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THE NONMINIMALITY OF THE DIFFERENTIAL CLOSURE

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The differential closure of a given ordinary differential field k is characterized to within (differential) k-isomorphism as a differentially closed (differential) extension field $\hat{k}$ of k which is k-isomorphic to a subfield of any differentially closed extension field of k. It has been conjectured that, in analogy to the cases of the algebraic closure of a field and the real closure of an ordered field, the differential closure of any differential field k is minimal, that is, not k-isomorphic to a proper subfield of itself. The conjecture is here shown to be false.

Let k be a differential field (ordinary, that is with one specified derivation) of characteristic zero and let $k\{y\}$ be the differential ring of differential polynomials over k in the differential indeterminate y. Recall that the *order* of a nonzero differential polynomial in $k\{y\}$ is simply the smallest integer $r \geqq -1$ such that the differential polynomial involves none of the derivatives $y^{(r+1)}$, $y^{(r+2)}$, $\cdots$. According to Lenore Blum's definition, k is *differentially closed* if, for any $f, g \in k\{y\}$ with g of smaller order than f, there is a zero of f in k that is not a zero of g. For any differential field k, a *differential closure* of k is a differential extension field $\hat{k}$ of k that is differentially closed and that can be k-embedded in any differentially closed differential extension field of k. Blum has used the methods of model theory to show the existence of $\hat{k}$ and to derive a number of its properties [2], appreciably extending and simplifying a theory initiated by Abraham Robinson [5]. The uniqueness of $\hat{k}$ to within differential k-isomorphism follows from a recent result of Shelah [7]. The differential closure $\hat{k}$ of k is called *minimal* if there is no (differential) k-isomorphism of $\hat{k}$ with a proper subfield of itself. One of the unsolved problems of the theory has been to determine whether or not $\hat{k}$ is always minimal. Sacks has conjectured [6] that $\hat{k}$ is minimal over k in the special case $k = \boldsymbol{Q}$. It is proved here, among other things, that this conjecture is false. It was learned after the completion of this paper that this result has also been proved by Kolchin [4] and announced by Shelah [8]. The author is greatly indebted to Lenore Blum for calling his attention to the problem and for numerous conversations on her work.

We begin by recalling some facts outlined in a recent paper of Ax [1]. Let $k \subset K$ be fields. There is a K-module $\Omega^1_{K/k}$, the space of differential forms of degree one of K/k, and a k-linear map $d: K \rightarrow \Omega^1_{K/k}$ such that $d(xy) = xdy + ydx$ for all $x, y \in K$ (and these can be constructed just by insisting on universality for these properties) which is the usual dual space of the K-module of k-derivations of K, a vector space over K of dimension tr. deg. K/k if the latter is finite and the field characteristic is zero. For any derivation D of K such that $Dk \subset k$, there is a map $D^1: \Omega^1_{K/k} \to \Omega^1_{K/k}$ (most easily constructed using the universal properties of $\Omega^1_{K/k}$) which is characterized by the following properties: for all $\omega, \eta \in \Omega^1_{K/k}$ and all $f \in K$ we have $D^1(\omega + \eta) = D^1\omega + D^1\eta$, $D^1(f\omega) = (Df)\omega + f(D^1\omega)$, $D^1(df) = d(Df)$.

The following generalizes a lemma in Ax's paper [1, Lemma 3].

LEMMA 1. *Let $k \subset K$ be fields of characteristic zero, D a derivation of K such that $Dk \subset k$, C the D-constants of k, u and t elements of K that are algebraically dependent over C. Consider the k-differential of K given by udt. Then $D^1(udt) = d(uDt)$.*

For $D^1(udt) = (Du)dt + udDt$, while $d(uDt) = (Dt)du + udDt$, so we have to show that $(Du)dt = (Dt)du$. Let U, T be indeterminates over C and let $F(U, T) \in C[U, T]$ be an irreducible polynomial such that $F(u, t) = 0$. If u is transcendental over C then t is algebraic over $C(u)$ and $F(u, T)$ is irreducible over $C(u)$, so that $(\partial F/\partial T)(u, t) \neq 0$. Similarly if t is transcendental over C then $(\partial F/\partial U)(u, t) \neq 0$. The relation $(Du)dt = (Dt)du$ follows from the equations

$$\frac{\partial F}{\partial U}(u, t)du + \frac{\partial F}{\partial T}(u, t)dt = 0 ,$$

$$\frac{\partial F}{\partial U}(u, t)Du + \frac{\partial F}{\partial T}(u, t)Dt = 0$$

unless $(\partial F/\partial U)(u, t)$ and $(\partial F/\partial T)(u, t)$ are both zero, which can happen only if u and t are both algebraic over C, in which case both du and dt are zero.

