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A FACTORIZATION THEOREM FOR p-CONSTRAINED GROUPS

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Suppose that G is a finite p-constrained group. For some prime $p \geqq 5$ let S be a Sylow p-subgroup. Assume that G admits a group of automorphisms A such that $(|A|, |G|) = 1$ and the fixed point subgroup of A does not involve PSL(2, p). In this paper it is shown that under these conditions

$$G = O_{p'}(G)N(Z(J(S))) .$$

Thompson proved in [8] that if G is a strongly p-solvable group and $O_{p'}(G) = 1$, then $G = N(J(S))C(Z(S))$, where S is a Sylow p-subgroup of G. Since his paper several other stronger results of this type have been proved by Glauberman [1], [2]. Specifically he proved his ZJ-theorem which states that if G is p-constrained and p-stable then $G = O_{p'}(G)N(Z(J(S)))$. This implies Thompson's conclusion by the Frattini argument. Recently Glauberman has proved that

$$G = N(J(S))C(Z(S))$$

for all p, provided that G is p-solvable and admits a group of automorphisms A such that $(|G|, |A|) = 1$ and A has no fixed points of order p.

In this paper our goal is a theorem related to these results.

THEOREM A. *Let G be a p-constrained group with $p \geqq 5$ and S a Sylow p-subgroup of G. Suppose that G admits a group of automorphisms A such that $(|A|, |G|) = 1$ and the fixed point subgroup of A does not involve* PSL(2, p). *Then $G = O_{p'}(G)N(Z(J(S)))$.*

Using Glauberman's ZJ-theorem, Theorem A is a corollary of Theorem B.

THEOREM B. *Let G be a p-constrained group with $p \geqq 5$. Suppose that A is a group of automorphisms of G such that $(|G|, |A|) = 1$ and that the fixed point subgroup of A does not involve* PSL(2, p). *Then G is p-stable.*

All the groups in this paper are finite. The notation, except for the definition of p-stability, is standard and can be found in [3]. If P is a p-group $J(P) = \langle A \subseteqq P \mid A \text{ is abelian and of maximal order} \rangle$. For simplicity we will write $Z(J(P)) = ZJ(P)$. If K is a group, we say G involves K if a section of G is isomorphic to K.

1. Assumed results and definitions.

DEFINITION 1.1. Let G be a group with $O_p(G) \supsetneqq 1$. Let S be a Sylow p-subgroup of G and set $P = S \cap O_{p',p}(G)$. G is p-constrained is $C_G(P) \subseteqq O_{p',p}(G)$.

DEFINITION 1.2. Let G be a group and suppose that S is a Sylow p-subgroup. G is p-stable if for any $R \subseteqq S$ such that $RO_{p'}(G) \trianglelefteq G$ and for any $A \subseteqq N_S(R)$ with the property that $[R, A, A] = 1$, we have

$$AC(R)/C(R) \subseteqq O_p(N(R)/C(R)) .$$

This definition of p-stability is taken from Gorenstein-Walter [4]. It is weaker than the definition given in Gorenstein [3]. However this definition is sufficient for Glauberman's ZJ-theorem as a check of the proof [3] will indicate.

The principal tools of this paper are two theorems of Thompson. We state these for want of an available reference.

DEFINITION 1.3. Let p be a prime. We say that (G, M) is a quadratic pair for p if G is a group and

(i) M is an irreducible F_pG-module,

(ii) G acts faithfully on M, and

(iii) $G = \langle Q \rangle$, where $Q = \{g \in G - \{1\} \mid M(g-1)^2 = 0\}$.

THEOREM 1.4 (Central Product Theorem, Thompson). *Suppose that (G, M) is a quadratic pair for p and $p \geqq 5$. Then for some natural number n, the following hold.*

(i) $G = G_1G_2 \cdots G_n$, $[G_j, G_j] = 1$ $(1 \leqq i < j \leqq n)$,

(ii) *$G_i/Z(G_i)$ is simple and (G_i, M_i) is a quadratic pair for $i = 1, 2, \cdots, n$,*

(iii) $Q = \bigcup_{i=1}^{n} (Q \cap G_i)$,

(iv) *M and $M_1 \otimes \cdots \otimes M_n$ are isomorphic F_pG-modules.*

THEOREM 1.5 (Thompson). *Suppose (G, M) is a quadratic pair for $p \geqq 5$, and $\bar{G} = G/Z(G)$ is simple. Then for some natural number e and $q = p^e$, $\bar{G}$ is isomorphic to one of the following groups:*

$$A_n(q),\ B_n(q),\ C_n(q),\ D_n(q),\ G_2(q),\ F_4(q),\ E_6(q)\ ,$$
$$E_7(q),\ {}^2A_n(q),\ {}^2D_n(q),\ {}^3D_4(q),\ {}^2E_6(q)\ .$$

Any group from the above list will be called a simple group of quadratic type.

