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COMMUTANTS OF SOME QUASI-HAUSDORFF MATRICES

B. E. RHOADES

Let $B(c)$ denote the Banach algebra of bounded linear operators over c, the space of convergent sequences, and Γ^* the subalgebra of conservative infinite matrices. Given an upper triangular matrix A in Γ^*, a sufficient condition is established for the commutant of A in Γ^* to be upper triangular. Also determined is the commutant, in $B(c)$, of certain quasi-Hausdorff matrices.

The spaces of bounded, convergent and null sequences will be denoted by m, c, c_0 respectively, and l will denote the set of sequences x satisfying $\sum_k |x_k| < \infty$. Let Δ^* denote the algebra of conservative upper triangular matrices; i.e., $A \in \Delta^*$ implies $A: c \to c$ and $a_{nk} = 0$ for $n > k$. $\mathscr{H}^*$ will denote the algebra of conservative quasi-Hausdorff transformations, and Γ the algebra of all conservative matrices. Γ_a^{1*} is the quasi-Hausdorff transformation generated by $\mu_n = a(n+a)^{-1}$, $a > 1$. For other specialized terminology the reader can consult [3] or [5].

One cannot answer commutant questions for upper or lower triangular matrices in $B(c)$ by taking transposes. For example, let C denote the Cesàro matrix of order 1. C^T is not conservative. On the other hand, the matrix $A = (a_{nk})$ defined by

$$a_{nk} = \begin{cases} 1 \text{ for } n = \binom{j+1}{2}, \binom{j}{2} + 1 \leqq k \leqq n ; & j = 1, 2, \cdots , \\ 0 \text{ otherwise} , \end{cases}$$

is conservative, but A^T is not. It is true that the transpose of any conservative quasi-Hausdorff matrix is a conservative Hausdorff matrix. C shows that the converse is false.

We begin with some results analogous to those of [3] and [5].

THEOREM 1. *Let $A \in \Delta^*$. If A has the property that*

(1) *for each $t \in m$, $n \geqq 0$, $(A - a_{nn}I)t = 0$ implies $t \in$ linear span $\{e^0, e^1, \cdots, e^n\}$, then every matrix B with finite norm which commutes with A is upper triangular.*

$B \leftrightarrow A$ implies

$$\sum_{j=0}^{k} b_{nj} a_{jk} = \sum_{j=n}^{\infty} a_{nj} b_{jk} ; \qquad n, k = 0, 1, 2, \cdots . \tag{2}$$

Set $k = 0$ to get

$$b_{n0} a_{00} = \sum_{j=n}^{\infty} a_{nj} b_{j0} ; \qquad n = 0, 1, 2, \cdots ,$$

which can be written in the form $(A - a_{00}I)t^0 = 0$, where $t^0 = \{b_{n0}\}_{n=0}^{\infty}$. By hypothesis, t belongs to the linear span of e^0, so that $b_{n0} = 0$ for all $n > 0$. By induction one can show that b_{nk} for all $n > k$ and B is upper triangular.

REMARKS. 1. The condition that A be conservative is not needed in the proof. All one needs are restrictions on A and B sufficient to guarantee that the summations in (2) exist for each n and k; for example, it would be sufficient to assume that each row of A is in l and each column of B is in m.

2. It is an open question whether condition (1) is necessary. (The proof of the necessity of Theorem 1 in [3] is faulty, because it fails to show that B has finite norm.)

An upper triangular matrix is called factorable if $a_{nk} = c_n d_k$, $n \leqq k$. Examples of upper triangular factorable matrices in $B(c)$ are the transposes of the weighted mean methods $(\bar{N}, p_n)$ with $p_n = a^n$, $a > 1$, and the Γ_a^{1*}, $a > 1$.

