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A CHARACTERIZATION OF NORMAL ANALYTIC SPACES BY THE HOMOLOGICAL CODIMENSION OF THE STRUCTURE SHEAF

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In 1951 K. Oka proved that *if X is a hypersurface in C^n whose singular set $\Sigma(X)$ has codimension at least 2 in X, then X is a normal analytic space.* **This result was subsequently generalized by S. Abhyankar and (independently) W. Thimm to the case of a complete intersection.**

The main result of the present work is the following criterion for normality: *If* $\dim[\Sigma(X) \cap \{x \in X: \operatorname{codh}_x \mathcal{O}_X \leqq k + 2\}] \leqq k$ *for all integers* $k \geqq -1$, *then X is a normal analytic space.* **This is the best possible criterion following the lines of Oka, Abhyankar, and Thimm, and is, in fact, a characterization; the converse is true. The criterion implies that whenever $\mathcal{O}_X$ has Cohen-Macaulay stalks and $\Sigma(X)$ has codimension at least 2, then X is normal. Finally, the techniques used in proving the criterion are used to obtain a vanishing theorem for the first cohomology group of the complement of a subvariety A of suitably high codimension in a Stein manifold, with coefficients in the ideal sheaf of a normal subvariety containing A.**

It is appropriate to sketch Oka's proof. Oka looked at the exact cohomology sequence (where $\mathcal{I}$ = ideal sheaf of X in C^n) $\Gamma(C^n - \Sigma(X), {}_n\mathcal{O}) \xrightarrow{\rho} \Gamma(X - \Sigma(X), \mathcal{O}_X) \xrightarrow{\partial} H^1(C^n - \Sigma(X), \mathcal{I}) \xrightarrow{\alpha} H^1(C^n - \Sigma(X), {}_n\mathcal{O})$. Since X is a hypersurface, $\mathcal{I} \approx {}_n\mathcal{O}$ and Cartan's three annuli theorem implies that $H^1(C^n - \Sigma(X), \mathcal{I})$ vanishes (since $\Sigma(X)$ is of codimension at least 3 in C^n). Thus ρ is surjective and the Riemann 2nd removable singularity theorem in C^n shows that holomorphic functions on $X - \Sigma(X)$ extend holomorphically to X. Therefore, X is normal.

Since I is not generally free for an arbitrary complete intersection, Abhyankar and Thimm do not use Cartan's three annuli theorem. In this paper however, the spirit of Oka's proof is followed by reintroducing Cartan's three annuli theorem in a very general version due to G. Trautmann. This result is Theorem T in §2.

An inspection of the above exact sequence shows that the best possible result using Oka's technique occurs when α is injective. It is surprising therefore that α is injective only when (and, of course, when) $H^1(C^n - \Sigma(X), \mathcal{I}) = 0$. This is an immediate consequence of the vanishing theorem (Theorem 4) presented in the following section.

The few results needed about normal analytic spaces may be found in R. Narasimhan [4]. In particular:

(1) *Riemann's 2nd removable singularity theorem.* If X is a normal analytic space and if A is a closed subvariety with codim $A \geqq 2$, the restriction $\Gamma(X, \mathcal{O}_X) \to \Gamma(X - A, \mathcal{O}_X)$ is isomorphic.

(2) If X is normal then codim $\Sigma(X) \geqq 2$.

In the sequel $\Sigma = \Sigma(X)$ will denote the singular subvariety of X and dimension or codimension statements will be taken pointwise (i.e., codim $A \geqq 2$ means $\operatorname{codim}_x A \geqq 2$ for all $x \in X$).

We require the notions of *homological codimension*, codh, *m*th *singularity subvariety of the analytic sheaf* $\mathscr{F}$, $S_m(\mathscr{F})$ and *profondeur* (*depth*), prof. The definition and basic properties of these concepts may be found in [6]. For the present it suffices to recall that $S_m(\mathscr{F}) = \{x \in X: \operatorname{codh}_x \mathscr{F} \leqq m\}$ and that $S_m(\mathscr{F})$ is a subvariety of X, if $\mathscr{F}$ is coherent.

