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A LOCAL ESTIMATE FOR TYPICALLY REAL FUNCTIONS

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In this paper it is shown that for each typically real function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ the local estimate $n - a_n \leqq (1/6)n(n^2 - 1)(2 - a_2)$ holds, $n = 2, 3, \cdots$. The constant $(1/6)n(n^2 - 1)$ is best possible.

Bombieri [1] proved the existence of constants γ_n such that for each function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the unit disk D,

$$(1) \qquad |\operatorname{Re}(n - a_n)| \leqq \gamma_n \operatorname{Re}(2 - a_2), \qquad n = 2, 3, \cdots .$$

Hummel [2] showed that if in addition f maps D onto a domain starlike with respect to the origin, then $|n - a_n| \leqq \gamma_n |2 - a_2|$ for the value

$$(2) \qquad \gamma_n = n(n^2 - 1)/6 ;$$

furthermore, this choice of γ_n is best possible. In this paper we shall show that (2) is also the best possible constant in (1) for the collection of univalent functions with real coefficients. More generally, we answer this question for the set T of typically real functions.

DEFINITION 1. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in D is said to be *typically real* provided $f(z)$ is real if and only if z is real.

The class T was introduced by Rogosinski [5], [6]. Among other things he showed that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T$, then a_n is real and $|a_n| \leqq n$, $n = 2, 3, \cdots$. Note that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in D and has real coefficients, then $f(\bar{z}) = \overline{f(z)}$. From this fact it easily follows that $f \in T$.

We now introduce a family of polynomials $P_n(t)$ closely related to the Chebyshev polynomials of the second kind.

DEFINITION 2. For each n, $n = 1, 2, \cdots$, set

$$r = \left[\frac{n-1}{2}\right], \quad P_n(t) = \sum_{k=0}^{r} (-1)^k \binom{n-k-1}{k} t^{n-2k-1} ,$$

where t is real.

DEFINITION 3. Let c_n be the largest critical point of $P_n(t)$, $n = 3, 4, \cdots$.

To solve our problem, we need the following properties of these polynomials:

LEMMA 1. *$P_n(2\cos\theta) = \sin n\theta/\sin\theta$ for each $\theta \in [-\pi, \pi]$, $n = 1, 2, \cdots$. (The righthand side is defined so as to be continuous at $\theta = 0$, $\theta = \pm\pi$).*

2. *$P_n(c_n) = \min_{t\in[0,2]} P_n(t)$, and $P_n'(t)$ is strictly increasing in $[c_n, \infty)$.*

3. *If $n \geqq 4$ is even, then $|P_n(t)| \leqq n|t|/2$ for all $t \in [-2, 2]$. Equality holds only for $t = 0$, $t = \pm 2$.*

4. *$P_n'(2) = \gamma_n$, $n = 1, 2, \cdots$.*

Proof. The first three properties follow from Lemma 1 of [3]. To prove part 4, we observe that the derivative of the function $\theta \to \sin n\theta/\sin\theta$ exists at $\theta = 0$, and we differentiate the identity in part 1 to arrive at

$$P_n'(2) = \lim_{\theta\to 0} \frac{\sin n\theta\cos\theta - n\sin\theta\cos n\theta}{2\sin^3\theta}$$

$$= \lim_{\theta\to 0} \frac{(n\theta - n^3\theta^3/6 + \cdots)(1-\theta^2/2+\cdots) - n(\theta - \theta^3/6+\cdots)(1-n^2\theta^2/2+\cdots)}{2\theta^3 + \cdots}$$

$$= \lim_{\theta\to 0} \frac{3n(n^2-1)\theta^3/3 + O(\theta^4)}{2\theta^3 + O(\theta^4)} = \gamma_n .$$

The proof of the lemma is now finished.

Let us remark that two interesting corollaries of the lemma are the combinatorial identities

$$\sum_{k=0}^{m-1} (-1)^k (m-k)\binom{2m-k}{k} 4^{m-k} = \binom{2m+2}{3}, \qquad m = 1, 2, \cdots$$

$$\sum_{k=0}^{m-1} (-1)^k (2m-2k-1)\binom{2m-k-1}{k} 4^{m-k-1} = \binom{2m+1}{3}, \qquad m = 1, 2, \cdots,$$

which result from part 4 and Definition 2.

We can now prove our main result.