2. p-Constrained groups which are not p-stable.

LEMMA 2.1. *Suppose that G acts on a vector space V over $GF(p)$ and assume that G is generated by elements which act quadratically. If G is not a p-group, then G contains a normal subgroup H such that G/H is a simple group of quadratic type.*

Proof. Let W be a nontrivial composition factor of V under G. Then $\bar{G} = G/C_G(W)$ acts faithfully and irreducibly on W. Since $(\bar{G}, W)$ is a quadratic pair, Theorem 1.5 implies the result.

THEOREM 2.2. *A p-constrained group G with $O_{p'}(G) = 1$ which is not p-stable has a composition factor of quadratic type.*

Proof. Since G is not p-*stable there exists* $R \trianglelefteq G$, $R \subseteqq S$ a Sylow p-subgroup, $A \subseteqq N_S(R)$ with the property that $[R, A, A] = 1$, and $AC(R)/C(R) \not\subset O_p(N(R)/C(R))$. Since $R \trianglelefteq G$, $\Phi(R) \trianglelefteq G$. Consider $\bar{G} = G = G/\Phi(R)$. $\bar{G}$ satisfies the hypotheses of the theorem so by induction $\Phi(R) = 1$ and R is elementary abelian. Let $L = C(R)\langle x \mid [R, x, x] = 1\rangle$. By assumption $C(R) \subset L \not\subset O_p(G \bmod C(R))$ and by definition $L \trianglelefteq G$. Lemma 2.1 implies that there exists $K \trianglelefteq L$ such that L/K is simple of quadratic type.

3. Automorphisms of semisimple groups.

DEFINITION 3.1. A semisimple group is the direct product of simple groups. The simple factors are called the components.

LEMMA 3.2. *Suppose that G is a semisimple group with no abelian components. If $K \trianglelefteq G$ and K is simple, then K is equal to one of the components.*

Proof. Standard result, [5].

We prove now a basic lemma about automorphisms of a semisimple group with isomorphic nonabelian components. Let G be the direct product of t copies of the simple group H. Define $H_i = \{(1, 1, \cdots, x_i, \cdots, 1) \mid x \in H\}$ for $1 \leqq i \leqq t$. Then G is the direct product of the H_i's. Two subgroups of $\mathrm{Aut}(G)$ are readily available. The first is $L = \Pi \mathrm{Aut}(H_i)$ where the action is the natural one. The second is K, the group of permutations of the H_i's.

LEMMA 3.3. Aut(G) *permutes the set* $\{H_i\}$.

Proof. This is an immediate consequence of Lemma 3.2.

THEOREM 3.4. *Suppose that* G *is the direct product of* t *copies of the simple nonabelian group* H. *Define* K *and* L *as above. Then* $L \trianglelefteq \mathrm{Aut}(G)$, $L \cap K = 1$, $LK = \mathrm{Aut}(G)$ *and* $K \cong \mathrm{Sym}(t)$.

Proof. By Lemma 3.3 we know that every $\sigma \in \mathrm{Aut}(G)$ permutes the set $\{H_1, \cdots, H_t\}$. In particular there is a homomorphism

$$\Psi\colon \mathrm{Aut}(G) \longrightarrow \mathrm{Sym}(t) .$$

Clearly $L = \ker(\Psi)$, and $K \cong \Psi(K) \cong \mathrm{Sym}(t)$. The result follows.

4. **Automorphisms of a group with a quadratic factor.** The main result of this section is the following.

THEOREM 4.1. *Let* G *be a group with a composition factor of quadratic type. If* $A \subseteq \mathrm{Aut}(G)$ *and* $(|A|, |G|) = 1$, *then the fixed point subgroup of* A *involves* PSL$(2, p)$.

We proceed via a series of lemmas.

LEMMA 4.2. *Suppose* H *is a simple nonabelian group of quadratic type with respect to the prime* $p \geqq 5$. *If* $A \subseteq \mathrm{Aut}(H)$ *and* $(|A|, |H|) = 1$, *then the fixed point subgroup of* A *involves* PSL$(2, p)$.

Proof. By the main result from Steinberg [7], $\mathrm{Aut}(H) = M$ contains a normal series $H \subseteq \tilde{H} \subseteq \tilde{M} \subseteq M$. Furthermore by the same theorem there are groups F and E, F the field automorphisms and E the graph automorphisms, such that $M = \tilde{H}EF$. Since every simple group of quadratic type is a finite Chevalley group they must all involve PSL$(2, p)$. Thus $(|A|, |H|) = 1$ and $p \geqq 5$ imply that $(|A|, 2.3, p) = 1$. By order considerations Steinberg's theorem implies that $A \cap \tilde{H} = 1$ and $A \subseteq \tilde{M}$, where $\tilde{M} = \tilde{H}F$.