PROPOSITION 1. *Let k be a differential field of characteristic zero, C its field of constants, x an indeterminate over C, and $f(x)$ a nonzero element of $C(x)$ such that $1/f(x)$ has the form*

$$\frac{1}{f(x)} = \sum_{i=1}^{n} c_i \frac{\partial u_i(x)/\partial x}{u_i(x)} + \frac{\partial v(x)}{\partial x} ,$$

where $c_1, \cdots, c_n \in C$ and $u_1(x), \cdots, u_n(x)$, $v(x) \in C(x)$. Let x_1, x_2 be elements of a differential extension field of k whose constants are all algebraic over k, each of x_1, x_2 being a solution of the differential equation $x' = f(x)$, and suppose that x_1, x_2 are algebraically dependent over k. Then either x_1 or x_2 is algebraic over k or $(v(x_1))' = (v(x_2))'$.

The field $K = k(x_1, x_2)$ is a differential extension field of k, so for $j = 1, 2$ we may apply the Lemma to $dx_j/f(x_j) \in \Omega^1_{K/k}$ and $D = {}'$ to get

$$D^1\Big(\frac{dx_j}{f(x_j)}\Big) = d\Big(\frac{Dx_j}{f(x_j)}\Big) = D(1) = 0 .$$

Assuming that neither x_1 nor x_2 is algebraic over k, each $dx_j/f(x_j)$ is a nonzero element of the one-dimensional K-module $\Omega^1_{K/k}$, so that we can write $dx_2/f(x_2) = c dx_1/f(x_1)$, for some nonzero $c \in K$. Hence

$$0 = D^1\Big(\frac{dx_2}{f(x_2)}\Big) = D^1\Big(c\frac{dx_1}{f(x_1)}\Big) = (Dc)\frac{dx_1}{f(x_1)} + cD^1\Big(\frac{dx_1}{f(x_1)}\Big) = (Dc)\frac{dx_1}{f(x_1)} ,$$

so that $Dc = 0$. Thus c is a constant of K, hence, by assumption, algebraic over k. Now for $j = 1, 2$,

$$\frac{dx_j}{f(x_j)} = \sum_{i=1}^{n} c_i \frac{\frac{\partial u_i}{\partial x}(x_j)}{u_i(x_j)} dx_j + \frac{\partial v}{\partial x}(x_j) dx_j = \sum_{i=1}^{n} c_i \frac{du_i(x_j)}{u_i(x_j)} + dv(x_j) ,$$

so that

$$\sum_{i=1}^{n} c_i \frac{du_i(x_2)}{u_i(x_2)} + dv(x_2) = c\Big(\sum_{i=1}^{n} c_i \frac{du_i(x_1)}{u_i(x_1)} + dv(x_1)\Big) .$$

From the well-known fact that a linear combination with constant coefficients of normal differentials of third kind can be exact only if it is zero (cf. [1, Prop. 2], which generalizes the usual residue considerations) we deduce

$$\sum_{i=1}^{n} c_i \frac{du_i(x_2)}{u_i(x_2)} = \sum_{i=1}^{n} c \frac{du_i(x_1)}{u_i(x_1)} , \qquad dv(x_2) = c dv(x_1) .$$

Thus

$$\begin{aligned}(v(x_2))' &= \frac{\partial v}{\partial x}(x_2)x_2' = \frac{\partial v}{\partial x}(x_2)f(x_2) = \frac{dv(x_2)}{dx_2/f(x_2)} = \frac{c dv(x_1)}{c(dx_1/f(x_1))} \\ &= \frac{dv(x_1)}{dx_1/f(x_1)} = (v(x_1))' .\end{aligned}$$

