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AN ELEMENTARY PROOF OF THE LIFTING THEOREM

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An elementary proof is given of the lifting theorem for a complete totally finite measure space, which does not use the martingale theorem or Vitali differentiation.

Introduction. In this paper we give a proof of the lifting theorem for a complete totally finite measure space, which involves only elementary properties of measure. The complicated isomorphism theorem of Maharam's original proof [4] is avoided. On the other hand, we do not use the concepts of martingale or of Vitali differentiation ([1] [2] [3] [5]). In fact, the entire construction takes place in the σ-field of measurable sets, without passing to the algebra of essentially bounded measurable functions. We feel this makes it easier to see what is involved.

Throughout what follows:

$(S, \mathscr{M}, \mu)$ is a complete measure space with $\mu(S) < \infty$;

$\mathscr{N} = \{A \in \mathscr{M}: \mu(A) = 0\}$;

N is the set of nonnegative integers;

For subsets A, B of S,

$$\begin{aligned} AB &= A \cap B ; \\ A \backslash B &= \{s \in A: s \notin B\} ; \\ A^c &= S \backslash A ; \\ A \Delta B &= AB^c \cup BA^c ; \\ A &\doteq B \text{ iff } A, B \in \mathscr{M} \text{ and } \mu(A \Delta B) = 0 . \end{aligned}$$

For a family $\mathscr{K}$ of subsets of S,

$$\bigcup \mathscr{K} = \bigcup_{E \in \mathscr{K}} E .$$

1. DEFINITIONS. For any field $\mathscr{A} \subset \mathscr{M}$,

(1) d is a (lower) density on $\mathscr{A}$ iff d is a mapping on $\mathscr{A}$ to $\mathscr{A}$ such that, for A, B in $\mathscr{A}$,

(i) $d(A) \doteq A$;

(ii) $A \doteq B$ implies $d(A) = d(B)$;

(iii) $d(\varnothing) = \varnothing$, $d(S) = S$;

(iv) $d(AB) = d(A)d(B)$.

(2) l is a lifting on $\mathscr{A}$ iff l is a density on $\mathscr{A}$ such that

(v) $l(A^c) = l(A)^c$, for A in $\mathscr{A}$.

For a detailed study of liftings and their applications, we refer to A. and C. Ionescu Tulcea [3].

2. REMARKS. Let l be a lifting on the σ-field $\mathscr{A} \subset \mathscr{M}$ and $\mathscr{F} = l[\mathscr{A}]$. Then:

(1) $\mathscr{F}$ is a field in S.

(2) $\mathscr{F} \subset \{E \in \mathscr{A}: 0 < \mu(E) < \mu(S)\} \cup \{\varnothing, S\}$.

(3) If, for each n in N, $E_n \in \mathscr{F}$, and $A = \bigcup_n E_n$, then $l(A) \supset A$. (Indeed, for each n, $E_n \backslash l(A) \subset A \backslash l(A) \doteq \varnothing$, so $E_n \backslash l(A) = \varnothing$, by (2).)

3. THEOREM. *If d is a density on a field $\mathscr{A}$ with $\mathscr{N} \subset \mathscr{A} \subset \mathscr{M}$, then there exists a lifting l on $\mathscr{A}$, with*

$$(*) \qquad d(A) \subset l(A) \subset d(A^c)^c \,, \qquad \text{for } A \text{ in } \mathscr{A} \,.$$

Proof. For each filterbase $\mathscr{B} \subset \mathscr{A}$, let $\hat{\mathscr{B}}$ denote an ultrafilter containing $\mathscr{B}$. We recall that for subsets A, B of S,

(a) $A \in \hat{\mathscr{B}}$ iff $A^c \notin \hat{\mathscr{B}}$, and

(b) $A \cap B \in \hat{\mathscr{B}}$ iff $A \in \hat{\mathscr{B}}$ and $B \in \hat{\mathscr{B}}$.

