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SOME REMARKS ON HIGH ORDER DERIVATIONS

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Let k, A and B be commutative rings such that A and B are k-algebras. In this paper it is shown that $\Omega_k^{(q)}(A \otimes_k B)$, the module of high order differentials of $A \otimes_k B$ can be expressed by making use of $\Omega_k^{(i)}(A)$ and $\Omega_k^{(j)}(B)$. On the other hand let K/k be a finite purely inseparable field extension. Sandra Z. Keith has given a criterion for a k-linear mapping of K into itself to be a high order derivation of K/k. The representation of $\Omega_k^{(q)}(A \otimes_k B)$ is used to show that Keith's result is valid for larger class of algebras.

Let k, A and B be commutative rings with identities such that A and B are k-algebras. $A \otimes_k B$ is an A-algebra (resp. a B-algebra) via the natural homomorphism f_A (resp. f_B) such that $f_A(a) = a \otimes 1$ (resp. $f_B(b) = 1 \otimes b$). In [5] Y. Nakai proved that there exists a direct sum decomposition

$$\Omega_k^{(q)}(A \otimes_k B) = \Omega_k^{(q)}(A) \otimes_k B \oplus A \otimes_k \Omega_k^{(q)}(B) \oplus U_{A\otimes B|k}^{(q)} .$$

The submodule $U_{A\otimes B|k}^{(q)}$ has the universal mapping property with respect to qth order derivations of $A \otimes_k B$ which vanish on $f_A(A)$ and $f_B(B)$. In this paper we shall investigate the structure of $U_{A\otimes B|k}^{(q)}$. In fact we can express $U_{A\otimes B|k}^{(q)}$ by making use of $\Omega_k^{(i)}(A)$ and $\Omega_k^{(j)}(B)$ when k is a field.

On the other hand Sandra Z. Keith proved

THEOREM ([4]). *Let K/k be a finite purely inseparable field extension and let φ be a k-linear mapping of K into itself. Then we have $\varphi \in D_0^{(q)}(K/k)$ if and only if $\delta\varphi \in D_0^{(1)}(K/k)\smile D_0^{(q-1)}(K/k) + D_0^{(2)}(K/k)\smile D_0^{(q-2)}(K/k) + \cdots + D_0^{(q-1)}(K/k)\smile D_0^{(1)}(K/k)$, where δ is the Hochschild coboundary operator* (cf. [2]) *and $\smile$ denotes the cup-product.*

This gives an alternative inductive definition of qth order derivations which is meaningful for not-necessarily commutative rings but which possibly differs from Nakai's for commutative rings in general. In this paper we shall use our representation of $U_{A\otimes B|k}^{(q)}$ to show that Keith's result is generalized to larger class of algebras.

Any ring in this paper is assumed to be commutative and contain 1. Let k and A be commutative rings. We say that A is a k-algebra if there exists a ring homomorphism f such that $f(1) = 1$. The readers are expected to refer the paper [5] for notations and terminologies.

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1. Representation of $U^{(q)}_{A\otimes B/k}$. Let k, A and B be rings such that A and B are k-algebras.

LEMMA 1. *Let D be an* mth *order derivation of A/k into an A-module M and let Δ be an* nth *order derivation of B/k into a B-module N. Then $D\otimes\Delta$ is an* $(m+n)$th *order derivation of $A\otimes_k B$ into $M\otimes_k N$.*

Proof. We consider the idealizations $A\oplus M$ and $B\oplus N$ of M and N respectively. Then D (resp. Δ) is regarded as an mth (resp. nth) order derivation of A (resp. B) into $A\oplus M$ (resp. $B\oplus N$). The mapping $D\otimes\Delta$ of $A\otimes_k B$ into $(A\oplus M)\otimes_k(B\oplus N)$ is decomposed as follows:

$$A\otimes B_k \xrightarrow{D\otimes 1_B} (A\oplus M)\otimes_k B \xrightarrow{1_{A\otimes M}\otimes\Delta} (A\oplus M)\otimes_k(B\oplus N)\,.$$

By Corollary 6.1 in [5], $D\otimes\Delta$ is an $(m+n)$th order derivation.

The following lemmas are immediate.