Note that if C is algebraically closed, then any element of $C(x)$ can be written in the form prescribed for $1/f(x)$ in Proposition 1, as is seen by looking at partial fractions with respect to $C[x]$. Note also that since $(v(x_j))' = (\partial v/\partial x)(x_j)x_j' = (\partial v/\partial x)(x_j)f(x_j)$, $j = 1, 2$, the conclusion of Proposition 1 can be written

$$\Big(\frac{\partial v}{\partial x}(x_1)\Big)^{-1} \sum_{i=1}^{n} c_i \frac{\frac{\partial u_i}{\partial x}(x_1)}{u_i(x_1)} = \Big(\frac{\partial v}{\partial x}(x_2)\Big)^{-1} \sum_{i=1}^{n} c_i \frac{\frac{\partial u_i}{\partial x}(x_2)}{u_i(x_2)} .$$

REMARK. The condition in Proposition 1 that x_1 and x_2 be elements of a differential extension field of k whose constants are algebraic over k will certainly be satisfied if all the constants of $k(x_1, x_2)$ are algebraic over C, and this latter condition will automatically hold for most $f(x)$ of interest, in virtue of Lemma 2 and Proposition 2 below. For the same reason, the condition on constants in the following Corollary is superfluous. But we do not need this information for the nonminimality proof.

COROLLARY. *Let k be a differential field of characteristic zero, and suppose that x_1, x_2 are elements of a differential extension field of k whose constants are all algebraic over k, both x_1 and x_2 being solutions of the differential equation $x' = f(x)$, where $f(x)$ is either $x/(x+1)$ or $x^3 - x^2$. Then if x_1 and x_2 are algebraically dependent over k, either x_1 or x_2 is algebraic over k, or $x_1 = x_2$.*

First note that Proposition 1 is applicable since $1/f(x)$ is of the correct form, namely either

$$\frac{x+1}{x} = \frac{1}{x} + 1 = \frac{\partial x/\partial x}{x} + \frac{\partial x}{\partial x}$$

or

$$\frac{1}{x^3 - x^2} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2} = \frac{\frac{\partial}{\partial x}\left(\frac{x-1}{x}\right)}{(x-1)/x} + \frac{\partial}{\partial x}\left(\frac{1}{x}\right).$$

For $j = 1, 2$, in the case $f(x) = x/(x+1)$ we have $(v(x_j))' = x_j' = x_j/(x_j + 1)$, while in the case $f(x) = x^3 - x^2$ we have $(v(x_j))' = (1/x_j)' = -x_j'/x_j^2 = 1 - x_j$, so the Corollary follows directly from the Proposition.