For each s in S, let

$$\mathscr{F}(s) = \{A \in \mathscr{A}: s \in d(A)\} \,.$$

Since d is a density, $\mathscr{F}(s)$ is a filterbase. Put

$$l(A) = \{s \in S: A \in \hat{\mathscr{F}}(s)\} \,, \qquad \text{for } A \text{ in } \mathscr{A} \,.$$

By the properties (a), (b) of an ultrafilter, for A, B in $\mathscr{A}$, we have (v) $l(A^c) = l(A)^c$ and (iv) $l(AB) = l(A)l(B)$. Moreover, if $s \in d(A)$, then $A \in \mathscr{F}(s) \subset \hat{\mathscr{F}}(s)$, so that $s \in l(A)$. Hence, $d(A) \subset l(A)$. Similarly $d(A^c) \subset l(A^c)$. Using (v) we find that (*) holds. Since $d(A) \doteq A \doteq d(A^c)^c$, we have (i) $l(A) \doteq A$. If $N \doteq \varnothing$, then $d(N) = d(\varnothing) = \varnothing$ and $d(N^c) = d(S) = S$, so that, by (*), $l(N) = \varnothing$. Hence, (iii) $l(\varnothing) = \varnothing$, $l(S) = S$ and (ii) if $A \doteq B$, then $l(A) \Delta l(B) = l(A \Delta B) = \varnothing$, so that $l(A) = l(B)$. This completes the proof.

The proof of the following theorem usually uses martingales or Vitali differentiation. We use neither. However, the reader familiar with Sion [5] will recognize the connection with his method. (See Remark 7 below.)

4. THEOREM. *Suppose that, for each n in N, $\mathscr{A}_n$ is a σ-field with $\mathscr{N} \subset \mathscr{A}_n \subset \mathscr{A}_{n+1} \subset \mathscr{M}$ and l_n is a lifting on $\mathscr{A}_n$ with $l_n = l_{n+1} \mid \mathscr{A}_n$. Put $\mathscr{A} = \sigma$-field $(\bigcup_n \mathscr{A}_n)$. Then there is a lifting l on $\mathscr{A}$ with $l_n = l \mid \mathscr{A}_n$, for each n in N.*

Proof. The result will follow immediately from Theorem 3 if we can construct a density d on $\mathscr{A}$ with $d(A) = l_n(A)$ for A in $\mathscr{A}_n$. To this end, for each k in N, let $\mathscr{F}_k$ denote $l_k[\mathscr{A}_k]$. For each A in $\mathscr{A}$, k in N, and $r < 1$, put

$$\mathscr{D}(A; k, r) = \{E \in \mathscr{F}_k: \mu(AF) \geqq r\mu(F), \text{ whenever } E \supset F \in \mathscr{F}_k\} ,$$
$$d(A; k, r) = \bigcup \mathscr{D}(A; k, r) , \text{ and}$$
$$d(A) = \bigcap_{r<1} \bigcup_{n \in N} \bigcap_{k \geqq n} d(A; k, r) .$$

We will show that d is a suitable density function on $\mathscr{A}$.

For fixed A, r, and k, let $\mathscr{K}$ be a maximal disjoint subfamily of $\mathscr{D}(A; k, r)$. Then $\mathscr{K}$ is countable. Put $B = l_k(\cup \mathscr{K})$. Clearly, $B \in \mathscr{D}(A; k, r)$. Moreover, if $E \in \mathscr{D}(A; k, r)$, $E \backslash B = \varnothing$, by Remark 2(3) and the maximality of $\mathscr{K}$. This shows that $d(A; k, r) = B$ is the largest element of $\mathscr{D}(A; k, r)$. In particular, $d(A; k, r) \in \mathscr{F}_k \subset \mathscr{A}$. If $r < s < 1$, we have $d(A; k, r) \supset d(A; k, s)$, so we need only consider rational r. Since $\mathscr{A}$ is a σ-field, we conclude that $d(A) \in \mathscr{A}$.