THEOREM 2. *If A is a factorable upper triangular matrix with $a_{nn} \neq 0$ for all n, then $B \leftrightarrow A$ implies B is upper triangular.*

Proof. Set $n = k = 0$ in (2) to get $\sum_{j=1}^{\infty} a_{0j} b_{j0} = 0$. From (2) with $k = 0$, $n = 1$, we have

$$b_{10} a_{00} = \sum_{j=1}^{\infty} a_{1j} b_{j0} = \frac{c_1}{c_0} \sum_{j=1}^{\infty} a_{0j} b_{j0} = 0 \ .$$

Since $a_{00} \neq 0$, $b_{10} = 0$. By induction, $b_{n0} = 0$ for all $n > 0$. Then by induction on k, we can show $b_{nk} = 0$ for all $n > k$, and B is upper triangular.

COROLLARY 1. *If $A \in \Delta^*$, A is factorable and has exactly one zero on the main diagonal, then $B \leftrightarrow A$ implies B is upper triangular.*

Proof. Let N be such that $a_{NN} = 0$. If $N > 0$, then the proof of Theorem 2 forces $b_{nk} = 0$ for $n > k$, $k < N$. For $n > N$, $k = N$ in (1) we have

$$\sum_{j=n}^{\infty} a_{nj} b_{jN} = \sum_{j=0}^{N} b_{nj} a_{jN} = b_{nN} a_{NN} = 0 \ ,$$

or, $-a_{nn} b_{nN} = \sum_{j=n+1}^{\infty} a_{nj} b_{jN}$; i.e., $-d_n b_{nN} = \sum_{j=n+1}^{\infty} b_{jN} d_j$, which leads to $d_n b_{nN} = 0$. Since $d_n \neq 0$, $b_{nN} = 0$. By induction, $b_{nk} = 0$ for $n > k > N$.

COROLLARY 2. *If $A \in \Delta^*$, is factorable, and has at least two* *nonadjacent zeros on the main diagonal, then there exists a matrix* $B \leftrightarrow A$, *B not upper triangular.*

Let M and N satisfy $a_{MM} = a_{NN} = 0$, $N > M + 1$. There are four possibilities: (i) $c_M = c_N = 0$, (ii) $c_M = d_N = 0$, (iii) $d_M = c_N = 0$, and (iv) $d_M = d_N = 0$.

If $d_N \neq 0$ the system (1) with $n = M$ has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_M = 0$, $t_k = -\sum_{j=k+1}^{N} a_{kj}t_j/a_{kk}$, $k \neq M$, $k < N$. If $d_N = 0$, then (1), with $n = M$, has the solution $t_k = 0$, $k > N$, $t_N = 1$, $t_{N-1} = 0$, $t_M = 0$,

$$t_K = -\sum_{j=k+1}^{N-1} a_{kj}t_j/a_{kk}\ , \quad k \neq M\ , \quad k < N - 1\ .$$

Define B by $b_{nM} = t_n$, $b_{n,m+1} = -c_M t_n/c_{M+1}$, $n \leqq N$, $b_{nk} = 0$ otherwise. Then $B \leftrightarrow A$, $B \in \Gamma$, but $B \notin \Delta^*$.

Suppose $A \in \Delta^*$, is factorable, and satisfies $a_{NN} = a_{N+1,N+1} = 0$, $a_{nn} \neq 0$ for $n \neq N$, $N + 1$. If $d_{N+1} = 0$ or $c_N = 0$, then an examination of the proof of Corollary 2 shows that we can find a matrix B which commutes with A and which is not upper triangular. If, however, $c_{N+1} = d_N = 0$, but $c_N d_{N+1} \neq 0$, then B must be upper triangular.

