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CONTINUOUS OPERATORS ON PARANORMED SPACES AND MATRIX TRANSFORMATIONS

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The concept of a paranormed β-space is defined and some theorems of Banach-Steinhaus type are proved for sequences of continuous linear functionals on such a space. For example, necessary and sufficient conditions are given for a sequence $(A_n(x))$ of continuous linear functionals to be in the space of generalized entire sequences, for each x belonging to a paranormed β-space. The general theorems are then used to characterize matrix transformations between generalized l_p spaces and generalized entire sequences.

1. In § 2 we present theorems which generalize some results in [10]. These theorems are applied in § 3 to characterize some classes of matrix transformations. By N, R and C we denote respectively, the sets of natural numbers, real numbers, and complex numbers. By a sequence (x_k) we mean $(x_1, x_2, \cdots)$, and by $\Sigma_k x_k$ we mean $\sum_{k=1}^{\infty} x_k$.

X will denote a nontrivial complex linear space of elements x, with zero element θ and with paranorm g, i.e. $g: X \rightarrow R$ satisfies $g(\theta) = 0$, $g(x) = g(-x)$ on X, g is subadditive, and, for $\lambda \in C$ and $x \in X$, $\lambda \rightarrow \lambda_0$ and $g(x - x_0) \rightarrow 0$ imply $g(\lambda x - \lambda_0 x_0) \rightarrow 0$, where $\lambda_0 \in C$ and $x_0 \in X$.

Extending the definitions of Sargent in [8], we can define a paranormed β-space as follows. Let (X_n) be a sequence of subsets of X such that $\theta \in X_1$ and such that if $x, y \in X_n$ then $x \pm y \in X_{n+1}$ for $n \in N$; then (X_n) is called an α-sequence in X. If we can write $X = \bigcup_{n=1}^{\infty} X_n$, where (X_n) is an α-sequence in X and each X_n is nowhere dense in X, then X is called an α-space; otherwise X is a β-space. Clearly, every α-space is of the first category, whence we see that any complete paranormed space is a β-space.

If $Y \subset X$ then we denote the closure of Y in X by $\bar{Y}$. We write, for any $a \in X$ and $\delta > 0$, $S(a, \delta) = \{x: x \in X \text{ and } g(x - a) < \delta\}$. A subset G of X is called a fundamental set in X if $l.$ hull(G), the set of all finite linear combinations of elements of G, is dense in X. A sequence (b_k) of elements of X is said to be a basis in X if for each $x \in X$ there is a unique complex sequence (λ_k) such that $g(x - \sum_{k=1}^{n} \lambda_k b_k) \rightarrow 0 (n \rightarrow \infty)$. Thus any basis in X is also a fundamental set in X.

We denote the set of continuous linear functionals on X by X^*. A linear functional A on X is an element of X^* if and only if

$$\|A\|_M \equiv \sup\left\{|A(x)|: g(x) \leqq \frac{1}{M}\right\} < \infty \text{ for some } M > 1 .$$

If X is a space of complex sequences $x = (x_k)$, then we denote the generalized Köthe-Toeplitz dual of X by $X^\dagger$, i.e.

$$X^\dagger = \{(\alpha_k): \Sigma_k \alpha_k x_k \text{ converges for every } x \in X\} .$$

We now list some sets of complex sequences due to Maddox [4]. If $p = (p_k)$ is a sequence of strictly positive real numbers, then

$$l_\infty(p) = \{x: \sup_k |x_k|^{p_k} < \infty\} ,$$

$$c_0(p) = \{x: \lim_k |x_k|^{p_k} = 0\} ,$$

$$c(p) = \{x: \lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in C\} ,$$

$$l(p) = \{x: \Sigma_k |x_k|^{p_k} < \infty\} .$$

We write $e^{(k)} = (0, 0, \cdots, 1, 0, 0, \cdots)$, the 1 occurring in the k^{th} place, for each $k \in N$, and $e = (1, 1, 1, \cdots)$, and we write $l_\infty = l_\infty(e)$, $c_0 = c_0(e)$, $c = c(e)$, and $l_1 = l(e)$.

The case $p = (1/k)$ of $c_0(p)$ is of particular interest, since the function defined by $\sum_{k=0}^\infty \alpha_k z^k$, $z \in C$, is an entire function if and only if $(\alpha_k) \in c_0(1/k)$. Work on the space of entire functions has been carried out, by V. Ganapathy Iyer in [2] and in other papers, and by other authors, using this correspondence with $c_0(1/k)$. It is shown in [2] that $c_0(1/k)^\dagger = l_\infty(1/k)$.

Now we collect some known results which will be useful in what follows.

LEMMA 1. *$l(p)$ is a linear space if and only if p is bounded.* (See [4], Theorem 1, and [7], Theorem 1.)

LEMMA 2. *If p is bounded with $H = \max(\sup p_k, 1)$, then $g(x) = (\Sigma_k |x_k|^{p_k})^{1/H}$ defines a paranorm on $l(p)$, $l(p)$ is complete under g, and $(e^{(k)})$ is a basis in $l(p)$.* (See [5], Theorem 1 and Corollary 1, and [7].)

