

COMPLETELY DECOMPOSABLE GROUPS WHICH ADMIT ONLY NILPOTENT MULTIPLICATIONS

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A triangle of size n is a collection $\{A_u\}$ of $n(n+1)/2$ (not necessarily distinct) rank one torsion-free abelian groups indexed by all integer sequences of the form $u = i, i+1, \dots, i+j$ with $1 \leq i \leq i+j \leq n$, satisfying $T(A_u) + T(A_s) \leq T(A_{us})$ for all consecutive sequences u, s . Here $T(A_v)$ denotes the type of the rank one torsion-free abelian group A_v . If $A = \bigoplus_{i \in I} A_i$ is a direct sum of rank one torsion-free abelian groups A_i , let $\Delta(A) = \sup \{n \mid \exists \text{ a triangle of size } n \text{ of groups chosen, possibly with repetitions, from } \{A_i \mid i \in I\}\}$, $\Delta'(A) = \sup \{n \mid \exists \text{ a triangle of size } n \text{ of groups chosen without repetition from } \{A_i \mid i \in I\}\}$. An abelian group $(G, +)$ is radical iff whenever $(R, +, \cdot)$ is a ring with $(R, +) \cong (G, +)$ there exists a positive integer n with $R^n = (0)$.

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$ be such that $\{T(A_i) \mid i \in I\}$ is an ordered set and $\Delta(A) < \infty$. Then A is radical.*

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$ be such that $\Delta'(A) = \infty$. Then A is not radical.*

THEOREM. *Let $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$ be such that $\Delta(A) < \infty$, $\Delta(B) < \infty$. Then if $\{T(A_i) \mid i \in I\} \cup \{T(B_j) \mid j \in J\}$ is an ordered set $A \oplus B$ is radical. A bound is given for the index of nilpotency of any multiplication on $A \oplus B$.*

1. Preliminaries. Several authors ([2], [3], [4], [5]) have studied the class of abelian groups $(A, +)$ which admit only a trivial ring structure; i.e., if $(R, +, \cdot)$ is a ring with $(R, +) \cong (A, +)$, then $R^2 = (0)$. These are called nil groups.

In [6] a larger class was introduced—abelian groups which admit only nilpotent multiplications. More precisely:

DEFINITION 1.1. An abelian group $(A, +)$ is a radical group iff whenever $(R, +, \cdot)$ is a ring with $(R, +) \cong (A, +)$, we have $R^n = 0$ for some positive integer n .

In [6], using the techniques of [1], the class of finite rank torsion free radical groups was studied, and it was shown that this class is closed under finite direct sums.

In this paper we study completely decomposable radical groups. We work toward the goals of characterizing such groups and of obtaining information on finite direct sums of such groups.

Throughout, the word “group” means torsion-free abelian group.

We let $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$ be arbitrary completely decompo-

sable groups; i.e., I, J are arbitrary index sets, $\text{rank } A_i = \text{rank } B_j = 1$ for all $i \in I, j \in J$.

The standard notions of height and type for torsion-free groups ([2], § 42) are used. If $g \in G$, then $h_g(g), T_g(g)$ denote the height vector and type respectively of g in G . The subscript is deleted when no confusion will result. Sums of height vectors and sums of types are defined in the obvious way. If $\text{rank } G = 1$, the type of G , $T(G)$, is $T(g)$ where $0 \neq g \in G$. A type is called nil iff it is of the form $\langle m_k \rangle$, where $0 < m_k < \infty$ for infinitely many k . ($\langle m_k \rangle$ denotes the equivalence class of the height vector $\langle m_k \rangle$.)

2. Triangles of rank one groups.

DEFINITION 2.1. A *triangle of size n* is a collection of groups of rank 1 indexed by all sequences of the form $(i, i+1, i+2, \dots, i+j)$ where $1 \leq i \leq n, 0 \leq j \leq n-i$, such that $T[A_{(i, i+1, \dots, i+j)}] + T[A_{(i+j+1, i+j+2, \dots, i+j+k)}] \leq T[A_{(i, i+1, \dots, i+j+k)}]$ for all i, j as above and k with $i+j+k \leq n$.

It is easiest to consider these triangles in terms of diagrams as in the following.

EXAMPLE. $n = 3$.

A triangle of size 3 looks like $A_1 \quad A_2 \quad A_3$
 $\quad \quad \quad A_{12} \quad A_{23}$
 $\quad \quad \quad \quad \quad A_{123}$

with $T(A_1) + T(A_2) \leq T(A_{12})$
 $T(A_2) + T(A_3) \leq T(A_{23})$
 $T(A_1) + T(A_{23}) \leq T(A_{123})$
 $T(A_{12}) + T(A_3) \leq T(A_{123})$.