COROLLARY 3. *Let A be a factorable upper triangular matrix such that, for some integer N,* $d_N = c_{N+1} = 0$, *and* $c_N d_{N+1} \neq 0$, *and* $a_{nn} \neq 0$ *for* $n \neq N$, $N + 1$. *Then* $B \leftrightarrow A$ *implies B is upper triangular.*

From the proof of Theorem 2, $b_{nk} = 0$ for each $k < N$, $n > k$. For $k = N$, $n \geqq N$, we have, from (2),

$$(3) \qquad \sum_{j=n}^{\infty} a_{nj}b_{jN} = \sum_{j=0}^{N} b_{nj}a_{jN} = b_{nN}a_{NN} = 0\ .$$

For $n > N + 1$, (3) becomes $c_n \sum_{j=n}^{\infty} d_j b_{jN} = 0$, which leads to $b_{nN} = 0$ since c_n, $d_n \neq 0$. With $n = N$, (3) now becomes $a_{NN}b_{NN} + a_{N,N+1}b_{N+1,N} = 0$. By induction it can be shown that $b_{nk} = 0$ for $n > k > N + 1$, so that B is upper triangular.

To determine the commutants of various quasi-Hausdorff matrices in the algebras Δ^*, Γ and $B(c)$, we shall use Γ_a^{1*}, which is a member of Δ^*.

COROLLARY 4. Com(Γ_a^{1*}) *in* Δ^* = Com(Γ_a^{1*}) *in* $\Gamma = \mathscr{H}^*$.

The first equality follows from Theorem 2, since Γ_a^{1*} is factorable. The second equality comes from the following Lemma and Theorem 4.1 of [2].

LEMMA. *Let H be a quasi-Hausdorff method with distinct diagonal entries, B any upper triangular matrix, $B \leftrightarrow H$. Then B is quasi-Hausdorff.*

Proof. From (2) we get

$$\sum_{j=n}^{k} h_{nj} b_{jk} = \sum_{j=n}^{k} b_{nj} h_{jk}\ , \qquad k \geqq n\ .$$

Denote the diagonal entries of B by λ_n. Then, it can be shown by induction that $b_{n,n+p} = \binom{n+p}{p} \Delta^p \lambda_n$, $p = 0, 1, \cdots$, and B is quasi-Hausdorff.

Leviatan [2] has shown that every matrix which commutes formally with the inverse of C^T is a quasi-Hausdorff matrix.

For any $T \in B(c)$ one can define continuous linear functionals χ and χ_i by $\chi(T) = \lim Te - \Sigma_k \lim (Te^k)$ and $\chi_i(T) = (Te)_i - \Sigma_k (Te^k)_i$, $i = 1, 2, \cdots$. Any $T \in B(c)$ has the representation $Tx = v \lim x + Bx$ for each $x \in c$, where B is the matrix representation of the restriction of T to c_0, and v is the bounded sequence $v = \{\chi_i(T)\}$. (See, e.g. [1].)

THEOREM 3. *For each $a > 1$, $\mathrm{Com}\,(\Gamma_a^{1*})$ in $B(c) = \{T \in B(c): v = v_1 e$ and $B \in \mathscr{H}^*\}$.*

Proof. From Corollary 1 of [5] we must have $Av = \chi(A)v$. Therefore, for each n, $\sum_{k=n}^{\infty} h_{nk}^* v_k = a v_n/(a-1)$. But

$$h_{nk}^* = \frac{a k!\, \Gamma(n+a)}{n!\, \Gamma(k+a+1)}\ .$$

Thus

$$v_n = \frac{(a-1)\Gamma(n+a)}{n!} \sum_{k=n}^{\infty} \frac{k!\, v_k}{\Gamma(k+a+1)}\ ,$$

which leads to $v_n = v_1$ for all $n > 1$.

That $B \in \mathscr{H}^*$ comes from the lemma.

Theorems 3 and 4 of [5] are not extendable to upper triangular matrices because the system of equations $Av = \chi(A)v$ is now much more complicated.

It is an open question whether having distinct diagonal entries is a sufficient condition for a conservative quasi-Hausdorff matrix H^* to have the same commutant in Δ^* and Γ.

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INDIANA UNIVERSITY