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SOME CONVERGENCE THEOREMS IN BANACH ALGEBRAS

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This paper is concerned with finding necessary and sufficient conditions for the convergence of the sequence $\{f_n(a)\}$ of elements of Banach algebra, where $\{f_n\}$ is a sequence of analytic functions imitating the behavior of the sequence of integral powers. In particular, it is shown that the sequence $\{a^n\}$ converges iff the spectrum of a (with the possible exception of the point $\lambda = 1$) lies in the open unit disc and $\lambda = 1$ is a pole of $(\lambda - a)^{-1}$ of order $\leqq 1$.

The spectral characterization of power convergent operators on Hilbert (or Banach) spaces given in [3] can be extended to elements of Banach algebras, however, the methods of [3], based on the direct decomposition of the underlying space are no longer applicable. The main purpose of this note is to prove certain convergence theorems in a complex unital Banach algebra $\mathscr{A}$, which will yield, as a special case, the following result (cf. [3] for operator formulation):

THEOREM 0. *Let $a \in \mathscr{A}$. The sequence $\{a^n\}$ converges iff*
(i) $\operatorname{Sp}(a) - \{1\}$ *lies in the open unit disc, and*
(ii) 1 *is a pole of* $(\lambda - a)^{-1}$ *of order* $\leqq 1$.

($\operatorname{Sp}(a)$ denotes the spectrum of the element $a \in \mathscr{A}$.) Rephrasing the theorem slightly, we may say that the sequence $\{f_n(a)\}$ converges in $\mathscr{A}$ iff $\{f_n(\lambda)\}$ converges uniformly to zero on $\operatorname{Sp}(a) - \{1\}$ and 1 is a pole of $(\lambda - a)^{-1}$ of order $\leqq 1$, where $f_n(\lambda) = \lambda^n$. In the sequel, we shall consider functions more general than $f_n(\lambda) = \lambda^n$, employing the operational calculus in a Banach algebra (cf. [2, Chapter V] *or* [1, Chapter VII]).

A complex function f of complex variable will be called (in this paper) *power-like* if the following two conditions are fulfilled:

(1) f is analytic in a disc $\Delta(f) = \{\lambda : |\lambda| < \delta\}, \delta > 1$,

(2) $(1 - f(\lambda))(1 - \lambda)^{-1}$ has a removable singularity at $\lambda = 1$.

A sequence $\{f_n\}$ of power-like functions will be called *admissible for* $\mathscr{A}$ if

(3) $(1 - x)f_n(x) \to 0$ for each $x \in \mathscr{A}$ with $\operatorname{Sp}(x) \subset \bigcap_n \Delta(f_n)$ and with $\{f_n(x)\}$ convergent,

and

$$(4) \qquad f_n(0) \longrightarrow 0 .$$

We offer some examples of sequences of power-like functions admissible for any algebra $\mathscr{A}$:

(i) The very prototype of such sequences, the sequence $\{\lambda^n\}$ of integral powers of λ.

(ii) The sequence of Cesàro means of the integral powers,

$$\frac{1}{n}(1 + \lambda + \cdots + \lambda^{n-1}) .$$

(iii) Let $\{\gamma_n\}$ be any sequence of complex numbers convergent to 0. We may define f_n inductively by one of the following formulae [5, Proposition 2.1]:

$$f_{n+1}(\lambda) = (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n , \qquad f_1(\lambda) \equiv 1 ,$$

$$f_{n+1}(\lambda) = (1 - \gamma_n)\lambda f_n(\lambda) + \gamma_n\lambda , \qquad f_1(\lambda) \equiv 1 ,$$

$$f_{n+1}(\lambda) = ((1 - \gamma_n)\lambda + \gamma_n)f_n(\lambda) , \qquad f_1(\lambda) \equiv 1 .$$

In each of the three formulae, f_n is a polynomial of the form

$$(5) \qquad f_n(\lambda) = 1 + (\lambda - 1)g_n(\lambda) ,$$

where g_n is a polynomial of degree $\leqq n - 2$.

We observe that, by virtue of (2), each power-like function f_n can be written in the form (5) with g_n analytic in $\Delta(f_n)$.

THEOREM 1. *Let $\{f_n\}$ be an admissible sequence of power-like functions, and let* $\mathrm{Sp}\,(a) \subset \bigcap_n \Delta(f_n)$. *Then $\{f_n(a)\}$ converges iff*

$$(6) \qquad a = p + c ,$$

where

$$(7) \qquad p^2 = p , \qquad pc = cp = 0 , \qquad f_n(c) \longrightarrow 0 .$$

Proof. Suppose first that $f_n(a) \to p$. Then $(1 - a)p = p(1 - a) = 0$ in view of (3), and $ap = pa = p$. More generally,