Now let $A = \bigoplus_{i \in I} A_i$. Clearly, if $T(A_i)$ is non-nil for some i , then A is not radical ([2]). Since we are interested in completely decomposable radical groups, we assume in what follows that $T(A_i)$ is nil for all i .

It is well known [3] that such an A is a nil group iff, for all $i, j, k \in I$, $T(A_i) + T(A_j) \not\leq T(A_k)$ — in other words no triangle of size 2 can be formed using $\{A_i \mid i \in I\}$. The following theorems generalize this result by relating multiplications on A which yield nonzero products of length n to triangles of size n .

THEOREM 2.1. Let $\{A_i \mid i \in I\}$ be a collection of rank one groups such that $\{T(A_i) \mid i \in I\}$ is an ordered set. Let $*$ be a multiplication associative on $A = \bigoplus_{i \in I} A_i$ with $x_1 * x_2 * \dots * x_n \neq 0$ for some

$\{x_1, \dots, x_n\} \subseteq A$. Then a triangle of size n can be formed by choosing (possibly with repetitions) groups from $\{A_i \mid i \in I\}$.

Proof. Let $x_1 * x_2 * \dots * x_n \neq 0$ and choose, for each i , some $0 \neq a_i \in A_i$. Write each x_j as a finite rational combination $x_j = \sum_{\text{finite}} \alpha_{ji} a_i$. It follows immediately that $a_{i_1} * a_{i_2} * \dots * a_{i_n} \neq 0$ for some (not necessarily distinct) indices $\{i, \dots, i_n\} \subseteq I$. Reindex, setting $A_1 = A_{i_1}$, $A_2 = A_{i_2}$, \dots , $A_n = A_{i_n}$. The groups A_1, \dots, A_n form the top row of the triangle.

For $1 \leq i < i + j \leq n$ write $a_i * \dots * a_{i+j} = \sum \alpha_s a_s$, and let $A_{i \dots i+j}$ be any A_s such that $T(A_s) = T_A(a_i * \dots * a_{i+j})$. Such A_s always exists since $\{T(A_i) \mid i \in I\}$ is an ordered set. It is easy to check that the types of the rank one groups thus chosen satisfy the additive requirements of Definition 2.1.

The requirement that $\{T(A_i) \mid i \in I\}$ be an ordered set cannot be deleted in the above theorem. One can construct a completely decomposable group $A = \bigoplus_{i=1}^{11} A_i$ such that no triangle of size 4 can be formed by choosing groups from $\{A_i \mid 1 \leq i \leq 11\}$ but such that $A^4 \neq (0)$ under an appropriate multiplication. (The numbers 11 and 4 are the minimum rank and index for such an example.)

THEOREM 2.2. *Let $\{A_i \mid i \in I\}$ be a set of rank one groups such that a triangle of size n can be formed by choosing $n(n + 1)/2$ distinct groups from the set. Then there is a multiplication on $A = \bigoplus_{i \in I} A_i$ with a nonzero product of length n .*

Proof. Reindex the groups chosen from $\{A_i \mid i \in I\}$ as in Definition 2.1. For each index, u , choose $0 \neq a_u \in A_u$ such that for consecutive indices $r = i \dots i + j$, $s = i + j + 1 \dots i + j + k$, $i + j + k \leq n$, $h(a_r) + h(a_s) \leq h(a_{i \dots i+j+k}) = h(a_{rs})$. Such a choice is possible because of the type relationships in the triangle. Now define $a_r \cdot a_s = a_{rs}$ for all consecutive indices r, s . If r and s are not of the above form, define $a_r \cdot a_s = 0$. If A_i is not a group in the triangle, let $A_i A = A A_i = (0)$. These products define a unique associative ring structure on A . Moreover, in this ring $a_1 \cdot a_2 \cdot \dots \cdot a_n \neq 0$.

Note that the above method (or simple variations of it) cannot be used to obtain multiplication on $\bigoplus A_i$ from triangles formed by groups arbitrarily chosen from $\{A_i \mid i \in I\}$. If the same group is allowed in repeated positions, it seems impossible to overcome the consistency problems arising in the definitions of the products $a_r \cdot a_s$. However, we can obtain a nonassociative multiplication in this case.

The results of this section can be used to obtain information on completely decomposable radical groups and on direct sums of such

groups. First we prove theorem on the existence of subtriangles of a given triangle.