Now let C be a differential field of constants. We shall show that its differential closure $\hat{C}$ is not minimal over C. Let x be an indeterminate over C, $f(x)$ a nonzero element of $C(x)$. For any $x_1, x_2, \cdots, x_n$ in $\hat{C}$, the differential equation $y' = f(y)$ has at least one solution in $\hat{C}$ not annulling $(y - x_1)(y - x_2)\cdots(y - x_n)$. Hence the differential equation $y' = f(y)$ has an infinity of solutions in $\hat{C}$. Since there are only a finite number of constant solutions of $y' = f(y)$, namely the zeros of $f(y)$, we can find distinct nonconstant elements $x_1, x_2, \cdots$ of $\hat{C}$ such that $x_i' = f(x_i)$ for all $i = 1, 2, \cdots$. We claim that in either of the special cases $f(x) = x/(x+1)$ or $f(x) = x^3 - x^2$, the set $\{x_1, x_2, \cdots\}$ is a set of indiscernibles over C (or, in the terminology of Sacks [4], a set of *conjugates over* C) and this fact will prove the nonminimality of $\hat{C}$ over C [6, p. 633]. What has to be shown is that for any $n = 1, 2, \cdots$ and any distinct positive integers $i_1, \cdots, i_n$, the differential isomorphism class of $(x_{i_1}, \cdots, x_{i_n})$ over C is independent of the choice of $i_1, \cdots, i_n$. Since $x_i' = f(x_i)$, $i = 1, 2, \cdots$, it suffices to prove that the algebraic isomorphism class of $(x_{i_1}, \cdots, x_{i_n})$ over C is independent of the choice of $i_1, \cdots, i_n$, which will certainly be true if $x_{i_1}, \cdots, x_{i_n}$ are always algebraically independent over C. Hence we are reduced to proving that $x_1, x_2, \cdots$ are algebraically independent over C. As a preliminary, note that the constants of $C(x_1, x_2, \cdots)$ are among the constants of $\hat{C}$, which are precisely the algebraic closure $\bar{C}$ of C, an easy consequence of Blum's theory [2]. We now assume that for a certain $n = 1, 2, \cdots$, the elements $x_1, x_2, \cdots, x_n$ are algebraically dependent over C, and we have to derive a contradiction. Taking n minimal and changing our notation, if necessary, we may assume that no proper subset of $\{x_1, \cdots, x_n\}$ is algebraically dependent over C. If $n > 1$, then x_{n-1} and x_n are algebraically dependent over the differential field $C(x_1, \cdots, x_{n-2})$ and are distinct solutions of the differential equation $x' = f(x)$, so the previous Corollary implies that either x_{n-1} or x_n is algebraic over $C(x_1, \cdots, x_{n-2})$, a contradiction of the minimality of n, while if $n = 1$ we have x_1 algebraic over C, therefore a constant, again a contradiction. This proves that $x_1, x_2, \cdots$ are algebraically independent over C, and hence that $\hat{C}$ is not minimal over C.

It is of interest to generalize somewhat the argument of the preceding paragraph. Let k be any differential field of characteristic zero and let $x_1, x_2, \cdots, x_n$ be distinct elements of a differential extension field of k, none algebraic over k, such that for each $i = 1, \cdots, n$ we have $x_i' = f(x_i)$, where $f(x)$ is either $x/(x+1)$ or $x^3 - x^2$. Then $x_1, \cdots, x_n$ are algebraically independent over k and the constant subfields of $k(x_1, \cdots, x_n)$ and of k are the same. To see this, we use the argument of the preceding paragraph, supplemented by Lemma 2 and Proposition 2 below. The Remark following Proposition 1 enables us to follow the above proof literally to get $x_1, \cdots, x_n$ algebraically independent over k, after which the equality of the constant subfields of $k(x_1, \cdots, x_n)$ and of k is a direct consequence of Proposition 2.

LEMMA 2. *Let K be a differential field, algebraic over its differential subfield k. Then the constants of K are algebraic over the subfield of constants of k.*

For let c be a constant of K, let $n = [k(c): k]$, and pick $a_1, \cdots, a_n \in k$ such that $c^n + a_1c^{n-1} + \cdots + a_n = 0$. Differentiation gives $a_1'c^{n-1} + \cdots + a_n' = 0$, from which we deduce that each $a_i' = 0$, so each a_i is a constant of k.

LEMMA 3. *Let $k \subset K$ be differential fields of characteristic zero,*

$C \subset \mathscr{C}$ their respective subfields of constants, and suppose that k is algebraically closed in K and that K is a finite field extension of k of transcendence degree one. Then if $C \neq \mathscr{C}$, C is algebraically closed in $\mathscr{C}$ and $\mathscr{C}$ is a finite field extension of C of transcendence degree one of genus at most that of K/k.