LEMMA 3. (i) *If $1 < p_k \leqq H$ and $p_k^{-1} + s_k^{-1} = 1$ for each $k \in N$, then*

$$l(p)^\dagger = \{(\alpha_k): \Sigma_k |\alpha_k|^{s_k} \cdot M^{-s_k} < \infty \text{ for some } M > 1\} .$$

(ii) *If $0 < p_k \leqq 1$ for all $k \in N$ then $l(p)^\dagger = l_\infty(p)$.* (See [6], Theorem 1, and [9], Theorem 7.)

LEMMA 4. *If either $1 < p_k \leqq H$ for all k, or $0 < p_k \leqq 1$ for all*

k, then every $A \in l(p)^*$ may be written as $A(x) \equiv \Sigma_k \alpha_k x_k$ on $l(p)$ for some $(\alpha_k) \in l(p)^\dagger$, and conversely $A(x) \equiv \Sigma_k \alpha_k x_k$ defines an element of $l(p)^*$ for each $(\alpha_k) \in l(p)^\dagger$. (See [6], Theorem 2, and [9], Theorem 7.)

Given sets Y and Z of sequences and a matrix $A = (a_{n,k})$ of complex numbers $(n, k = 1, 2, \cdots)$ we say that $A \in (Y, Z)$ if and only if $\Sigma_k a_{n,k} y_k$ converges for every $y = (y_k) \in Y$ and $n \in N$, and $(\Sigma_k a_{n,k} y_k) \in Z$ for every $y \in Y$.

We shall frequently use the following inequalities. Take $x, y \in C$; if $0 < p \leqq 1$ then

$$|x|^p - |y|^p \leqq |x + y|^p \leqq |x|^p + |y|^p ,$$

and if $p > 1$ and $p^{-1} + s^{-1} = 1$ then

$$|xy| \leqq |x|^p + |y|^s .$$

2. For the remainder of this paper, $q = (q_n)$ will denote a sequence of strictly positive real numbers. If q is bounded with $H = \max(\sup q_n, 1)$ then it follows by Lemma 1 of [4] that $c_0(q) = c_0(H^{-1}q)$; similarly $l_\infty(q) = l_\infty(H^{-1}q)$ and $c(q) = c(H^{-1}q)$.

THEOREM 1. *Let X be a paranormed space and let (A_n) be a sequence of elements of X^*, and suppose q is bounded. Then*

$$\sup_n (\| A_n \|_M)^{q_n} < \infty \text{ for some } M > 1 \tag{1}$$

implies

$$(A_n(x)) \in l_\infty(q) \text{ for every } x \in X , \tag{2}$$

and the converse is true if X is a β-space.

Proof. In view of the remarks at the beginning of this section, we may without loss of generality assume that $q_n \leqq 1$ fore all $n \in N$. First let (1) hold, and choose any $x \in X$. By the continuity of scalar multiplication in a paranormed space, there is a $K \geqq 1$ such that $g(K^{-1}x) \leqq 1/M$, where the M is that of (1). Then we have for any n, since $q_n \leqq 1$,

$$\begin{aligned} |A_n(x)|^{q_n} = |KA_n(K^{-1}x)|^{q_n} &\leqq K^{q_n}(\| A_n \|_M)^{q_n} \\ &\leqq K \sup_n (\| A_n \|_M)^{q_n} , \end{aligned}$$

so that (2) holds.

Now let (2) hold, with X a β-space, and define for any $m \in N$,

$$X_m = \{x : x \in X \text{ and } |A_n(x)|^{q_n} \leqq 2^m \text{ for all } n \in N\} .$$

Then (X_m) is an α-sequence in X, for obviously $\theta \in X_1$, and if for some $m \geqq 1$, $x, y \in X_m$ then, since $q_n \leqq 1$ for every n,

$$| A_n(x \pm y) |^{q_n} \leqq | A_n(x) |^{q_n} + | A_n(y) |^{q_n} \leqq 2^{m+1}$$

for any $n \in N$. Also $X = \bigcup_{m=1}^{\infty} X_m$, so since X is a β-space there exists a $B \in N$ such that X_B is not nowhere dense. Using the continuity of the A_n, it is not difficult to show that $\bar{X}_m = X_m$ for every m, whence there is a sphere $S(a, \delta) \subset X_B$. Thus if $g(x - a) < \delta$ we have $| A_n(x) |^{q_n} \leqq 2^B$ for all n, so if $g(x) < \delta$ we have

$$| A_n(x) |^{q_n} \leqq | A_n(x + a) |^{q_n} + | A_n(a) |^{q_n} \leqq 2^{B+1} \text{ for all } n.$$

Taking $M > \delta^{-1}$ we obtain (1).