3. Subtriangles.

DEFINITION 3.1. Let $\{A_u\}$ be a triangle of size n , indexed as in Definition 2.1. For $1 \leq m \leq n$, a subtriangle of size m is an array of $m(m + 1)/2$ (not necessarily distinct) groups chosen from $\{A_u\}$ such that the resulting array, after reindexing, satisfies the requirements of Definition 2.1.

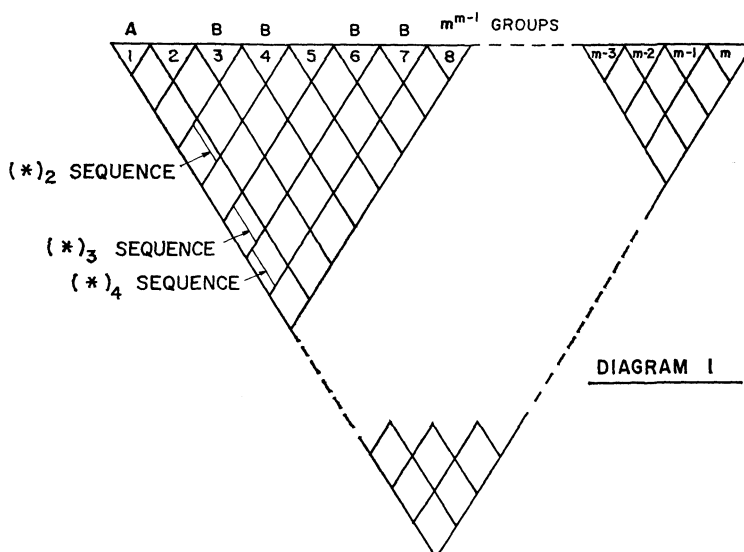
EXAMPLE. The array $A_1 \quad A_{34} \quad A_5$ is a subtriangle of any

$$\begin{array}{cc} A_{1234} & A_{345} \\ & A_{123456} \end{array}$$

triangle of size $n \geq 6$.

THEOREM 3.1. Let $A = \{A_i\}$, $B = \{B_i\}$ and let n be any positive integer. If m is any integer such that $m \geq f(n) \equiv \max \{3n - 1, n(n + 1)/2\}$, then any triangle of size m^m of groups from $A \cup B$ contains a subtriangle of size n of groups from A alone or from B alone.

Proof. Use induction on n . The result is easy for $n = 1$. Now consider a triangle of size m^m of groups from $A \cup B$, when $m \geq f(n)$, (n an integer not less than 2). As in Diagram 1, divide the first m^{m-1} rows into m disjoint subtriangles of size m^{m-1} . Since $m^{m-1} \geq$



$(m - 1)^{m-1}$ and $m - 1 \geq f(n - 1) = \max \{3(n - 1) - 1, (n - 1)n/2\}$, by the induction hypothesis, each of these disjoint triangle of size m^{m-1} contains a subtriangle of size $n - 1$ of groups from A alone or from B alone.

Case I. Both triangles of size $n - 1$ of groups from A and triangles of size $n - 1$ of groups from B appear as subtriangles of the triangles of size m^{m-1} . In this case, there occurs a subtriangle of size $n - 1$ of groups from one collection, say A , such that either to the right or to the left of this subtriangle there are n disjoint triangles of size m^{m-1} , each of which contains subtriangles of size $n - 1$ of groups from B . This is due to the fact that in any arrangement of $3n - 1$ objects which are of two types, one of the types must have n objects of the other type either to the right or to the left.

Without loss of generality, assume that the left-most triangle (1) contains a subtriangle of size $n - 1$ of groups from A , and that there are n triangles to the right which contain subtriangles of groups from B . The technique is to expand one of these subtriangles to a triangle of size n .

Let $G_{i_k, i_{k+1}, \dots, i_{k+j_k}}$ be the rightmost group in the k^{th} row of the subtriangle of groups from A contained in triangle 1, $1 \leq k \leq n - 1$ (see Diagram 1). If triangle r_k , $1 \leq k \leq n$, contains the k^{th} subtriangle to the right of triangle 1 of groups from B , consider, for all $2 \leq k \leq n$, the sequence of groups

$$(*)_k \quad G_{i_k, i_{k+1}, \dots, (r_k, -1)^{m^{m-1}+1}} G_{i_k, i_{k+1}, \dots, (r_{k-1})^{m-1}+2}, \dots, G_{i_k, i_{k+1}, \dots, r_k m^{m-1}} .$$