LEMMA 2. *In $A\otimes_k A$ we have*

$$\begin{aligned}&(1\otimes a_1-a_1\otimes 1)\cdots(1\otimes a_q-a_q\otimes 1)\\ &\quad=(1\otimes a_1\cdots a_q-a_1\cdots a_q\otimes 1)\\ &\quad+\sum_{s=1}^{q-1}(-1)^s\sum_{i_1<\cdots<i_s}a_{i_1}\cdots a_{i_s}(1\otimes a_1\cdots\hat{a}_{i_1}\cdots\hat{a}_{i_s}\cdots a_q\\ &\quad-a_1\cdots\hat{a}_{i_1}\cdots\hat{a}_{i_s}\cdots a_q\otimes 1)\,.\end{aligned}$$

LEMMA 3. *Let D be a* qth *order derivation of $A\otimes_k B$ into an $A\otimes_k B$-module M vanishing on $f_A(A)$ and $f_B(B)$, where f_A* (resp. f_B) *is the homomorphism of A* (resp. B) *into $A\otimes_k B$ such that $f_A(a)=a\otimes 1$* (resp. $f_B(b)=1\otimes b$). *Then we have*

$$\begin{aligned}&D(a_1\cdots a_i\otimes b_1\cdots b_{q+1-i})\\ &\quad=\sum_{s=1}^{i-1}(-1)^{s-1}\sum_{\alpha_1<\cdots<\alpha_s}(a_{\alpha_1}\cdots a_{\alpha_s}\otimes 1)\\ &\qquad\times D(a_1\cdots\hat{a}_{\alpha_1}\cdots\hat{a}_{\alpha_s}\cdots a_i\otimes b_1\cdots b_{q+1-i})\\ &\quad+\sum_{t=1}^{q-i}(-1)^{t-1}\sum_{\beta_1<\cdots<\beta_t}(1\otimes b_{\beta_1}\cdots b_{\beta_t})\\ &\qquad\times D(a_1\cdots a_i\otimes b_1\cdots\hat{b}_{\beta_1}\cdots\hat{b}_{\beta_t}\cdots b_{q+1-i})\\ &\quad+\sum_{s,t=1}^{s\leq i-1,t\leq q-i}(-1)^{s+t-1}\sum_{\substack{\alpha_1<\cdots<\alpha_s\\ \beta_1<\cdots<\beta_t}}(a_{\alpha_1}\cdots a_{\alpha_s}\otimes b_{\beta_1}\cdots b_{\beta_t})\\ &\qquad\times D(a_1\cdots\hat{a}_{\alpha_1}\cdots\hat{a}_{\alpha_s}\cdots a_i\otimes b_1\cdots\hat{b}_{\beta_1}\cdots\hat{b}_{\beta_t}\cdots b_{q+1-i})\,.\end{aligned}$$

We denote by $\delta^{(q)}_{A/k}$ the canonical qth order derivation of A into $\Omega^{(q)}_k(A)$. Unless any confusion arises, $\delta^{(q)}_{A/k}$ is denoted by $\delta^{(q)}_A$ or $\delta^{(q)}$ simply. If $i \leqq j$, we have the canonical epimorphism φ_{ij} of $\Omega^{(j)}_k(A)$ onto $\Omega^{(i)}_k(A)$ given by $\varphi_{ij}(\delta^{(j)}a) = \delta^{(i)}a$. Let ψ_{ij} be the homomorphism of $\Omega^{(j)}_k(B)$ onto $\Omega^{(i)}_k(B)$ defined as above. We define the homomorphism Φ_q of $\bigoplus_{i=1}^{q-1} \Omega^{(i)}_k(A) \otimes_k \Omega^{(q-i)}_k(B)$ into $\bigoplus_{i=1}^{q-2} \Omega^{(i)}_k(A) \otimes_k \Omega^{(q-1-i)}_k(B)$ as follows: for $x \otimes y \in \Omega^{(i)}_k(A) \otimes_k \Omega^{(j)}_k(B)$,

$$\Phi_q(x \otimes y) = \begin{cases} \varphi_{q-2,q-1}(x) \otimes y & \text{if } i = q-1, j = 1 \\ \varphi_{i-1,i}(x) \otimes y - x \otimes \psi_{j-1,j}(y) & \text{if } i, j > 1 \\ -x \otimes \psi_{q-2,q-1}(y) & \text{if } i = 1, j = q-1 . \end{cases}$$

Obviously Φ_q is surjective.