THEOREM 2. *Let X be a paranormed space and let (A_n) be a sequence of elements of X^*.*

(i) *If X has fundamental set G and if q is bounded, then the following propositions*

$$(3) \qquad (A_n(b)) \in c_0(q) \text{ for every } b \in G ,$$

$$(4) \qquad \lim_M \limsup_n (\| A_n \|_M)^{q_n} = 0 ,$$

together imply

$$(5) \qquad (A_n(x)) \in c_0(q) \text{ for every } x \in X .$$

(ii) *If $q_n \to 0 (n \to \infty)$ then (4) implies (5).*

(iii) *Let X be a β-space; then (5) implies (4) even if q is unbounded.*

Proof. (i) Again, we may without loss of generality assume that $q_n \leqq 1$ for every $n \in N$. Let X have fundamental set G, and suppose (3) and (4) hold. Choose any $x \in X$ and any $\varepsilon > 0$. There exist $M > 1$ and n_0 such that $(\| A_n \|_M)^{q_n} < \varepsilon/2$ for all $n \geqq n_0$, by (4). Since $l \cdot \text{hull}\ (G)$ is dense in X there exist $\lambda_1, \lambda_2, \cdots, \lambda_m \in C$ and $b_1, b_2, \cdots, b_m \in G$ such that $g(x - \sum_{k=1}^{m} \lambda_k b_k) < 1/M$, and we write $L = \max(|\lambda_1|, \cdots, |\lambda_m|, 1)$. Then by (3) there is an $n_1 \geqq n_0$ such that $| A_n(b_k) |^{q_n} < \varepsilon/(2Lm)$, $k = 1, 2, \cdots, m$, if $n \geqq n_1$, whence if $n \geqq n_1$, we have

$$\begin{aligned} | A_n(x) |^{q_n} &= \left| A_n\left(x - \sum_{k=1}^{m} \lambda_k b_k\right) + \sum_{k=1}^{m} \lambda_k A_n(b_k) \right|^{q_n} \\ &\leqq \left| A_n\left(x - \sum_{k=1}^{m} \lambda_k b_k\right|^{q_n} + L \sum_{k=1}^{m} | A_n(b_k) |^{q_n} \\ &\leqq (\| A_n \|_M)^{q_n} + mL \cdot \varepsilon/(2Lm) < \varepsilon ; \end{aligned}$$

thus (5) holds.

(ii) Suppose (4) holds and $q \in c_0$, and choose any $x \in X$ and any $\varepsilon > 0$. There is an $M > 1$ and an n_0 such that $(\| A_n \|_M)^{q_n} < \varepsilon/2$ if $n \geqq n_0$, and since scalar multiplication is continuous on X there is a $K \geqq 1$ such that $g(K^{-1}x) \leqq 1/M$. Then we can choose $n_1 \geqq n_0$ such that $K^{q_n} \leqq 2$ if $n \geqq n_1$ whence if $n \geqq n_1$

$$| A_n(x) |^{q_n} = K^{q_n} | A_n(K^{-1}x) |^{q_n} < \varepsilon ,$$

so that (5) is true.

(iii) Let X be a β-space and suppose (5) is true. We define sequences (B_n), (C_n) of elements of X^* and sequences $r = (r_n)$, $s = (s_n)$ of strictly positive real numbers as follows. If $q_n \geqq 1$ then define $B_n = A_n$, $C_n = 0$, $r_n = q_n$, and $s_n = 1$; if $q_n < 1$ write $B_n = 0$, $C_n = A_n$, $r_n = 1$, and $s_n = q_n$. Then $(B_n(x)) \in c_0(r)$ and $(C_n(x)) \in c_0(s)$ on X; $\sup_n s_n \leqq 1$, and $r_n \geqq 1$ for all $n \in N$. Also, $(\| A_n \|_M)^{q_n} = (\| B_n \|_M)^{r_n} + (\| C_n \|_M)^{s_n}$ for all large enough M, $n = 1, 2, \cdots$, whence

$$\lim_M \limsup_n (\| A_n \|_M)^{q_n} \leqq \lim_M \limsup_n (\| B_n \|_M)^{r_n} + \lim_M \limsup_n (\| C_n \|_M)^{s_n} .$$

Choose any $\varepsilon > 0$, and define for each $m \in N$

$$X_m = \{x : x \in X \text{ and } | 2^{-m} C_n(x) |^{s_n} \leqq \frac{\varepsilon}{2} \text{ for all } n \geqq m\} .$$