Start the proof by noting that since $C = k \cap \mathscr{C}$ and k is algebraically closed in K, we have C algebraically closed in $\mathscr{C}$. Suppose that $C \neq \mathscr{C}$ and let $t \in \mathscr{C}$, $t \notin C$. Then t is transcendental over C, and indeed over k. If also $u \in \mathscr{C}$, then t and u are algebraically dependent over k, so there exists an irreducible $f(T, U) \in k[T, U]$, T and U being indeterminates over k, such that $f(t, u) = 0$. The minimal polynomial of u over $k(t)$ is $f(t, U)$, up to a factor in $k(t)$, and $f(T, U)$ is unique, up to a factor in k, with the degree in U of $f(T, U)$ at most $[K: k(t)]$. Let $f(T, U) = \sum_{i,j} a_{ij} t^i u^j$, with each $a_{ij} \in k$, and with at least one of the a_{ij}'s equal to 1. Applying the derivation D of K, we get $\sum_{i,j} (Da_{ij}) t^i u^j = 0$. Now $\sum_{i,j} (Da_{ij}) T^i U^j$ must equal a multiple of $f(T, U)$, necessarily by an element of k, and this element of k must be 0 since one of the a_{ij}'s is 1. Thus $Da_{ij} = 0$ for all i, j, so that each $a_{ij} \in k \cap \mathscr{C} = C$. Therefore u is algebraic over $C(t)$, of degree at most $[K: k(t)]$. Therefore $\mathscr{C}$ is algebraic over $C(t)$, with $[\mathscr{C}: C(t)] \leqq [K: k(t)]$. It remains to prove the genus statement, and here we give two proofs, each relying on well-known facts about ground field extensions of algebraic function fields that may be found in [3]. First, if $\omega = f dg$ is a differential of first kind of $\mathscr{C}/C$, with $f, g \in \mathscr{C}$, then ω can also be considered a differential of K/k; in fact we have a natural injection of differentials $\Omega^1_{\mathscr{C}/C} \rightarrow \Omega^1_{K/k}$. For any k-place P of K, if f, g are finite at P then ω, considered as a differential of K/k, is also finite at P. If either f or g is not finite at P, then P induces a C-place p of $\mathscr{C}$, and since ω is finite at p we can write $\omega = f_1 dg_1$, with $f_1, g_1 \in \mathscr{C}$ both finite at p, so that again ω is finite at P. Therefore ω, considered as a differential of K/k, is of the first kind. Let $\omega_1, \cdots, \omega_g$ be a C-basis for the space of differentials of first kind of $\mathscr{C}/C$ ($g =$ genus of $\mathscr{C}/C$). If $\omega_1, \cdots, \omega_g$, considered as differentials of K/k, are linearly dependent over k, then there exist $a_1, \cdots, a_g \in k$, not all zero, such that $a_1\omega_1 + \cdots + a_g\omega_g = 0$. Suppose that we have such $a_1, \cdots, a_g$, with a minimal number of nonzero a_i's, one of which is 1. Since each $\omega_i/\omega_1 \in \mathscr{C}$, applying D to $a_1(\omega_1/\omega_1) + \cdots + a_g(\omega_g/\omega_1) = 0$ we get $(Da_1)(\omega_1/\omega_1) + \cdots + (Da_g)(\omega_g/\omega_1) = 0$. At least one Da_i is 0, so that each $Da_i = 0$, so each $a_i \in \mathscr{C}$. Thus $a_i \in \mathscr{C} \cap k = C$, contradicting the linear independence of $\omega_1, \cdots, \omega_g$ over C. Therefore $\omega_1, \cdots, \omega_g$ are k-linearly independent differentials of first kind of K/k, so that the genus of K/k is at least g. For the second proof of the genus statement, consider what happens when we extend the ground field C of the function field $\mathscr{C}/C$ from C to k. Since C is algebraically closed in k, $\mathscr{C} \otimes_C k$ is an integral domain, isomorphic to $\mathscr{C}[k] \subset K$, and so the ground field extension, which preserves the genus of $\mathscr{C}/C$, gives us $\mathscr{C}(k)/k$. Since $\mathscr{C}(k)$ is a subfield of K that contains k, its genus is at most that of K/k. This completes the second proof.

PROPOSITION 2. *Let k be a differential field of characteristic zero, with derivation D and constants C. Let $k(x)$ be a pure transcendental extension field of k, let $f(x)$ be a nonzero element of $k(x)$, and make $k(x)$ a differential extension field of k by setting $Dx = f(x)$. Suppose that $1/f(x)$ is of neither of the forms*

$$(\textit{element of } C)\ \frac{\partial u(x)/\partial x}{u(x)} \quad \textit{nor} \quad \frac{\partial v(x)}{\partial x}\,,$$

for $u(x)$, $v(x) \in C(x)$. Then every constant of $k(x)$ is in C.