THEOREM 1. *There exists a natural isomorphism*
(1) $U^{(2)}_{A\otimes B/k} \cong \operatorname{Ker} \Phi_2 = \Omega^{(1)}_k(A) \otimes_k \Omega^{(1)}_k(B)$,
(2) *for* $q \geqq 3$, $U^{(q)}_{A\otimes B/k} \cong \operatorname{Ker} \Phi_q$ *if k is a field.*

Proof. We consider the mapping δ of $A \otimes_k B$ into $\bigoplus_{i=1}^{q-1} \Omega^{(i)}_k(A) \otimes_k \Omega^{(q-i)}_k(B)$ defined by

$$\delta(a \otimes b) = \sum_{i=1}^{q-1} \delta^{(i)}_A a \otimes \delta^{(q-i)}_B b .$$

By Lemma 1 we see that δ is a qth order derivation. Since the image of δ is contained in $\operatorname{Ker} \Phi_q$, δ induces a qth order derivation of $A \otimes_k B$ into $\operatorname{Ker} \Phi_q$. The induced one is also denoted by δ. Clearly δ vanishes on $f_A(A)$ and $f_B(B)$. We have only to prove that the pair $\{\operatorname{Ker} \Phi_q, \delta\}$ satisfies the universal mapping property with respect to qth order derivations of $A \otimes_k B$ which vanish on $f_A(A)$ and $f_B(B)$ ([5]). Let I_A (resp. I_B) be the kernel of the contraction mapping: $A \otimes_k A \to A$ (resp. $B \otimes_k B \to B$). We regard $I_A \otimes_k I_B$ as an $A \otimes_k B$-module via

$$(a \otimes b)\{(x \otimes y) \otimes (u \otimes v)\} = (ax \otimes y) \otimes (bu \otimes v) .$$

Under our assumption it will be shown that we have a natural isomorphism of $A \otimes_k B$-modules

$$\operatorname{Ker} \Phi_q \cong I_A \otimes_k I_B \Big/ \sum_{i=1}^{q} I^i_A \otimes I^{q+1-i}_B ,$$

where $I^i_A \otimes I^j_B$ denotes the image of the canonical homomorphism of $I^i_A \otimes_k I^j_B$ into $I_A \otimes_k I_B$. For $q = 2$ our assertion is obvious. For $q \geqq 3$ we assume that k is a field. We define the $A \otimes_k B$-linear mapping Ψ of $I_A \otimes_k I_B$ into $\bigoplus_{i=1}^{q-1} \Omega^{(i)}_k(A) \otimes_k \Omega^{(q-i)}_k(B)$ by

$$\Psi((1 \otimes a - a \otimes 1) \otimes (1 \otimes b - b \otimes 1)) = \sum_{i=1}^{q-1} \delta_A^{(i)} a \otimes \delta_B^{(q-i)} b \ .$$