There is no difficulty in showing that $A \doteq B \in \mathscr{A}$ implies $d(A) = d(B)$, or that $d(A) = l_n(A)$, for A in $\mathscr{A}_n$. In particular, $d(\varnothing) = \varnothing$ and $d(S) = S$. We have left to check conditions (i) and (iv) of the definition of a density.

To check condition (iv), let $A, B \in \mathscr{A}$, $k \in N$, $r < 1$. For each F in $\mathscr{F}_k$ contained in $d(A; k, (r+1)/2) \cap d(B; k, (r+1)/2)$, we have

$$\begin{aligned}\mu(ABF) &= \mu(AF) + \mu(BF) - \mu((A \cup B)F) \\ &\geqq ((r+1)/2)\mu(F) + ((r+1)/2)\mu(F) - \mu(F) \\ &= r\mu(F) .\end{aligned}$$

Hence, $d(A; k, (r+1)/2) \cap d(B; k, (r+1)/2) \subset d(AB; k, r)$. By direct computation, this yields $d(A)d(B) \subset d(AB)$. On the other hand, for each k and r, $d(AB; k, r) \subset d(A; k, r) \cap d(B; k, r)$, so that $d(AB) \subset d(A)d(B)$, establishing (iv).

To verify condition (i), let $A \in \mathscr{A}$ and put

$$d'(A) = \bigcup_{0<r<1} \bigcap_{n \in N} \bigcup_{k \geqq n} d(A; k, r) .$$

We will show that

(a) $d'(A)A^c \doteq \varnothing$,

(b) $Ad'(A^c) \doteq \varnothing$, and

(c) $d'(A^c) \supset d(A)^c$, $d'(A) \supset d(A)$,

from which we get

$$d(A) \Delta A = d(A)A^c \cup Ad(A)^c \subset d'(A)A^c \cup Ad'(A^c) \doteq \varnothing ,$$

as required.

Fix r in $(0, 1)$ and write $D_k = d(A; k, r)$, for k in N. Since $D_k \in \mathscr{D}(A; k, r)$, we have for each B in $\mathscr{A}_k$,

$$\mu(ABD_k) = \mu(Al_k(B)D_k) \geqq r\mu(l_k(B)D_k) = r\mu(BD_k)\,.$$

Suppose $B \in \bigcup_n \mathscr{A}_n$. Then there exists an n in N such that $B \in \mathscr{A}_n$. For $m \geqq n$, $\mathscr{A}_m \supset \mathscr{A}_n$, and putting $C_m = BD_m \backslash \bigcup_{n \leqq k < m} D_k$, we have

$$\begin{aligned}\mu(AB\bigcup_{k\geqq n} D_k) &= \sum_{m\geqq n} \mu(AC_m) \\ &\geqq \sum_{m\geqq n} r\mu(C_m) \\ &= r\mu(B\bigcup_{k\geqq n} D_k)\,.\end{aligned}$$

Taking intersections over n we have

$$\mu(AB\bigcap_n \bigcup_{k\geqq n} D_k) \geqq r\mu(B\bigcap_n \bigcup_{k\geqq n} D_k)\,.$$

By considering monotone sequences of such B we see that this holds for all B in $\mathscr{A}$, the σ-field generated by the field $\bigcup_n \mathscr{A}_n$. In particular, putting $B = A^c$ we have $0 \geqq r\mu(A^c \bigcap_n \bigcup_{k\geqq n} D_k)$. But $r > 0$, so $\mu(A^c \bigcap_n \bigcup_{k\geqq n} D_k) = 0$. Taking the union over rational r in $(0, 1)$ we have $A^c d'(A) \doteq \varnothing$. This proves (a). Replacing A by A^c we have (b).