THEOREM. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T$, then $n - a_n \leqq \gamma_n(2 - a_2)$, $n = 2, 3, \cdots$, where γ_n is given by (2). Equality can hold only when $f(z) = z/(1-z)^2$.*

Proof. Let $a_2 = x$, and fix $n > 2$; we seek the smallest $t = t(n) > 0$ such that

$$n - a_n - t(2 - x) \leqq 0 . \tag{3}$$

If n is odd, we may assume $x \geqq 0$, since $-f(-z) \in T$. Apply Theorem 2 of [3] to obtain the sharp inequalities

$$tx - a_n \leqq \begin{cases} tc_n - P_n(c_n) & \text{if } 0 \leqq x \leqq c_n\,, \\ tx - P_n(x) & \text{if } c_n \leqq x \leqq 2\,, \end{cases} \tag{4}$$

where equality can hold when $c_n \leqq x \leqq 2$ only for the function

$$f(z) = z/(1 - xz + z^2)\,. \tag{5}$$

To satisfy (3) we must have

$$t \geqq \max\left(\frac{n - P_n(c_n)}{2 - c_n}, \max_{c_n \leqq x \leqq 2} \frac{n - P_n(x)}{2 - x}\right) = \max_{c_n \leqq x \leqq 2} \frac{P_n(2) - P_n(x)}{2 - x}\,,$$

by (4) and part 1 of the lemma. Using parts 2 and 4 and the mean-value theorem, we conclude that $t \geqq P_n'(2) = \gamma_n$; hence $t = \gamma_n$ is the smallest constant for which (3) holds. Furthermore, strict inequality holds in (3) unless $x = 2$, that is, $f(z) = z/(1 - z)^2$, by (5).

Next let n be even, and put $F_n(t) = (P_n(t) + n)/(t + 2)$, $0 \leqq t \leqq 2$. By part 2 of the lemma F_n attains its minimum at only one point r_n, with $c_n \leqq r_n < 2$. Again Theorem 2 of [3] gives

$$tx - a_n \leqq \begin{cases} tx + n - (x + 2)F_n(r_n) & \text{if } -2 \leqq x \leqq r_n\,, \\ tx - P_n(x) & \text{if } r_n \leqq x \leqq 2\,, \end{cases}$$

with equality for the case $r_n \leqq x \leqq 2$ only when f is as in (5). Hence

$$t \geqq \max_{-2 \leqq x \leqq r_n} \frac{2n - (x + 2)F_n(r_n)}{2 - x}, \quad t \geqq \max_{r_n \leqq x \leqq 2} \frac{n - P_n(x)}{2 - x}\,. \tag{6}$$

Now it follows from the definition of F_n, part 3 of the lemma, and direct algebraic manipulations that the maximum of the first term in (6) occurs at $x = r_n$ only. Consequently, as earlier we get $t \geqq P_n'(2) = \gamma_n$, and the rest of the argument proceeds as before. The proof of the theorem is thus complete.

COROLLARY. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T$, *then* $n - |a_n| \leqq \gamma_n(2 - |a_2|)$, *where* γ_n *is given by* (2). *Equality holds only for* $f(z) = z/(1 \pm z)^2$.

Proof. By substituting $-f(-z)$ for $f(z)$ if necessary, we may assume that $a_2 \geqq 0$. Then the theorem yields

$$n - |a_n| \leqq n - a_n \leqq \gamma_n(2 - a_2) = \gamma_n(2 - |a_2|)\,,$$

with equality only if $f(z)$ or $-f(-z)$ is the function $z \to z/(1 - z)^2$.

In conclusion, we show that the statement of our theorem is in general false if T is replaced by the class of normalized univalent functions. To produce a counterexample we employ the theory of Löwner [4], which says in particular that if K is a piecewise continuous function from $[0, \infty)$ into the unit circle ∂D, then there exists a univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that

$$a_2 = 2\int_0^\infty e^{-t}K(t)dt,\ a_3 = a_2^2 - \int_0^\infty e^{-2t}K(t)^2 dt\ . \tag{7}$$

We set

$$K(t) = \begin{cases} 1 & \text{if } 0 \leqq t \leqq \log 2\ , \\ (1 + \sqrt{3}\, i)/2 & \text{if } \log 2 < t < \infty\ , \end{cases}$$

and find from (7) that $a_2 = (3 + \sqrt{3}\, i)/2$, $a_3 = (5 + 11\sqrt{3}\, i)/8$, and $\operatorname{Re}(3 - a_3) > 4 \operatorname{Re}(2 - a_2)$.

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