}_n(\pi(x^*x))$; if and only if $\psi(x^*x) = 0$; if and only if $xe = 0$. Therefore, we have $\pi^{-1}(0) = \tilde{A}(1-e)$. Hence π is an ismorphism of $\tilde{A}e$ onto $\mathscr{M}$, so that $\mathscr{M}$ is atomic. The minimal projections in $\mathscr{M}$ belong to A by the first part of the proof.

Therefore, we conclude that the possibility of characterizing the faithfulness of a normal state on a von Neumann algebra by means of the faithfulness of the restricted state of a given dense C^*-subalgebra is very limited, actually it is possible only for a von Neumann algebra of type I and a very nice C^*-subalgebra.

PROPOSITION 2. *For a separable C^*-algebra A the following two statements are equivalent*:

(i) *A is post-liminal;*

(ii) *If the left kernel of a factor state φ on A is a two sided ideal, then the corresponding cyclic vector ξ_φ is separating for $\pi_\varphi(A)''$, where $\{\pi_\varphi, \mathfrak{H}_\varphi, \xi_\varphi\}$ is the cyclic representation of A induced by φ.*

Proof. (ii) $\Rightarrow$ (i): Let ω be a pure state on A, and $\{\pi_\omega, \mathfrak{H}_\omega, \xi_\omega\}$ be the irreducible cyclic representation of A corresponding to ω. Then $\mathfrak{H}_\omega$ is separable and $\pi_\omega(A)'' = \mathscr{L}(\mathfrak{H}_\omega)$, the von Neumann algebra of all bounded operators on $\mathfrak{H}_\omega$. For any normal state $\tilde{\varphi}$ on $\mathscr{L}(\mathfrak{H}_\omega)$, $\varphi = {}^t\pi_\omega(\tilde{\varphi})$ gives rise to a cyclic representation $\{\pi_\varphi, \mathfrak{H}_\varphi, \xi_\varphi\}$ which is quasi-equivalent to π_ω. Hence φ is a factor state on A. Let ρ denote the normal isomorphism of $\mathscr{L}(\mathfrak{H}_\omega)$ onto $\pi_\varphi(A)''$ such that $\pi_\varphi = \rho\cdot\pi_\omega$. Suppose the restriction of $\tilde{\varphi}$ to $\pi_\omega(A)$ is faithful. Then the left kernel $\mathfrak{m}_\varphi$ of φ is the same as the kernel $\pi_\omega^{-1}(0)$ of π_ω, so that $\mathfrak{m}_\varphi$ is a two sided ideal. Hence by assumption, the cyclic vector ξ_φ is separating for $\pi_\varphi(A)'$. But we have, for any $x \in A$,

$$\tilde{\varphi}\cdot\pi_\omega(x) = \varphi(x) = (\pi_\varphi(x)\xi_\varphi \,|\, \xi_\varphi) = (\rho\cdot\pi_\omega(x)\xi_\varphi \,|\, \xi_\varphi) \,.$$

Hence we get

$$\tilde{\varphi}(x) = (\rho(x)\xi_\varphi \,|\, \xi_\varphi)\,, \quad x \in \mathscr{L}(\mathfrak{H}_\omega) \,.$$

Therefore, $\tilde{\varphi}$ is faithful on $\mathscr{L}(\mathfrak{H}_\omega)$. Thus the C^*-subalgebra $\pi_\omega(A)$ has the property in Theorem 1. Hence $\pi_\omega(A)$ contains the C^*-algebra $\mathscr{L}\mathscr{C}(\mathfrak{H}_\omega)$ of all compact operators on $\mathfrak{H}_\omega$, which means that A is post-liminal.

(i) $\Rightarrow$ (ii): Suppose A is post-liminal. Then every factor representation is quasi-equivalent to an irreducible representation, and the image of A under each irreducible representation contains the C^*-algebra of all compact operators on the representation space. Therefore, if a normal state on the weak closure $\pi(A)''$ of the image of A under any factor representation π is faithful on $\pi(A)$, then it is faithful on $\pi(A)''$. Thus, statement (ii) follows.

Returning to the original question how to guarantee the faithfulness of the extended normal states on the weak closure, we introduce the following:

DEFINITION. A state ω on a C^*-algebra A is said to satisfy the *quasi Kubo-Martin-Schwinger condition* if $\lim_{n\to\infty}\omega(x_n x_n^*) = 0$ for every sequence $\{x_n\}$ in A with the property that $\lim_{n\to\infty}\omega(x_n^* x_n) = 0$ and $\lim_{n,m\to\infty}\omega((x_n - x_m)(x_n - x_m)^*) = 0$.

Making use of Tomita's theory of left Hilbert algebras, one can prove the following, though the presentation of the full proof is beyond the scope of this paper, see [5] and [6].

THEOREM 3. *If a state φ on a C^*-algebra A satisfies the quasi KMS-condition, then*

(i) *the canonically induced normal state $\tilde{\varphi}$ on the von Neumann algebra $\pi_\varphi(A)''$ generated by $\pi_\varphi(A)$ is faithful;*

(ii) *there exists a one parameter automorphism group σ_t of $\pi_\varphi(A)''$ for which $\tilde{\varphi}$ satisfies the KMS-condition.*

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UNIVERSITY OF CALIFORNIA, LOS ANGELES