Now let $N = F \cap \tilde{H}A$. Then

$$\tilde{H}N = \tilde{H}(F \cap \tilde{H}A) = \tilde{H}F \cap \tilde{H}A = M \cap \tilde{H}A = \tilde{H}A .$$

Since $A \cap \tilde{H} = N \cap \tilde{H} = 1$, $(|\tilde{H}|, |A|) = 1$ and N is solvable; the Schur-Zassenhaus theorem implies that A is conjugate to N in M. If we prove the result for a conjugate of A it is certainly true for A. Therefore we may assume that $A = N \subseteq F$.

Now the field automorphisms have a fixed point subgroup which contains the corresponding Chevalley group over the prime field $GF(p)$. In particular this subgroup involves PSL (2, p). Since $A \subseteqq F$, certainly the fixed point subgroup of A involves PSL (2, p) as desired.

LEMMA 4.3. *Let G be the direct product of t copies of H, a simple group of quadratic type with respect to the prime $p \geqq 5$. Suppose that $A \subseteqq$ Aut (G) and that $(|A|, |G|) = 1$. Then the fixed point subgroup of A involves* PSL (2, p).

Proof. We adopt the notation presented in §3. Let A^* be the subgroup of A stabilizing H_1. Then $A^*/C_{A^*}(H_1)$ is a subgroup of Aut $(H_1) \cong$ Aut (H). Therefore by Lemma 4.2 there exists subgroups U_1 and V_1 contained in the fixed point subgroup of A^* on H_1 such that $V_1/U_1 \cong$ PSL (2, p).

Now let T be a transversal of A^* in A. Suppose that t and u are distinct elements of T. By Lemma 3.3 $H_1^t = H_i$ and $H_1^u = H_j$ for some i and j. If $i = j$, then $H_1^t = H_1^u$ and tu^{-1} stabilizes H_1 contrary to assumption. Thus $i \neq j$ and $[H_1^t, H_1^u] = 1$. This fact implies that the set $V = \{\prod_{t \in T} x^t \mid x \in V_1\}$ is a group. Furthermore it implies that the elements of V are fixed by A. If $U = \{\prod_{t \in T} x^t \mid x \in U_1\}$, then $V/U \cong V_1/U_1 \cong$ PSL (2, p) and we conclude that the fixed point subgroup of A involves PSL (2, p).

As a consequence of Lemma 4.3 we get the following corollary.

COROLLARY 4.4. *Suppose that X is the direct product of t copies of a simple group of quadratic type with respect to the prime $p \geqq 5$. Assume that G is a group, $A \subseteqq$ Aut (G) and G contains a factor isomorphic to X that is normalized by A. If $(|A|, |G|) = 1$, then the fixed point subgroup of A involves* PSL (2, p).

Proof. Suppose that $K \trianglelefteq L \trianglelefteq G$ and that A normalizes $L/K = X$. By Lemma 4.3 there exist subgroups S and T such that $K \trianglelefteq S \trianglelefteq T \trianglelefteq L$, $T/S \cong$ PSL (2, p) and A fixes T/K. Suppose that q is a prime divisor of $|$PSL (2, p)$|$ and let Q be a Sylow q-subgroup of T normalized by A. Then since $Q = C_Q(A)[Q, A]$ and A fixes T/S, $Q = C_Q(A)(Q \cap S)$. Pick such a Q for each prime divisor of $|$PSL (2, p)$|$ and call this set of Sylow subgroups $\mathscr{S}$. Then

$$T = \langle C_Q(A) \mid Q \in \mathscr{S} \rangle S$$

and consequently $C_G(A)$ involves PSL (2, p).

LEMMA 4.5. *Suppose G is a group with a composition factor isomorphic to K, then G contains a semisimple factor X normalized by A such that every component of X is isomorphic to K.*

Proof. Let F be the semidirect product of G and suppose that $\{F_i\}$ is a chief series of F containing G. Then there exists i such that $F_{i-1} \subset F_i \subset G$ and F_i/F_{i-1} has K as a composition factor. Since F_i/F_{i-1} is a direct product of isomorphic simple group, it is the product of copies of K.

Proof of Theorem 4.1. Theorem 4.1 is now a consequence of Lemma 4.5 and Corollary 4.4.

5. Proof of Theorem B. Theorem B is a consequence of the following result.

THEOREM 5.1. *Let G be a p-constrained group with $p \geqq 5$. Suppose that $A \subseteqq \mathrm{Aut}(G)$ and $(|G|, |A|) = 1$. Then if G is not p-stable the fixed point subgroup of A involves* PSL (2, p).

Proof. Suppose that $O_{p'}(G) \supset 1$ and set $\bar{G} = G/O_{p'}(G)$. $\bar{G}$ is not p-stable and induction implies the result. Thus we may assume that $O_{p'}(G) = 1$.

Theorem 2.2 implies that G contains a composition factor of quadratic type. Then Theorem 4.1 implies that the fixed point subgroup of A involves PSL (2, p).

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