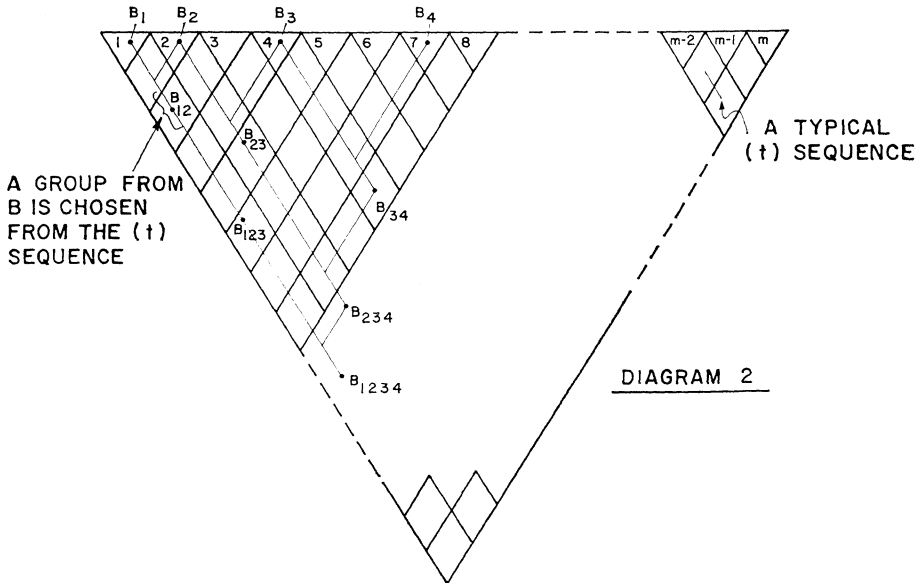
(Each of these sequences is represented by a diagonal in Diagram 1.)

Claim: Either $(*)_k$ contains a group from A or the subtriangle of groups from B in triangle r_k can be expanded to a triangle of size n of groups from B . Suppose $(*)_k$ contains no group from A . Let B^l be any group from B in triangle r_{k-1} . Now let B^l , $2 \leq l \leq n$, be the group in the sequence $(*)_k$ whose index ends in the same number as the index of the left-most group in the $l-1$ row of the subtriangle of groups from B in triangle r_k . By adjoining each B^l as the left-most group in the l^{th} row of the triangle, the subtriangle of groups from B in triangle r_k is expanded to a triangle of size n of groups from B . In this case we are done, so assume now that each sequence $(*)_k$, $2 \leq k \leq n$, contains a group from A -call it A^k . Let A^1 be any group from A in triangle r_1 (if none exists, we're done). Now by adjoining A^k as the right-most group in the k^{th} row of the A -subtriangle of triangle 1 we obtain an A -subtriangle of size n .

Case II. There exist only A -subtriangles of size $n - 1$ in the triangles 1, 2, \dots , m . If this is the case, then *any* subtriangle of

size m^{m-1} contained in the first m^{m-1} rows most contain only A -subtriangles of size $n - 1$. If there were a B -subtriangle of size $n - 1$, the techniques of Case I could be applied to construct an A - or B -subtriangle of size n .

Now consider any diagonal sequence of the form $(+): G_{i, \dots, i+j}, G_{i, \dots, i+j+1}, \dots, G_{i, \dots, i+j+m^{m-1}-1}$ where $2 \leq i \leq m^m - m^{m-1} - n - 1, n \leq j \leq m^m - m^{m-1} - i$. These $(+)$ sequences are illustrated in Diagram 2. If all of the groups in $(+)$ are in A then consider the A -subtriangle of size $n - 1$ contained in the triangle of size m^{m-1} whose top row is $G_{i+j}, G_{i+j+1}, \dots, G_{i+j+m^{m-1}-1}$. (Refer to Diagram 2.) Again using the techniques of Case I, this A -subtriangle can be expanded to a triangle of size n using the sequence $(+)$. Therefore, without loss of generality, assume that every sequence $(+)$ contains a group from B . This assumption can be used to build a B -subtriangle of size n .



Choose from triangle 1 any group B_1 from B . Say $B_1 = G_{i, \dots, i+j}$ where $2 \leq i \leq m^{m-1}$ and $0 \leq j \leq m^{m-1} - i$. If such a B_1 does not exist, we are done, for triangle 1 is then an A -subtriangle of size $m^{m-1} \geq n$. Similarly choose from triangle 2 a group $B_2 = G_{i', \dots, i'+j'}$ from B , where $m^{m-1} < i' \leq 2m^{m-1}, 0 \leq j' \leq 2m^{m-1} - i'$. Now choose a group B_{12} from B from the sequence $G_{i, i+1, \dots, i'+j'}, \dots, G_{i, i+1, \dots, i'+j'+m^{m-1}-1}$. (See Diagram 2.) This is a $(+)$ sequence unless $i' + j' - i < n$. In this case replace $i' + j'$ with $i + n$ in the above sequence. Now let B_3 be any group from B found in triangle 4. The group B_{23} is obtained from the $(+)$ sequence which begins where the diagonals from B_2 and B_3 intersect (see Diagram 2). The group B_{123} is obtained