To prove this, first assume that C is algebraically closed. Suppose that not all constants of $k(x)$ are in C. By Lemma 3, the subfield of constants of $k(x)$ is an algebraic function field of one variable over C of genus zero, hence, since C is algebraically closed, of the form $C(t)$, for some $t \in k(x)$, $t \notin k$. Now consider the nonzero differentials dt and $dx/f(x)$ of $k(x)/k$. We can write $dx/f(x) = \alpha dt$, for some $\alpha \in k(x)$. Applying the operator D^1 on $\Omega^1_{k(x)/k}$, we get $D^1(dx/f(x)) = D^1(\alpha dt) = (D\alpha)dt + \alpha dDt = (D\alpha)dt$. By Lemma 1, $D^1(dx/f(x)) = d(Dx/f(x)) = d(1) = 0$, so $D\alpha = 0$, so that $\alpha \in C(t)$. That is, $dx/f(x) = \alpha dt$, with $\alpha \in C(t)$. Now write $dx/f(x)$ in the form

$$\frac{dx}{f(x)} = \sum_{i=1}^{n} c_i \frac{du_i(x)}{u_i(x)} + dv(x)\,,$$

with $c_1, \cdots, c_n \in C$ and $u_1(x), \cdots, u_n(x)$, $v(x) \in C(x)$, which can be done immediately by looking at the partial fraction expansion of $1/f(x)$ with respect to $C[x]$. Using the logarithmic derivative identities

$$\frac{d(ab)}{ab} = \frac{da}{a} + \frac{db}{b}\,, \qquad \frac{da^\nu}{a^\nu} = \nu\frac{da}{a}\,,$$

we can, if necessary, modify $n, c_1, \cdots, c_n, u_1(x), \cdots, u_n(x)$ so that $c_1, \cdots, c_n$ are linearly independent over the rational numbers $\boldsymbol{Q}$. Looking at the partial fraction decomposition of α with respect to $C[t]$, we get an expression

$$\alpha dt = \sum_{i=1}^{m} \gamma_i \frac{dw_i}{w_i} + dy\,,$$

where $\gamma_1, \cdots, \gamma_m \in C$ and $w_1, \cdots, w_m, y \in C(t)$. Extend $c_1, \cdots, c_n$ to a basis $c_1, \cdots, c_n, c_{n+1}, c_{n+2}, \cdots, c_N$ of the $\boldsymbol{Q}$-vector space $\boldsymbol{Q}c_1 + \cdots + \boldsymbol{Q}c_n + \boldsymbol{Q}\gamma_1 + \cdots + \boldsymbol{Q}\gamma_m$. Using the logarithmic derivative identities, we can modify $m, \gamma_1, \cdots, \gamma_m, w_1, \cdots, w_m$, so that the same expression for αdt holds with $m = N$, and $\gamma_1 = c_1/M, \cdots, \gamma_N = c_N/M$ for some positive integer M. The above expression for $dx/f(x)$ remains true if we replace n by N, taking $u_{n+1}(x) = u_{n+2}(x) = \cdots = 1$. Hence we may assume that in the displayed expressions for $dx/f(x)$ and αdt we have $m = n$, $c_1, \cdots, c_n$ linearly independent over $\boldsymbol{Q}$, and $M\gamma_1 = c_1, \cdots, M\gamma_n = c_n$, for some positive integer M. From the equation $dx/f(x) = \alpha dt$ we now infer

$$\sum_{i=1}^{n} c_i \frac{d((u_i(x))^M/w_i)}{(u_i(x))^M/w_i} + Md(v(x) - y) = 0 .$$