1. Characterization of normal analytic spaces.

The following theorem is a modification of a result due to G. Trautmann [8]. It follows easily from Theorem 1.14 [6] and will not be proved here.

THEOREM T. *Let* $(X, \mathcal{O}_X)$ *be an analytic space, A a closed subvariety of X, $\mathscr{F}$ a coherent analytic sheaf and q an integer* $\geqq 0$. *Then the following conditions are logically equivalent*:

(1) *For every Stein open set* $U \subset X$ *the restriction*

$$\Gamma(U, \mathscr{F}) \longrightarrow \Gamma(U - A, \mathscr{F})$$

is isomorphic and

$$H^i(U - A, \mathscr{F}) = 0$$

for $1 \leqq i \leqq q - 1$.

(2) $\dim [A \cap S_{k+q+1}(\mathscr{F})] \leqq k$ *for all integers* $k \geqq -1$.

(3) $\operatorname{prof}_A \mathscr{F} \geqq q + 1$.

An implication of the form (2) $\Rightarrow$ (1) can be considered as a combination of Riemann's 2nd removable singularity theorem with a generalization of Cartan's three annuli theorem (or as a general version of Frenkel's lemma).

The following theorem, which characterizes normal analytic spaces, is the most general version of the results of Oka, Abhyankar, and Thimm, [5], [1], and [7], on the normality of subvarieties of manifolds.

THEOREM 1. *A reduced analytic space X is normal if and only if*

$$\dim [\Sigma(X) \cap S_{k+2}(\mathcal{O}_X)] \leqq k$$

for all integers $k \geqq -1$.

Proof. If the dimension estimate holds, then (2) of Theorem T is fulfilled for $q = 1$. Hence the 2nd Riemann removable singularity theorem holds and, as in the introduction, X is normal.

Conversely, if X is normal, the second Riemann removable singularity theorem holds, so that (1) of Theorem T is verified for $q = 1$. Hence (2) of Theorem T holds, giving the dimension estimate.

There is a characterization of normal Noetherian rings due to J. P. Serre [cf. 3, page 125]: *A Noetherian ring* $\mathcal{O}$ *is normal if and only if* $\mathscr{P} \in \operatorname{Spec}(\mathcal{O})$ *implies that* $\operatorname{prof}(\mathcal{O}_{\mathscr{P}}) \geqq 2$ *when* $ht\,\mathscr{P} \geqq 2$ *and that* $\mathcal{O}_{\mathscr{P}}$ *is regular when* $ht\mathscr{P} < 2$. An application of (3) in Theorem T and Theorem 1 gives the following analytic version of Serre's characterization.

COROLLARY. *An analytic space* X *is normal if and only if* $\operatorname{prof}_{\Sigma}\mathcal{O}_X \geqq 2$.

Another immediate consequence of Theorem 1 is the classical result that the nonnormal points of an analytic space form an analytic subvariety.

THEOREM 2. *If* X *is a reduced analytic space then* $\{x \in X\colon X$ *is not normal at* $x\}$ *is an analytic subvariety* X.

Proof. This follows immediately from Theorem 1 and the fact that for a subvariety V of X $\{x \in X\colon \dim_x V \geqq k\}$ is a subvariety of X, too.

A local ring $\mathcal{O}$ is said to be a *Cohen-Macaulay ring* if $\operatorname{codh}_{\mathfrak{m}} \mathcal{O} =$ Krull $\dim \mathcal{O}$ (where $\mathfrak{m} =$ maximal ideal of $\mathcal{O}$). The next result is a stronger generalization of the Oka-Abhyankar-Thimm theorems than Theorem 1.

THEOREM 3. *If* X *is a reduced analytic space and* $x \in X$ *is such that* $\mathcal{O}_{X,x}$ *is a Cohen-Macaulay ring and* $\operatorname{codim}_x \Sigma(X) \geqq 2$, *then* X *is normal at* x.