from the $(+)$ sequence which begins where the diagonals from B_{12} and B_{23} intersect. The group B_i is then chosen from triangle 7 and the process continued. Since each new group requires a new $(+)$ sequence, it is not difficult to see that $n(n+1)/2$ $(+)$ sequences are needed in this process. Hence the requirement $m \geq n(n+1)/2$. Thus a B -triangle of size n can be constructed and the proof is complete.

REMARK. The number $f(n) = \max \{3n - 1, n(n+1)/2\}$ is probably much too large as is the triangle size m^m used in the proof. These in fact could be lowered slightly, but are retained in the interest of a less complicated proof.

4. Applications. In this section, the results of the previous sections are consolidated to give some information on when completely decomposable groups are radical. We begin with

DEFINITION 4.1. Let $A = \bigoplus_{i \in I} A_i$ be any completely decomposable group, and assume $T(A_i)$ is nil for all $i \in I$. Let $\Delta(A) = \sup \{n \mid \exists \text{ a triangle of size } n \text{ chosen, possibly with repetitions, from } \{A_i \mid i \in I\}\}$. Let $\Delta'(A) = \sup \{n \mid \exists \text{ a triangle of size } n \text{ chosen without repetition from } \{A_i \mid i \in I\}\}$.

THEOREM 4.1. Let $A = \bigoplus_{i \in I} A_i$ be as above. If $\{T(A_i) \mid i \in I\}$ is an ordered set and $\Delta(A)$ is finite, then A is radical.

Proof. If A is not radical, then for any $n \geq 0$, there exists a multiplication on A which yields a nonzero product of length greater than n . By Theorem 2.1, $\Delta(A) > n$. This contradicts $\Delta(A)$ finite.

A sort of converse to Theorem 4.1 is provided by

THEOREM 4.2. If $A = \bigoplus_{i \in I} A_i$ is completely decomposable with $\Delta'(A) = \infty$ then A is not radical.

Proof. If $\Delta'(A)$ is infinite, we show there exists a multiplication on A which produces nonzero products of arbitrary length. Let n be any positive integer. Since $\Delta'(A)$ is infinite, there exists a triangle T_1 of size $m > n$ consisting of distinct groups from $\{A_i\}$. Now let $k = f(m+1)$ and form a triangle of size $l \geq k^k$ consisting of distinct groups from $\{A_i\}$. By applying Theorem 3.1 to the disjoint collections of groups T_1 and $\{A_i\} \setminus T_1$, we obtain a subtriangle T_2 of size $m+1$ consisting of distinct groups from $\{A_i\} \setminus T_1$. Continuing this process we obtain a disjoint sequence of collections of groups, T_1, T_2, T_3, \dots , such that distinct groups in T_j may be used to form a triangle of

size at least $m + j - 1$. Thus by Theorem 2.1 a ring may be defined on the direct sum of the groups in T_j which contains a nonzero product of length $m + j - 1$. The direct sum of these rings is then non-nilpotent and is a group direct summand of A . Therefore A cannot be radical.

The final result concerns the direct sum of two completely decomposable radical groups.

THEOREM 4.3. *Let $A = \bigoplus_{i \in I} A_i$, $B = \bigoplus_{j \in J} B_j$ be completely decomposable radical groups such that $\Delta(A) = n$ and $\Delta(B) = m$ are finite. Then if $\{T(A_i) \mid i \in I\} \cup \{T(B_j) \mid j \in J\}$ is an ordered set, $A \oplus B$ is radical. Moreover, for any multiplication on $A \oplus B$, $(A \oplus B)^{k^k} = 0$ for $k = \max\{f(n + 1), f(m + 1)\}$.*

Proof. If $A \oplus B$ is not radical, then $\Delta(A \oplus B)$ is infinite by Theorem 4.1. But Theorem 3.1 then implies that either $\Delta(A)$ or $\Delta(B)$ is infinite, a contradiction. In fact, if $(A \oplus B)^{k^k} \neq (0)$ for some multiplication on $A \oplus B$, then $\Delta(A \oplus B) \geq k^k$. But by Theorem 3.1, this implies that $\Delta(A) \geq n + 1$ or $\Delta(B) \geq m + 1$, a contradiction.

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