At this point we again apply, in more precise form than was necessary for the proof of Proposition 1, the argument about when a linear combination of normal differential forms of third kind is exact [1, Prop. 2] to deduce that each $d((u_i(x))^M/w_i)$ and $d(v(x) - y)$ are zero. (This conclusion can be directly verified in the present case by expressing each $(u_i(x))^M/w_i$ as a power product of irreducible elements of $k[x]$ and $v(x) - y$ in terms of partial fractions.) Therefore $(u_1(x))^M/w_1, \cdots, (u_n(x))^M/w_n, v(x) - y \in k$, so that also $D((u_1(x))^M/w_1), \cdots, D((u_n(x))^M/w_n), D(v(x) - y) \in k$. Since $w_1, \cdots, w_n, y$ are constants, we deduce that

$$(Du_1(x))/u_1(x), \cdots, (Du_n(x))/u_n(x) , \quad Dv(x) \in k .$$

But $u_1(x), \cdots, u_n(x), v(x)$ are in the differential field $C(x)$, so that $(Du_1(x))/u_1(x), \cdots, (Du_n(x))/u_n(x), Dv(x) \in k \cap C(x) = C$. Now for any $\phi(x) \in C(x)$ we have $D\phi(x) = (\partial\phi(x)/\partial x)Dx = (\partial\phi(x)/\partial x)f(x)$. At least one of the quantities $u_1(x), \cdots, u_n(x), v(x)$ is not in k, for otherwise $dx = 0$, so at least one of

$$\frac{\partial u_1(x)/\partial x}{u_1(x)} f(x), \cdots , \frac{\partial u_n(x)/\partial x}{u_n(x)} f(x) , \frac{\partial v(x)}{\partial x} f(x)$$

is a nonzero element of C, implying that $1/f(x)$ is of one of the excluded forms. It remains to prove the Proposition when C is not algebraically closed. Suppose that there are constants of $k(x)$ that are not in C. The differential field structures on k and $k(x)$ extend uniquely to differential field structures on $k(\bar{C})$ and $(k(\bar{C}))(x)$, $\bar{C}$ being the algebraic closure of C, and we get constants of $(k(\bar{C}))(x)$ that are not in the subfield of constants $\bar{C}$ of $k(\bar{C})$, since $k(x) \cap \bar{C} = C$. Hence $1/f(x)$ is of the form $a(\partial u/\partial x)/u$ for some $a \in \bar{C}$, $u \in \bar{C}(x)$, or of the form $1/f(x) = \partial v/\partial x$, for some $v \in \bar{C}(x)$. Suppose first that $1/f(x) =$

$a(\partial u/\partial x)/u$, with a and u as above. Take u, as we may, to be a quotient of monic elements of $\bar{C}[x]$. We shall be done if we show that $a \in C$, $u \in C(x)$. For any $\sigma \in \mathrm{Aut}\,(\bar{C}(x)/C(x)) \approx \mathrm{Aut}\,(\bar{C}/C)$ we get $1/f(x) = a^\sigma(\partial u^\sigma/\partial x)/u^\sigma$, so that $a(\partial u/\partial x)/u = a^\sigma(\partial u^\sigma/\partial x)/u^\sigma$, or

$$(\partial(u^\sigma/u)/\partial x)/(u^\sigma/u) = a/a^\sigma \in \bar{C}\ .$$

Writing u^σ/u as a power product of distinct monic linear elements of $\bar{C}[x]$, we see that we get a nonconstant function on the left of the equation for a/a^σ unless $u^\sigma/u = 1$. Hence $u^\sigma = u$. Since this is true for all $\sigma \in \mathrm{Aut}\,(\bar{C}/C)$, we get $u \in C(x)$, hence also $a \in C(x) \cap \bar{C} = C$, showing $1/f(x)$ to be of the desired form. Suppose, finally, that we have $1/f(x) = \partial v/\partial x$, for some $v \in \bar{C}(x)$. We may take v such that its partial fraction expansion with respect to $\bar{C}[x]$ has constant term zero. We wish to show $v \in C(x)$. For any $\sigma \in \mathrm{Aut}\,(\bar{C}/C)$ we get $1/f(x) = (\partial v/\partial x)^\sigma = \partial v^\sigma/\partial x$, so that $\partial v^\sigma/\partial x = \partial v/\partial x$. Hence $v^\sigma = v$, and since this is true for all $\sigma \in \mathrm{Aut}\,(\bar{C}/C)$ we get $v \in C(x)$, as desired.

Clearly neither of the two special values for $f(x)$ of which we have made so much use, namely $x/(x+1)$ and $x^3 - x^2$, is of the special form indicated in Proposition 2.

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