Proof. If $\mathcal{O}_{X,x}$ is Cohen-Macaulay, then this is true at nearby points also, therefore, $\operatorname{codh}_y \mathcal{O}_X = \dim_y X$ for y near x. Since $y \mapsto \operatorname{codh}_y \mathcal{O}_X$ is lower semi-continuous and $y \mapsto \dim_y X$ is upper semi-continuous, it follows that $\operatorname{codh}_y \mathcal{O}_X$ is constant in a neighborhood of x. Let $n = \dim_x X$.

Then $\Sigma \cap S_{k+2}(\mathcal{O}_X)$ is empty if $k < n - 2$ and is Σ if $k = n - 2$. But we observe that by hypothesis, $\dim \Sigma \leqq n - 2$, so the dimension estimate of Theorem 1 holds.

COROLLARY (*theorems of Oka-Abhyankar-Thimm*). *If X is a complete intersection such that* $\operatorname{codim} \Sigma \geqq 2$, *then X is normal.*

Proof. A complete intersection is Cohen-Macaulay ([2], Proposition 3, page 200).

Oka [5] proved Theorem 3 in the case that X is a hypersurface. The key idea in the proof is to show that $H^1(C^n - \Sigma(X), \mathscr{I}) = 0$, where I is the ideal sheaf of X in C^n. We now establish a general vanishing theorem for normal analytic spaces embedded in Stein manifolds.

THEOREM 4. *If Z is a Stein manifold and X a proper normal analytic subvariety of Z with ideal sheaf $\mathscr{I}$, then for any subvariety A of X with*

$$\dim A \leqq \dim X - 2$$

we have

$$H^1(Z - A, \mathscr{I}) = 0 .$$

Proof. We have $\mathscr{O}_X = \mathscr{O}_Z/I$ and hence we have the following exact sequence

$$\Gamma(Z - A, \mathscr{O}_Z) \xrightarrow{\rho} \Gamma(X - A, \mathscr{O}_X) \xrightarrow{\partial} H^1(Z - A, \mathscr{I}) \longrightarrow H^1(Z - A, \mathscr{O}_Z) .$$

Therefore, it suffices to prove that ρ is surjective and that $H^1(Z - A, \mathscr{O}_Z) = 0$.

To show ρ surjective, consider the commutative diagram

$$\begin{array}{ccc} \Gamma(Z - A, \mathscr{O}_Z) & \xrightarrow{\rho} & \Gamma(X - A, \mathscr{O}_X) \\ \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle r'} \\ \Gamma(Z, \mathscr{O}_Z) & \xrightarrow[r'']{} & \Gamma(X, \mathscr{O}_X) \end{array}$$

where r, r', and r'' are restrictions.

Since X and Z are normal, the 2nd Riemann removable singularity theorem implies that r and r' are isomorphic. Also Cartan's Theorem B implies that r'' is surjective. Hence ρ is surjective.

Next, $H^1(Z - A, \mathscr{O}_Z) = 0$. To see this we observe that since the stalks of $\mathscr{O}_Z$ are regular rings, $\operatorname{codh}_x (\mathscr{O}_Z) = \dim_x Z$ for all x. Hence $A \cap S_{k+3}(\mathscr{O}_Z)$ is empty if $k < \dim Z - 3$ and is A if $k = \dim Z - 3$. Since A is of codimension $\geqq 2$ in X and since X is a proper subvariety of Z, $\dim A \leqq \dim Z - 3$. Therefore, the dimension estimate in (2) of Theorem T holds for $q = 2$. Hence (1) of Theorem T implies that $H^1(Z - A, \mathcal{O}_Z) = 0$, completing the proof.

The next result gives a characterization of embedded normal analytic spaces. The proof is similar to the proof of Theorem 1, so we omit the derivation.

THEOREM 5. *If Z is a Stein manifold and X is a proper analytic subvariety with $\mathcal{I}$ the ideal sheaf of X in Z, then X is normal if and only if*

$$\dim [\Sigma(X) \cap S_{k+3}(\mathcal{I})] \leqq k$$

for all $k \geqq -1$.

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