Clearly $\theta \in X_1$, and if for some $m \in N$ we have $x, y \in X_m$ then for $n \geqq m + 1$

$$\begin{aligned} | 2^{-(m+1)} C_n(x \pm y) |^{s_n} &\leqq (| 2^{-(m+1)} C_n(x) | + | 2^{-(m+1)} C_n(y) |)^{s_n} \\ &\leqq (2 \max (| 2^{-(m+1)} C_n(x) |, | 2^{-(m+1)} C_n(y) |))^{s_n} \\ &= \max (| 2^{-m} C_n(x) |^{s_n}, | 2^{-m} C_n(y) |^{s_n}) \leqq \frac{\varepsilon}{2} ; \end{aligned}$$

thus (X_m) is an α-sequence in X. Also $X = \bigcup_{m=1}^{\infty} X_m$ and $X_m = \bar{X}_m$ for all $m \in N$ whence, since X is a β-space, some X_B contains a sphere $S(a, \delta)$. Then if $g(x) < \delta$ we deduce that $| 2^{-B} C_n(x) |^{s_n} \leqq \varepsilon$ for $n \geqq B$. Write $\rho = 2^{-B}\delta$ and choose $M > \rho^{-1}$; then by the subadditivity of g we have $g(2^B x) < \delta$ if $g(x) < \rho$. Hence if $g(x) \leqq 1/M$ we have

$$| C_n(x) |^{s_n} = | 2^{-B} C_n(2^B x) |^{s_n} \leqq \varepsilon \text{ if } n \geqq B ,$$

and since $\varepsilon > 0$ was arbitrary we obtain $\lim_M \limsup_n (\| C_n \|_M)^{s_n} = 0$.

Now $(B_n(x)) \in c_0(r)$ on X implies $(B_n(x)) \in c_0$ on X. For suppose if possible that for some sequence $(n(i))$ of integers and some $x \in X$ $\inf | B_{n(i)}(x) | = \alpha > 0$; then $| B_{n(i)}(\alpha^{-1}x) |^{r_{n(i)}} \geqq 1$ for all i, contrary to hypothesis. By the argument used above we deduce that

$$\lim_M \limsup_n \| B_n \|_M = 0 ,$$

whence since $r_n \geqq 1$ for all n, $\lim_M \limsup_n (\| B_n \|_M)^{r_n} = 0$. By our earlier remarks, (4) now follows.

THEOREM 3. *Let X be a paranormed space and let (A_n) be a sequence of element of X^* and suppose q is bounded.*

(i) *If X has fundamental set G, and if there is an $l \in X^*$ such that $(A_n(b) - l(b)) \in c_0(q)$ for all $b \in G$ and*

$$(6) \qquad \lim_M \limsup_n (\| A_n - l \|_M)^{q_n} = 0 ,$$

then

$$(7) \qquad (A_n(x)) \in c(q) \text{ on } X .$$

(ii) *If $q_n \to 0 (n \to \infty)$ and if there is an $l \in X^*$ such that* (6) *holds, then* (7) *is true.*

(iii) *If X is a β-space and if* (7) *is true, then there is an $l \in X^*$ such that* (6) *holds.*

Proof. (i) If the hypotheses hold, then $A_n - l \in X^*$ for every $n \in N$ whence by part (i) of Theorem 2 $((A_n - l)(x)) \in c_0(q)$ on X; thus (7) is true.

(ii) Follows similarly from Theorem 2(ii).

(iii) Suppose (7) holds; then for some l we have $| A_n(x) - l(x) |^{q_n} \to 0 (n \to \infty)$ on X. We deduce that $l(x) = \lim_n A_n(x)$ on X and $\sup_n | A_n(x) | < \infty$ on X. Then by Theorem 1 we have $\sup_n \| A_n \|_M < \infty$ for some $M > 1$, whence $\| l \|_M < \infty$. Clearly l must be linear, so that $l \in X^*$. Thus $A_n - l \in X^*$ for each $n \in N$, and by hypothesis $((A_n - l)(x)) \in c_0(q)$ on X, so by Theorem 2(iii), (6) must be true.

3. We now apply the theorems above in characterizing the classes $(l(p), l_\infty(q))$, $(l(p), c_0(q))$, and $(l(p), c(q))$ in the case when both p and q are bounded. Throughout, $A = (a_{n,k})$ will denote an infinite matrix of complex numbers. As a preliminary, we state Theorem 1 of [3]:

THEOREM 4. (i) *Let $1 < p_k \leqq H < \infty$ and $p_k^{-1} + s_k^{-1} = 1$ for every k. Then $A \in (l(p), l_\infty)$ if and only if there exists an integer $B > 1$ such that $\sup_n \Sigma_k | a_{n,k} |^{s_k} \cdot B^{-s_k} < \infty$.*

(ii) *Let $0 < p_k \leqq 1$ for every k. Then $A \in (l(p), l_\infty)$ if and only if $\sup_{n,k} | a_{n,k} |^{p_k} < \infty$.*

In the proofs of the following results, as in earlier ones, we may without loss of generality assume that $q_n \leqq 1$ for all $n \in N$, and we shall do so when convenient.

We first consider the case when $0 < p_k \leqq 1$ for all $k \in N$.