$$(8) \qquad a^k p = pa^k = p , \qquad k \geqq 0 .$$

For each complex $\lambda \notin \mathrm{Sp}\,(a) \cup \{1\}$,

$$(9) \qquad (\lambda - a)^{-1}p = (\lambda - 1)^{-1}p .$$

This shows that $p = 0$ whenever $\lambda = 1$ is a regular point for $(\lambda - a)^{-1}$. Let C_n be a contour in $\Delta(f_n)$ enclosing $\mathrm{Sp}\,(a) \cup \{1\}$. ($C_n$ is a boundary of an open set $U_n(\supset \mathrm{Sp}\,(a) \cup \{1\})$ consisting of a finite number of closed rectifiable Jordan curves positively oriented with respect to U_n.) Then

$$pf_n(a) = \frac{1}{2\pi i}\int_{C_n} f_n(\lambda)(\lambda - a)^{-1}p d\lambda$$
$$= \frac{p}{2\pi i}\int_{C_n} f_n(\lambda)(\lambda - 1)^{-1}d\lambda = pf_n(1) = p\ ;$$

we have used (9), and then (5) to get $f_n(1) = 1$. Consequently,

$$p^2 = p \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} pf_n(a) = p\ .$$

More generally, $p^k = p$ for each $k \geqq 1$, and induction (utilizing (8)) yields

$$(a - p)^k = a^k - p\ , \qquad k \geqq 1\ . \tag{10}$$

Let us write α_{nk} for $f_n^{(k)}(0)/k!$, and set $c = a - p$. Then

$$f_n(a) - (1 - f_n(0))p = \sum_{k=0}^{\infty} \alpha_{nk}a^k - \left[\sum_{k=1}^{\infty} \alpha_{nk}\right]p$$
$$= \sum_{k=0}^{\infty} \alpha_{nk}(a - p)^k = f_n(a - p) = f_n(c)\ ,$$

using the analyticity of f_n on $\Delta(f_n)(\supset \mathrm{Sp}\,(a))$, and the identity (10). Therefore, $f_n(c)$ is defined, and

$$f_n(c) = (f_n(a) - p) + f_n(0)p \longrightarrow 0$$

by virtue of (4). Finally,

$$cp = pc = p(a - p) = pa - p^2 = 0\ .$$

Assume, conversely, that (6) and (7) hold. Then

$$a^k = (p + c)^k = p + c^k\ ,$$

and

$$f_n(a) = f_n(p + c) = \sum_{k=0}^{\infty} \alpha_{nk}(p + c)^k = f_n(0) + \sum_{k=1}^{\infty} \alpha_{nk}c^k + \left[\sum_{k=1}^{\infty} \alpha_{nk}\right]p$$
$$= f_n(c) + (1 - f_n(0))p \longrightarrow p \quad \text{as} \quad n \longrightarrow \infty\ .$$

If $f_n(\lambda) = \lambda^n$ in the preceding theorem, we obtain the following result.

COROLLARY. *$\{a^n\}$ converges iff $a = p + c$, where*

$$p^2 = p\ , \qquad pc = cp = 0\ , \qquad \lim_{n\to\infty} \| c^n \|^{1/n} < 1\ .$$

The following theorem gives a sufficient condition for the convergence of $\{f_n(a)\}$ if $\{f_n\}$ is an admissible sequence of power-like functions. A brief glance at Theorem 3 will tell the reader how far this condition is from being also necessary. The proof of the theorem could be based on our Theorem 1, on Theorem 5.5.1 [1, p. 174], and on Theorem VII.3.22 [2, p. 576]. We give a direct proof which appears to be fairly simple and straightforward.

THEOREM 2. *Let $\{f_n\}$ be an admissible sequence of power-like functions. If*

(i) *all f_n are analytic and uniformly convergent to zero on a fixed open neighborhood Ω of* $\mathrm{Sp}(a) - \{1\}$,
and

(ii) *1 is a pole of $(\lambda - a)^{-1}$ of order $\leqq 1$,*
then $\{f_n(a)\}$ converges.

Proof. For a certain $\delta > 0$,

$$(\lambda - a)^{-1} = (\lambda - 1)^{-1}p + h(\lambda)\,, \qquad 0 < |\lambda - 1| < \delta\,, \tag{11}$$

where h is analytic in an open neighborhood of $\mathrm{Sp}(a)$. We can select a contour C in Ω enclosing $\mathrm{Sp}(a) - \{1\}$, and for each n we can find a positively oriented circle $C_n = \{\lambda: |\lambda - 1| = \varepsilon < \delta\}$ that misses C and such that f_n is analytic in an open neighborhood of C_n. Using (11), we get

$$\begin{aligned} f_n(a) - p &= \frac{1}{2\pi i}\int_{C+C_n} f_n(\lambda)(\lambda - a)^{-1}d\lambda - \frac{1}{2\pi i}\int_{C_n}(\lambda - a)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda + \frac{1}{2\pi i}\int_{C_n}(f_n(\lambda) - 1)(\lambda - a)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda + \frac{p}{2\pi i}\int_{C_n} g_n(\lambda)d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{C_n}(f_n(\lambda) - 1)h(\lambda)d\lambda \\ &= \frac{1}{2\pi i}\int_C f_n(\lambda)(\lambda - a)^{-1}d\lambda\,, \end{aligned}$$

where g_n is specified in (5). Hence

$$||f_n(a) - p|| \leqq \frac{1}{2\pi}\sup_{\lambda\in C}||f_n(\lambda)(\lambda - a)^{-1}||\cdot l(C) \leqq K\sup_{\lambda\in\Omega}|f_n(\lambda)|\,,$$

with

$$K = \frac{l(C)}{2\pi}\sup_{\lambda\in C}||(\lambda - a)^{-1}|| < +\infty\,, \quad l(C) \text{ the length of } C\,.$$

This gives $f_n(a) \to p$, and completes the proof.