To prove (c) we let $k \in N$, $0 < r < 1$ and show

$$d(A; k, r)^c \subset d(A^c; k, 1 - r)\,.$$

To this end suppose $\varnothing \neq E \in \mathscr{F}_k$ and $E \subset d(A; k, r)^c$. Then $E \notin \mathscr{D}(A; k, r)$, so there exists F in $\mathscr{F}_k$ contained in E with $\mu(AF) < r\mu(F)$. Let $\mathscr{K}$ be a maximal disjoint collection of such F. By Remark 2(3) and maximality of $\mathscr{K}$ we have $E \backslash l_k(\bigcup \mathscr{K}) = \varnothing$, so $E = l_k(\bigcup \mathscr{K})$. Moreover, $\mu(AE) = \sum_{F\in\mathscr{K}} \mu(AF) \leqq \sum_{F\in\mathscr{K}} r\mu(F) = r\mu(E)$. In other words, $\mu(A^cE) \geqq (1 - r)\mu(E)$. This shows that $d(A; k, r)^c \in \mathscr{D}(A^c; k, 1 - r)$, so $d(A; k, r)^c \subset d(A^c; k, 1 - r)$. Hence,

$$\begin{aligned}d(A)^c &= \bigcup_{r\in(0,1)} \bigcap_n \bigcup_{k\geqq n} d(A; k, r)^c \\ &\subset \bigcup_{r\in(0,1)} \bigcap_n \bigcup_{k\geqq n} d(A^c; k, 1 - r) \\ &= d'(A^c)\,.\end{aligned}$$

Since it is clear that $d(A) \subset d'(A)$, this proves (c) and completes the proof of the theorem.

To prove the lifting theorem, we need one more lemma, due to A. and C. Ionescu Tulcea [2]. For completeness, we include a proof here.

5. LEMMA. *Let $\mathscr{A}$ be a σ-field with $\mathscr{N} \subset \mathscr{A} \subset \mathscr{M}$, l a lifting*

on $\mathscr{A}$. If $A \in \mathscr{M} \backslash \mathscr{A}$ and $\mathscr{A}' = $ field $(\mathscr{A} \cup \{A\})$, then there exists a lifting on $\mathscr{A}'$ extending l.

Proof. Let $\mathscr{F} = l[\mathscr{A}]$, $\mathscr{E} = \{E \in \mathscr{F}: \mu(EA^c) = 0\}$. Let $\mathscr{K}$ be a maximal disjoint subfamily of $\mathscr{E}$ and let $A_1 = l(\bigcup \mathscr{K})$. Then $A_1 \in \mathscr{E}$ and, by maximality of $\mathscr{K}$ and Remark 2(3), $E \backslash A_1 = \varnothing$, for all E in $\mathscr{E}$, so that A_1 is the largest element of $\mathscr{E}$. Similarly, let A_2 be the largest E in $\mathscr{F}$ with $\mu(EA) = 0$. Put $\bar{A} = (A \cup A_1) \backslash A_2$. Then $\bar{A} \doteq A$. (Indeed, $\bar{A} \varDelta A \subset A_1 A^c \cup A_2^c A \doteq \varnothing$.) Thus, $\mathscr{A}' =$ field $(\mathscr{A} \cup \{\bar{A}\})$ $(= \{(C\bar{A} \cup D\bar{A}^c: C, D \in \mathscr{A}\})$. For E, F in $\mathscr{F}$,

(a) $E\bar{A} \doteq F\bar{A}$ implies $E\bar{A} = F\bar{A}$, and

(b) $E\bar{A}^c \doteq F\bar{A}^c$ implies $E\bar{A}^c = F\bar{A}^c$.

Indeed, $E\bar{A} \doteq F\bar{A}$ implies $\mu((E \varDelta F)A) = \mu((E \varDelta F)\bar{A}) = 0$, so that, by definition of A_2, $E \varDelta F \subset A_2 \subset \bar{A}^c$. Thus, $(E \varDelta F)\bar{A} = \varnothing$, so $E\bar{A} = F\bar{A}$. The proof of (b) is similar.