THEOREM 5. *Suppose* $0 < p_k \leqq 1$ *for all* $k \in N$, *and* $q = (q_n)$ *is bounded. Then,*

(i) $A \in (l(p), l_\infty(q))$ *if and only if*

$$\sup_n (\sup_k |a_{n,k}| M^{-1/p_k})^{q_n} < \infty \text{ for some } M > 1 . \tag{8}$$

(ii) $A \in (l(p), c_0(q))$ *if and only if*

$$|a_{n,k}|^{q_n} \to 0 (n \to \infty) \text{ for every } k \in N \tag{9}$$

and

$$\lim_M \limsup_n (\sup_k |a_{n,k}| M^{-1/p_k})^{q_n} = 0 . \tag{10}$$

(iii) $A \in (l(p), c(q))$ *if and only if* $\sup_n \sup_k |a_{n,k}| M^{-1/p_k} < \infty$ *for some* $M > 1$ *and there exist* $\alpha_1, \alpha_2, \cdots$ *such that*

$$|a_{n,k} - \alpha_k|^{q_n} \to 0 (n \to \infty) \text{ for each } k \in N \tag{11}$$

and

$$\lim_M \limsup_n (\sup_k |a_{n,k} - \alpha_k| M^{-1/p_k})^{q_n} = 0 . \tag{12}$$

Proof. Write, for each $x \in l(p)$ and each $n \in N$

$$A_n(x) \equiv \Sigma_k a_{n,k} x_k . \tag{13}$$

(i) Let $A \in (l(p), l_\infty(q))$; then for each n, $(a_{n,1}, a_{n,2}, \cdots) \in l(p)^\dagger = l_\infty(p)$, by Lemma 3(ii). Also, by Lemma 4, $A_n \in l(p)^*$ for each $n \in N$. We show that for each n, $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ for all M such that $\|A_n\|_M$ is defined. Choose any $n \in N$. First, if M is such that, for some sequence $(k(i))$ of integers, $|a_{n,k(i)}| M^{-1/p_{k(i)}} \geqq i$ for each $i \in N$, then by defining $x^{(k(i))} = (M^{-1/p_{k(i)}} \operatorname{sgn} a_{n,k(i)}) e^{(k(i))}$, $i = 1, 2, \cdots$, we see that $\|A_n\|_M$ is undefined. Since $(a_{n,1}, a_{n,2}, \cdots) \in l_\infty(p)$ there is an $M_n \geqq 1$ such that $|a_{n,k}|^{p_k} \leqq M_n$ for all k. Choose $M \geqq M_n$. We have if $g(x) = \Sigma_k |x_k|^{p_k} \leqq 1/M$, since $M^{1/p_k} |x_k| \leqq 1$ for all k and since $\sup_k p_k \leqq 1$,

$$\begin{aligned} |A_n(x)| &\leqq \Sigma_k |a_{n,k}| M^{-1/p_k} \cdot M^{1/p_k} |x_k| \\ &\leqq \Sigma_k |a_{n,k}| M^{-1/p_k} \cdot M |x_k|^{p_k} \\ &\leqq M g(x) \sup_k |a_{n,k}| M^{-1/p_k} , \end{aligned}$$

whence $\|A_n\|_M \leqq \sup_k |a_{n,k}| M^{-1/p_k}$. Given $\varepsilon > 0$ we can choose an m such that $|a_{n,m}| M^{-1/p_k} > \sup_k |a_{n,k}| M^{-1/p_k} - \varepsilon$. Defining $x = (M^{-1/p_k} \operatorname{sgn} a_{n,m}) e^{(m)}$ we have $g(x) \leqq 1/M$ and $A_n(x) > \sup_k |a_{n,k}| M^{-1/p_k} - \varepsilon$, whence $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ as required. By Lemma 2, $l(p)$ is complete, so it is a β-space; thus by Theorem 1 we must have (8).

Conversely let (8) hold. Then as above it follows that for each n, $A_n \in l(p)^*$ with $\|A_n\|_M = \sup_k |a_{n,k}| M^{-1/p_k}$ for all M such that

$\| A_n \|_M$ is defined. Then using Theorem 1 we obtain $(A_n(x)) \in l_\infty(q)$ on $l(p)$, i.e. $A \in (l(p), l_\infty(q))$.

We remark that (8) reduces to $\sup_{n,k} | a_{n,k} |^{p_k} < \infty$ if $0 < \inf q_n \leqq \sup q_n < \infty$, corresponding to the condition given for $A \in (l(p), l_\infty)$ in Theorem 4(ii).

(ii) If $A \in (l(p), c_0(q)) \subset (l(p), l_\infty(q))$ then as above we have $A_n \in X^*$ and $\| A_n \|_M = \sup_k | a_{n,k} | M^{-1/p_k}$ whenever $\| A_n \|_M$ is defined, for each $n \in N$. Then, by Theorem 2(iii), (10) must hold. Also taking $x = e^{(k)} \in l(p)$ $(k = 1, 2, \cdots)$ we obtain (9). Conversely if (9) and (10) hold we can show that $A_n \in l(p)^*$ with $\| A_n \|_M = \sup_k | a_{n,k} | M^{-1/p_k}$ whenever $\| A_n \|_M$ is defined, for each $n \in N$; also $(e^{(k)})$ is a basis in $l(p)$ by Lemma 2. Then by Theorem 2(i) we can deduce that $A \in (l(p), c_0(q))$.