Theorem 2 has a partial converse which will be proved after the following two auxiliary results.

LEMMA 1. *If* $x_n x \to 1$ *and* $x x_n \to 1$, *then* x *is invertible, and* $x_n \to x^{-1}$.

Proof. Let N be a fixed positive integer such that

$$\|1 - x_N x\| < \frac{1}{2}.$$

For each $\varepsilon > 0$ we can find a positive integer n_0 such that

$$\|x x_n - x x_m\| < \varepsilon/(2\|x_N\|) \qquad \text{whenever } n, m > n_0.$$

Since

$$x_n - x_m = (1 - x_N x)(x_n - x_m) + x_N(x x_n - x x_m),$$

we get

$$\|x_n - x_m\| < \frac{1}{2}\|x_n - x_m\| + \frac{1}{2}\varepsilon,$$

and

$$\|x_n - x_m\| < \varepsilon \qquad \text{whenever } n, m > n_0.$$

Hence $x_n \to y$ for some $y \in \mathscr{A}$, and $yx = xy = 1$.

LEMMA 2. *Let* $\{f_n\}$ *be an arbitrary sequence of power-like functions with* $\bigcap_n \Delta(f_n) \supset \mathrm{Sp}(c)$. *If* $f_n(c) \to 0$, *then* 1 *is a regular point for* $(\lambda - c)^{-1}$, *and*

$$g_n(c) \longrightarrow (1 - c)^{-1},$$

with g_n *defined in* (5).

Proof. If $f_n(c) \to 0$, then

$$g_n(c)(1 - c) = (1 - c)g_n(c) \longrightarrow 1.$$

The result follows on taking $x_n = g_n(c)$ and $x = 1 - c$ in Lemma 1.

A special case of Lemma 1 for the algebra of bounded linear operators on a Banach space and with f_n polynomials of a certain form has been proved in [4, Proposition 5]. A particularly simple form of Lemma 2 is the following well known result: If $c^n \to 0$, then the series $\sum_n c^n$ converges to $(1 - c)^{-1}$. Also:

$$n^{-1}c^n + 1 \longrightarrow 0 \Longrightarrow n^{-1}(1 + c + \cdots + c^{n-1}) \longrightarrow (1 - c)^{-1},$$

$$n^{-1}(1 + c + \cdots + c^{n-1}) \longrightarrow 0 \Longrightarrow \sum_{k=0}^{n-2} \frac{n-k-1}{n} c^k \longrightarrow (1-c)^{-1} ,$$

etc.

THEOREM 3. *Let* $\{f_n\}$ *be an admissible sequence of power-like functions with* $\bigcap_n \Delta(f_n) \supset \mathrm{Sp}\,(a)$. *If* $\{f_n(a)\}$ *converges, then*

(i) $f_n(\lambda) \to 0$ *uniformly on* $\mathrm{Sp}\,(a) - \{1\}$,

and

(ii) 1 *is a pole of* $(\lambda - a)^{-1}$ *of order* $\leqq 1$.

Proof. Suppose $f_n(a) \to p$. The elements p and $c = a - p$ satisfy the conditions (6) and (7), in particular, $f_n(c) \to 0$. By Lemma 2, 1 is a regular point for $(\lambda - c)^{-1}$, and hence the function

$$h(\lambda) = (\lambda - c)^{-1}(1 - p)$$

is analytic in a certain open neighborhood of 1. The function

$$u(\lambda) = h(\lambda) + (\lambda - 1)^{-1}p$$

has a pole of order $\leqq 1$ at $\lambda = 1$. The elements $\lambda - a$ and $u(\lambda)$ commute (whenever the latter is defined). Moreover,

$$\begin{aligned}(\lambda - a)u(\lambda) &= (\lambda - a)(\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}(\lambda - a)p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c) + p \\ &= (\lambda - c)^{-1}(\lambda(1 - p) - c + (\lambda - c)p) \\ &= (\lambda - c)^{-1}(\lambda - c) \\ &= 1 .\end{aligned}$$

Hence, $u(\lambda) = (\lambda - a)^{-1}$, and

$$(\lambda - a)^{-1} = (\lambda - c)^{-1}(1 - p) + (\lambda - 1)^{-1}p . \tag{12}$$

The identity (12) shows that

$$\mathrm{Sp}\,(a) - \{1\} = \mathrm{Sp}\,(c) .$$

Finally, $f_n(c) \to 0$ implies $f_n(\lambda) \to 0$ uniformly on $\mathrm{Sp}\,(c)$ [1, p. 584], and the proof is complete.

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