Now define l' on $\mathscr{A}'$ by

$$l'(C\bar{A} \cup D\bar{A}^c) = l(C)\bar{A} \cup l(D)\bar{A}^c, \quad \text{for } C, D \text{ in } \mathscr{A}.$$

Using (a) and (b) we see that l' is well-defined and that for M_1, M_2 in $\mathscr{A}'$, $M_1 \doteq M_2$ implies $l'(M_1) = l'(M_2)$. The other properties of a lifting are easily verified. Moreover, for C in $\mathscr{A}$, $l'(C) = l(C)\bar{A} \cup l(C)\bar{A}^c = l(C)$, so l' extends l.

We can now prove *the lifting theorem*:

6. THEOREM. *Let $(S, \mathscr{M}, \mu)$ be a complete measure space with $\mu(S) < \infty$. Then, there exists a lifting on $\mathscr{M}$.*

Proof. Let $\mathscr{H}$ be the set of pairs $(\mathscr{A}, l)$ where $\mathscr{A}$ is a σ-field with $\mathscr{N} \subset \mathscr{A} \subset \mathscr{M}$ and l is a lifting on $\mathscr{A}$, with the ordering: $(\mathscr{A}, l) \leqq (\mathscr{A}', l')$ iff $\mathscr{A} \subset \mathscr{A}'$ and $l = l' \mid \mathscr{A}$. We show that $\mathscr{H}$ has a maximal element. Indeed, suppose $\mathscr{H}' = \{(\mathscr{A}_i, l_i): i \in I\}$ is a totally ordered subfamily of $\mathscr{H}$. We distinguish two cases:

(a) If $\mathscr{H}'$ has no countable cofinal subfamily, put $\mathscr{A} = \bigcup_{i \in I} \mathscr{A}_i$ and $l(A) = l_i(A)$, for A in $\mathscr{A}_i$, i in I. Then $(\mathscr{A}, l)$ is an upper bound for $\mathscr{H}'$ in $\mathscr{H}$.

(b) If $\mathscr{H}'$ has a countable cofinal subfamily $\mathscr{H}'' = \{(\mathscr{A}_{i_n}, l_{i_n}): n \in N\}$, then by Theorem 4, $\mathscr{H}''$ (and hence $\mathscr{H}'$) has an upper bound in $\mathscr{H}$. By Zorn's lemma, we conclude that $\mathscr{H}$ has a maximal element, $(\mathscr{A}, l)$.

By Lemma 3, and maximality, $\mathscr{A} = \mathscr{M}$, and the theorem is proved.

7. REMARKS.

(1) To see the relationship of our method to that of Sion [5],

for each k in N and s in S, let $\hat{\mathscr{F}}_k(s) = \{F \in \mathscr{F}_k : s \in F\}$, directed downward by inclusion. Then,

$$d(A; k, r) = l_k\left(\left\{s \in S: \varliminf_{F \in \hat{\mathscr{F}}_k(s)} \frac{\mu(AF)}{\mu(F)} \geqq r\right\}\right).$$

(One inclusion is obvious, the other follows from Sion's Theorem 2′.)

(2) As several authors have pointed out (see, for example, Sion [5], and for more references, Sion [6]), liftings provide very special Vitali differentiation system, even when no others are available. (If l is a lifting on $\mathscr{M}$, such a system is obtained by assigning to each s in S, $\{F: s \in F \in l[\mathscr{M}]\}$, directed downward by inclusion.) Apart from our desire for an elementary proof, this was our main motivation in looking for a construction of a lifting without using differentiation concepts.

(3) *Added in proof.* S. Graf [On the existence of strong liftings in second countable topological spaces, (to appear)] has noticed that one may change the word "lifting" to "density" in the statement of Theorem 4. The proof is essentially contained in our proof. Graf has independently obtained a proof of this result (using Radon-Nikodým derivatives).

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