(iii) Let $A \in (l(p), c(q))$; then as in (i) and (ii) above we have for each n that $A_n \in X^*$. By Theorem 3(iii) there is an $l \in X^*$ such that $\lim_M \limsup_n (\| A_n - l \|_M)^{q_n} = 0$, and by Lemmas 3(ii) and 4 we can write $l(x) = \Sigma_k \alpha_k x_k$ on $l(p)$ for some $(\alpha_k) \in l_\infty(p)$. We deduce that $\| A_n - l \|_M = \sup_k | a_{n,k} - \alpha_k | M^{-1/p_k}$ for large enough M, $n = 1, 2 \cdots$, whence (12) is true, and (11) must hold since $(A_n - l)(e^{(k)}) = a_{n,k} - \alpha_k$ for each n and k. Also $c(q) \subset l_\infty$ whence $(l(p), (c(q)) \subset (l(p), l_\infty)$; thus by (i) we must have $\sup_k | a_{n,k} | M^{-1/p_k} < \infty$ for some $M > 1$.

Finally, if $\sup_k | a_{n,k} | M^{-1/k_k} < \infty$ for some $M > 1$ then $A_n \in l(p)^*$ for all n. If in addition (11) and (12) hold then for any k we have, if n and M are large enough,

$$\begin{aligned} | \alpha_k | M^{-1/p_k} &\leqq | a_k - a_{n,k} | M^{-1/p_k} + | a_{n,k} | M^{-1/p_k} \\ &\leqq 1 + \sup_n (\sup_k | a_{n,k} | M^{-1/p_k}) = B \text{ say ;} \end{aligned}$$

hence $| \alpha_k |^{p_k} \leqq B^{p_k} \cdot M \leqq BM$ for all k, i.e. $(\alpha_k) \in l_\infty(p) = l(p)^\dagger$. By Lemma 4, $l(x) \equiv \Sigma_k \alpha_k x_k$ defines an element of $l(p)^*$, and the result now follows if we employ the methods used above together with Theorem 3(i).

THEOREM 6. *Suppose* $0 < p_k \leqq 1$ *for all* $k \in N$ *and* $q_n \to 0 (n \to \infty)$. *Then* $A \in l(p), c_0(q))$ *if and only if* (12) *is true.*

Proof. This follows from Theorem 2, parts (ii) and (iii), on using the methods of Theorem 5.

COROLLARY. (i) $A \in (l_1, c_0(1/n))$ *if and only if* $| a_{n,k} |^{1/n} \to 0$ *uniformly in* k *as* $n \to \infty$.

(ii) $A \in (l_1, l_\infty(1/n))$ *if and only if* $\sup_{n,k} | a_{n,k} |^{1/n} < \infty$.

Proof. These characterizations were given in Theorems 1 and 2 of [1], and follow readily on taking $p = e$ and $q = (1/n)$ in Theorems 5(i) and 6.

Now we consider the case when $1 < p_k \leqq H < \infty$ for all k.

THEOREM 7. *Let* $1 < p_k \leqq H$ *and* $p_k^{-1} + s_k^{-1} = 1$ *for each* $k \in N$, *and let* q *be bounded. Then* $A \in (l(p), l_\infty(q))$ *if and only if*

$$(14) \qquad T(B) \equiv \sup_n \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k/q_n} < \infty \text{ for some } B > 1 .$$

Proof. Define A_n by (13) on $l(p)$, for each $n \in N$. For the sufficiency, let (14) hold. Then if $x \in l(p)$ we have for each n, assuming $q_n \leqq 1$ for all n,

$$\begin{aligned} |A_n(x)|^{q_n} &\leqq (\Sigma_k |a_{n,k}x_k|)^{q_n} = (\Sigma_k |a_{n,k}| B^{-1/q_n} \cdot B^{1/q_n} |x_k|)^{q_n} \\ &\leqq (\Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k/q_n} + \Sigma_k B^{p_k/q_n} |x_k|^{p_k})^{q_n} \\ &\leqq (T(B))^{q_n} + B^H (g^H(x))^{q_n} \\ &\leqq T(B) + 1 + B^H(g^H(x) + 1) \end{aligned}$$

which implies $A \in (l(p), l_\infty(q))$.

Now let $A \in (l(p), l_\infty(q))$; then $(a_{n,1}, a_{n,2}, \cdots) \in l(p)^\dagger$ for each n and so, by Lemmas 3(i) and 4, $A_n \in l(p)$ for all n. By Theorem 1 there exist $M > 1$ and $G \geqq 1$ such that $|A_n(x)|^{q_n} \leqq G$ for all n and all $x \in l(p)$ with $g(x) \leqq 1/M$. Then $|\Sigma_k G^{-1/q_n} \cdot a_{n,k}x_k| \leqq 1$, $n = 1, 2, \cdots$, if $g(x) \leqq 1/M$. Write $\Gamma = (G^{-1/q_n}a_{n,k})$, and choose any $x \in l(p)$. By the continuity of scalar multiplication on $l(p)$ there is a $K \geqq 1$ such that $g(K^{-1}x) \leqq 1/M$, whence $|\Sigma_k G^{-1/q_n} \cdot a_{n,k}x_k| \leqq K$ for all n. Thus we see that $\Gamma \in (l(p), l_\infty)$ and so by Theorem 4(i) there is a $D > 1$ such that $\sup_n \Sigma_k |G^{-1/q_n} \cdot a_{n,k}|^{s_k} \cdot D^{-s_k} < \infty$. Writing $B = GD$ and using the fact that $D^{q_n} \leqq D$ for all n, we obtain (14).