Obviously we have $\operatorname{Im} \Psi \subset \operatorname{Ker} \Phi_q$. We shall show that Ψ is an epimorphism of $I_A \otimes_k I_B$ onto $\operatorname{Ker} \Phi_q$ with kernel $\sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$. Let $f \in I_A \otimes_k I_B$ and let $\pi_i(f)$ denote the canonical image of f in $\Omega_k^{(i)}(A) \otimes_k \Omega_k^{(q-i)}(B)$. We assume that $\sum_{i=1}^{q-1} \pi_i(f_i) \in \operatorname{Ker} \Phi_q$ for $f_i \in I_A \otimes_k I_B$ $(1 \leqq i \leqq q-1)$. From the definition of Φ_q we see that $f_i - f_{i+1} \in I_A^{i+1} \otimes I_B + I_A \otimes I_B^{q-i}$ $(1 \leqq i \leqq q-2)$. Hence we have $f_i + \alpha_i = f_{i+1} + \beta_{i+1}$ for some $\alpha_i \in I_A^{i+1} \otimes I_B$ and $\beta_{i+1} \in I_A \otimes I_B^{q-i}$ $(1 \leqq i \leqq q-2)$, and so it follows that $f_1 + \alpha_1 + \cdots + \alpha_{q-2} = f_2 + \beta_2 + \alpha_2 + \cdots + \alpha_{q-2} = \cdots = f_{q-1} + \beta_2 + \cdots + \beta_{q-1}$. Let f be this equal element of $I_A \otimes_k I_B$. Then we have $\pi_i(f) = \pi_i(f_i)$ and therefore Ψ is surjective. Next we prove $\operatorname{Ker} \Psi = \sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$. Let us consider an element g of $I_A \otimes_k I_B$. If g is in $\operatorname{Ker} \Psi$, we have $g \in I_A^{i+1} \otimes I_B + I_A \otimes I_B^{q+1-i}$ $(1 \leqq i \leqq q-1)$ and so $g = \varepsilon_i + \zeta_i$ for suitable $\varepsilon_i \in I_A^{i+1} \otimes I_B$ and $\zeta_i \in I_A \otimes I_B^{q+1-i}$. On the other hand we get $\varepsilon_i - \varepsilon_{i+1} = \zeta_{i+1} - \zeta_i \in (I_A^{i+1} \otimes I_B) \cap (I_A \otimes I_B^{q-i}) = I_A^{i+1} \otimes I_B^{q-i}$ since k is a field. This implies easily $g \in \sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$. We wish to show that the pair $\{\operatorname{Ker} \Phi_q, \delta\}$ has the universal mapping property. Let D be a qth order derivation of $A \otimes_k B$ into an $A \otimes_k B$-module M vanishing on $f_A(A)$ and $f_B(B)$. Then it suffices to prove that there is an $A \otimes_k B$-homomorphism Θ of $I_A \otimes_k I_B / \sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$ into M satisfying

$$\Theta(\pi\{(1 \otimes a - a \otimes 1) \otimes (1 \otimes b - b \otimes 1)\}) = D(a \otimes b) \ ,$$

where π is the canonical homomorphism of $I_A \otimes_k I_B$ onto $I_A \otimes_k I_B / \sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$. We consider the mapping Λ of $(A \otimes_k A) \otimes_k (B \otimes_k B)$ into M defined by

$$\Lambda((x \otimes y) \otimes (u \otimes v)) = (x \otimes u) D(y \otimes v) \ .$$

Since D vanishes on $f_A(A)$ and $f_B(B)$, Λ induces the mapping of $I_A \otimes_k I_B$ into M sending $(1 \otimes a - a \otimes 1) \otimes (1 \otimes b - b \otimes 1)$ to $D(a \otimes b)$. Now it follows from Lemmas 2 and 3 that Λ vanishes on $\sum_{i=1}^{q} I_A^i \otimes I_B^{q+1-i}$, and so Λ induces the desired mapping Θ. This completes our proof.

REMARK. If $\Omega_k^{(i)}(A) = I_A / I_A^{i+1}$ (resp. $\Omega_k^{(i)}(B) = I_B / I_B^{i+1}$) is k-flat for every i, we have $(I_A^{i+1} \otimes I_B) \cap (I_A \otimes I_B^{q-i}) = I_A^{i+1} \otimes I_B^{q-i}$ by [1] (§1, n°6, Proposition 7). In this case our proof shows that we have $U_{A \otimes B / k}^{(q)} \cong \operatorname{Ker} \Phi_q$ for $q \geqq 3$.

2. A generalization of the result due to Keith. Let k and A be rings such that A is a k-algebra. Let M and N be A-modules. We consider the homomorphism ω of $\operatorname{Hom}_A(M, A) \otimes_k \operatorname{Hom}_A(N, A)$ into $\operatorname{Hom}_{A \otimes_k A}(M \otimes_k N, A)$ given by

$$[\omega(f \otimes g)](m \otimes n) = f(m)g(n)$$

for $f \in \mathrm{Hom}_A(M, A)$, $g \in \mathrm{Hom}_A(N, A)$, $m \in M$ and $n \in N$. Now A is regarded as an $A \otimes_k A$-module via the contraction mapping: $A \otimes_k A \to A$.

LEMMA 4. *If M is a finite projective A-module, then ω is an epimorphism.*

Proof. When M is a finite free A-module, our assertion is obvious. If M is finite A-projective, M is a direct summand of a finite free A-module and hence we see easily that ω is an epimorphism.