Looking at Theorem 4, one might except the necessary and sufficient condition for $A \in (l(p), l_\infty(q))$ to be

$$(15) \qquad \sup_n (\Sigma_k |a_{n,k}|^{s_k} \cdot M^{-s_k})^{q_n} < \infty \text{ for some } M > 1 .$$

Using the method above we can show that (15) implies $A \in (l(p), l_\infty(q))$. In fact it can be shown that (15) implies (14) directly. For let (15) hold; then for some $B > 1$, $(\Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k})^{q_n} \leqq H$ for all n, and we may suppose that $H > 1$. If $q_n \leqq Q$ for all n then

$$(16) \qquad (\Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n})^{q_n/Q} \leqq 1 \text{ for all } n .$$

Put $M = HB^Q$; then $M^{s_k} = H^{s_k} \cdot B^{Qs_k} \geqq H \cdot B^{q_n s_k}$, whence $M^{s_k/q_n} \geqq H^{1/q_n} \cdot B^{s_k}$ for all k and n. Thus by (16) we obtain

$$\begin{aligned} \Sigma_k |a_{n,k}|^{s_k} \cdot M^{-s_k/q_n} &\leqq \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n} \\ &\leqq (\Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} \cdot H^{-1/q_n})^{q_n/Q} \leqq 1 \text{ for all } n , \end{aligned}$$

whence $T(M) \leqq 1$, i.e. (14) holds.

Clearly, (14) implies (15) if $\inf_n q_n > 0$ or if $\inf_k p_k > 1$. However, (15) is not necessary for $A \in (l(p), l_\infty(q))$ if $\inf_n q_n = 0$ and $\inf_k p_k = 1$. For choose bounded p and q, with $p_k > 1$ for all k, and suppose there exist sequence $(n(i), (k(j))$ of integers such that $q_{n(i)} \leqq 1/i$, $i = 1, 2 \cdots$, and $p_{k(j)} \leqq 1 + 1/j$, $j = 1, 2, \cdots$; then $s_{k(j)} \geqq j + 1$ for each j. Define $a_{n(i),k(j)} = i$, $i, j = 1, 2, \cdots$, and $a_{n,k} = 0$ for all other n and k. Then $A = (a_{n,k}) \in (l(p), l_\infty(q))$ since for all $i \in N$.

$$\Sigma_j \mid a_{n(i),k(j)} \mid^{s_{k(j)}} \cdot 2^{-s_{k(j)}/q_{n(i)}} \leqq \Sigma_j (i/2^i)^{j+1} \leqq 1 ,$$

but for any $M > 1$ we have if $i \geqq M$,

$$(\Sigma_j \mid a_{n(i),k(j)} \mid^{s_{k(j)}} \cdot M^{-s_{k(j)}})^{q_{n(i)}} \geqq (\Sigma_j \mid i/M \mid^{j+1})^{q_{n(i)}} ,$$

which diverges.

THEOREM 8. *Let q be bounded, and let $1 < p_k \leqq H$ and $p_k^{-1} + s_k^{-1} = 1$ for all $k \in N$. Then $A \in (l(p), c_0(q))$ if and only if* (9) *holds and, for every $D \geqq 1$,*

$$\lim_B \limsup_n (\Sigma_k \mid a_{n,k} \mid^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} = 0 . \tag{17}$$

Proof. Again, define A_n on $l(p)$ by (13). First we prove the necessity: let $A \in (l(p), c_0(q))$. Obviously we must have (9), and as in Theorem 7 we see that $A_n \in l(p)^*$ for all n. If $A \in (l(p), c_0(q))$ then $(D^{1/q_n} \cdot a_{n,k}) \in (l(p), c_0(q))$ for all $D > 1$, so it is enough to show that (17) holds for $D = 1$. Since $c_0(q) \subset l_\infty$ and using Theorem 4(i) there is a $B > 1$ such that $T_n \equiv \Sigma_k \mid a_{n,k} \mid^{s_k} \cdot B^{-Hs_k} \leqq 1$ for every $n \in N$. Choose any n, and define $x_k^{(n)} = B^{-Hs_k} \mid a_{n,k} \mid^{s_k - 1} \operatorname{sgn} a_{n,k}$ for each k; then

$$g^H(x^{(n)}) = \Sigma_k B^{-Hs_k - Hp_k} \mid a_{n.k} \mid^{s_k} \leqq B^{-H} T_n \leqq B^{-H}$$

and $A_n(x^{(n)}) = T_n$, whence $\| A_n \|_B \geqq T_n$ for each n. By Theorem 2(iii) we must have $\lim_B \limsup_n (\| A_n \|_B)^{q_n} = 0$, whence (17) holds with $D = 1$.