Let φ and ψ be k-linear mappings of A into itself. The Hochschild coboundary $\delta\varphi$ of φ is given by $(\delta\varphi)(a, b) = \varphi(ab) - a\varphi(b) - b\varphi(a)$ for $a, b \in A$ (cf. [2]). On the other hand the cup-product $\varphi \smile \psi$ of φ and ψ is the k-bilinear mapping of $A \oplus A$ into A such that $(\varphi \smile \psi)(a, b) = \varphi(a)\psi(b)$ for $a, b \in A$. Let P and Q be A-submodules of $\mathrm{Hom}_k(A, A)$, the set of k-linear mappings of A into itself. Then the cup-product $P \smile Q$ is the set of k-bilinear mappings of $A \oplus A$ into A which are finite sums of mappings of form $\varphi \smile \psi$ for $\varphi \in P$ and $\psi \in Q$.

THEOREM 2. *Let A be an algebra over a field k such that $\Omega_k^{(i)}(A)$ is a finite projective A-module for every $i \geqq 1$. Let φ be a k-linear mapping of A into iteslf. Then we have $\varphi \in D_0^{(q)}(A/k)$ if and only if $\delta\varphi \in D_0^{(1)}(A/k) \smile D_0^{(q-1)}(A/k) + D_0^{(2)}(A/k) \smile D_0^{(q-2)}(A/k) + \cdots + D_0^{(q-1)}(A/k) \smile D_0^{(1)}(A/k)$.*

Proof. By Theorem 1 we have an exact sequence

$$0 \longrightarrow U^{(q)}_{A\otimes A/k} \longrightarrow \bigoplus_{i=1}^{q-1} \Omega_k^{(i)}(A) \otimes_k \Omega_k^{(q-i)}(A)$$

$$\xrightarrow{\Phi_q} \bigoplus_{i=1}^{q-2} \Omega_k^{(i)}(A) \otimes_k \Omega_k^{(q-1-i)}(A) \longrightarrow 0 \ .$$

Our assumption implies that $\Omega_k^{(i)}(A) \otimes_k \Omega_k^{(j)}(A)$ is a projective $A \otimes_k A$-module, and so the above sequence splits. Hence we have an epimorphism of $\bigoplus_{i=1}^{q-1} \mathrm{Hom}_{A\otimes_k A}(\Omega_k^{(i)}(A) \otimes_k \Omega_k^{(q-i)}(A), A)$ onto $\mathrm{Hom}_{A\otimes_k A}(U^{(q)}_{A\otimes A/k}, A)$, where A is considered as an $A \otimes_k A$-module via the contraction mapping: $A \otimes_k A \to A$. Since $\Omega_k^{(i)}(A)$ is finite A-projective, Lemma 4 is applicable to see that $\mathrm{Hom}_A(\Omega_k^{(i)}(A), A) \otimes_k \mathrm{Hom}_A(\Omega_k^{(j)}(A), A)$ is mapped onto $\mathrm{Hom}_{A\otimes_k A}(\Omega_k^{(i)}(A) \otimes_k \Omega_k^{(j)}(A), A)$. Thus we get an epimorphism: $\bigoplus_{i=1}^{q-1} \mathrm{Hom}_A(\Omega_k^{(i)}(A), A) \otimes_k \mathrm{Hom}_A(\Omega_k^{(q-i)}(A), A) \to \mathrm{Hom}_{A\otimes_k A}(U^{(q)}_{A\otimes A/k}, A)$. Let us consider an element φ of $D_0^{(q)}(A/k)$. The contraction mapping of $A \otimes_k A$ into A followed by φ is a qth order derivation of $A\otimes_k A/k$ into A. From the direct sum decomposition of $\Omega_k^{(q)}(A\otimes_k A)$ it follows that $\delta\varphi$ gives an element of $\mathrm{Hom}_{A\otimes_k A}(U^{(q)}_{A\otimes A/k}, A)$. Now only if part is immediate. On the other hand if part is obvious by Proposition 3 of [5].

REMARK. The assumption in Theorem 2 is satisfied in the following two cases, and so in these cases Theorem 2 holds.

(1) A/k is a finitely generated field extension.

(2) A is a smooth algebra over a field k ([3] 16.10.1, 16.10.2).

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