For the sufficiency, let (9) be true and let (17) hold for all $D \geqq 1$. It follows that $A_n \in l(p)^*$ for all $n \in N$. Since $(e^{(k)})$ is a basis in $l(p)$ and using Theorem 2(i) it is enough to show that $\lim_B \limsup_n (\| A_n \|_B)^{q_n} = 0$. Choose ε, $0 < \varepsilon \leqq 1$, and $D > 2/\varepsilon$. There exist $B > 1$ and m such that $(\Sigma_k \mid a_{n,k} \mid^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} < \varepsilon/2$ if $n \geqq m$. Then if $g(x) \leqq 1/B$ and if $n \geqq m$ we have

$$\begin{aligned} \mid A_n(x) \mid^{q_n} &\leqq (\Sigma_k \mid a_{n,k} \mid D^{1/q_n} \cdot B^{-1} \cdot BD^{-1/q_n} \mid x_k \mid)^{q_n} \\ &\leqq (\Sigma_k \{ \mid a_{n,k} \mid^{s_k} D \cdot {}^{s_k/q_n} \cdot B^{-s_k} + D^{-p_k/q_n} \cdot B^{p_k} \mid x_k \mid^{p_k} \})^{q_n} \\ &< \varepsilon/2 + (D^{-1/q_n} \cdot B^H g^H(x))^{q_n} < \varepsilon , \end{aligned}$$

and this completes the proof.

One may show that if (9) is true and if (17) holds for $D = 1$, and if either $\inf_n q_n > 0$ or $\inf_k p_k > 1$, then $A \in (l(p), c_0(q))$, but that these conditions are not sufficient for $A \in (l(p), c_0(q))$ if $\inf_n q_n = 0$ and $\inf_k p_k = 1$.

THEOREM 9. *Let q be bounded, and let $1 < p_k \leqq H$ and $p_k^{-1} + s_k^{-1} = 1$ for all $k \in N$. Then $A \in (l(p), c(q))$ if and only if* $\sup_n \Sigma_k \times |a_{n,k}|^{s_k} B^{-s_k} < \infty$ *for some $B > 1$ and there exist $\alpha_1, \alpha_2, \cdots$ such that (11) holds and* $\lim_B \limsup_n (\Sigma_k |a_{n,k} - \alpha_k|^{s_k} \cdot D^{s_k/q_n} \cdot B^{-s_k})^{q_n} = 0$ *for all $D \geqq 1$.*

Proof. As usual, define A_n on $l(p)$ by (13) for each $n \in N$. First let $A \in (l(p)), c(q) \subset (l(p), l_\infty)$; then $\sup_n \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} < \infty$ for some $B > 1$. Also by Theorem 3 there is an $l \in l(p)^*$ such that $|A_n(e^{(k)}) - l(e^{(k)})|^{q_n} \to 0$ $(n \to \infty)$ for each k and such that $\lim_B \limsup_n (\| A_n - l \|_B)^{q_n} = 0$. By Lemma 4 we can write $l(x) = \Sigma_k \alpha_k x_k$ on $l(p)$ for some sequence $(\alpha_k) \in l(p)^\dagger$, and the necessity now follows using the method of Theorem 8.

For the sufficiency, we show that the conditions of this theorem imply $\Sigma_k |\alpha_k|^{s_k} \cdot M^{-s_k} < \infty$ for some $M > 1$; then $l(x) \equiv \Sigma_k \alpha_k x_k$ defines an element of $l(p)^*$. We have for suitably large B and n

$$\begin{aligned}\Sigma_k |\alpha_k|^{s_k}(2B)^{-s_k} &= \Sigma_k |\alpha_k - a_{n,k} + a_{n,k}|^{s_k} \cdot (2B)^{-s_k}\\ &\leqq \Sigma_k \max(|a_{n,k} - \alpha_k|, |a_{n,k}|)^{s_k} \cdot B^{-s_k}\\ &\leqq \Sigma_k |a_{n,k} - \alpha_k|^{s_k} \cdot B^{-s_k} + \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k}\\ &\leqq 1 + \sup_n \Sigma_k |a_{n,k}|^{s_k} \cdot B^{-s_k} < \infty .\end{aligned}$$

Then by Theorem 8, $(a_{n,k} - \alpha_k) \in (l(p), c_0(q))$ whence $|A_n(x) - l(x)|^{q_n} \to 0$ $(n \to \infty)$ on $l(p)$, and the proof is complete.

We note that $(l(p), c)$ was characterized, for bounded p, in the corollary to Theorem 1 of [3].

The conditions for $A \in (l(p), l_\infty(q)$, $(l(p), c_0(q))$ or $(l(p), c(q))$ in the general case $0 < p_k \leqq \sup p_k < \infty$ and q bounded may be obtained by combining the separate cases $0 < p_k \leqq 1$ and $1 < p_k \leqq H$ above.

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