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SPECTRAL DISTRIBUTION OF THE SUM OF SELF-ADJOINT OPERATORS

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Using the techniques of noncommutative integration theory, classical results of Hermann Weyl concerning the positive eigenvalues of the sum of two self-adjoint compact operators are extended to self-adjoint operators which are measurable with respect to a gage space. Let (H, A, m) be a gage space and let K and L be self-adjoint operators which are measurable with respect to (H, A, m). Let $P_K[\lambda, \infty)$ be the spectral projection of K for the interval $[\lambda, \infty)$ and let $\Lambda_K(x) = \sup\{\lambda \mid m(P_K[\lambda, \infty)) \geqq x\}$. Then $\Lambda_{K+L}(x + r) \leqq \Lambda_K(x) + \Lambda_L(r)$. If $K \leqq L$, then $\Lambda_K(x) \leqq \Lambda_L(x)$. If L is bounded, then $\Lambda_{LKL}(x) \leqq \|L\|^2 \Lambda_K(x)$ for $x \leqq m(P_K[0, \infty))$. If $q = m(\text{support } (L))$ and $q < \infty$, then $\Lambda_K(x + q) \leqq \Lambda_{K+L}(x)$; if $\mu = \Lambda_{|K|}(q)$, then $\|K + L\|_p \geqq \|KP_K(-\mu, \mu)\|_p$ for $1 \leqq p \leqq \infty$.

1. **Notation.** We specifically work in the context of a gage space. [See 5 for definitions and notation.] We will always require that an operator be measurable [5, Definition 2.1]. This is a technical consideration which is necessary to avoid the pathologies which can occur with unbounded operators. Any one of the following conditions implies that a self-adjoint operator T is measurable with respect to the gage space (H, A, m):

1. $T \in A$.
2. $T\eta A$ and m is a finite gage. ($T\eta A$ means that $UT = TU$ for every unitary operator U in the commutant of A.)
3. $T\eta A$ and $m(\text{support } (T)) < \infty$, where support (T) is the orthocomplement of the nullspace of T.
4. $T\eta A$ and A is abelian.

If P is a projection operator, P will be identified with the range of P. If T is an operator, $R(T)$ denotes the range of T and $\bar{R}(T)$ denotes the closure of $R(T)$. If T is self-adjoint, note that support $(T) = \bar{R}(T)$.

(H, A, m) is a gage space. If S and T are self-adjoint operators which are measurable with respect to (H, A, m), then $S + T(ST)$ will denote the strong sum (product) of S with T; this is the closure of the ordinary sum (product) and is self-adjoint and measurable [5, Corollary 5.2]. T has spectral decomposition $T = \int_{-\infty}^{\infty} \lambda dP_T(\lambda)$; the function $P_T(\lambda)$ is chosen to be continuous from the right. If $\mathscr{I}$ is an interval, $P_T(\mathscr{I})$ is the spectral projection of T for the interval $\mathscr{I}$. The function Λ_T is defined, for $x > 0$, by $\Lambda_T(x) = \sup\{\lambda \mid m(P_T[\lambda, \infty)) \geqq x\}$.

Note that $m(P_T[\Lambda_T(x), \infty)) < x$ is possible if $m(P_T(\Lambda_T(x) - \varepsilon, \Lambda_T(x))) = \infty$ for every $\varepsilon > 0$. $\Lambda_T(x)$ is a nonincreasing function of x and is continuous from the left. If $x > m(I)$, where I is the identity operator on H, then $\Lambda_T(x) = \sup(\phi) = -\infty$; we will not explicitly mention this pathology in order to avoid excessive technicality and splitting into cases.

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2. Statement of the results. Below we state the theorems and corollaries and prove the corollaries. The theorems are proved in the next section. For the remainder of this paper, K and L are self-adjoint operators which are measurable with respect to the gage space (H, A, m).

THEOREM 1. *Let* $q = m(\text{support }(L))$ *and assume* $q < \infty$. *Then* $\Lambda_K(x + q) \leqq \Lambda_{K+L}(x)$ *for* $x > 0$; *equivalently,* $\Lambda_K(x) \leqq \Lambda_{K+L}(x - q)$ *for* $x > q$.

THEOREM 2. $\Lambda_{K+L}(x + r) \leqq \Lambda_K(x) + \Lambda_L(r)$, *for* $x > 0$, $r > 0$.

If (H, A, m) is the algebra of all bounded operators on H, m is the usual trace, and K is a compact operator and has (counting multiplicity) at least $j + 1$ positive eigenvalues, then $\Lambda_K(j)$ is the jth positive eigenvalue of K and $\Lambda_K(j + 1/2)$ is the $(j + 1)$st positive eigenvalue of K. If in addition L is compact and has at least $k + 1$ positive eigenvalues, then Theorem 2 implies $\Lambda_{K+L}(j + k + 1) \leqq \Lambda_K(j + 1/2) + \Lambda_L(k + 1/2) = \Lambda_K(j + 1) + \Lambda_L(k + 1)$, which is Weyl's result [6, Satz 1]. Similarly, Theorem 1 reduces to [6, Satz 2] if L has finite rank.

By $K \leqq L$ is meant $L - K \geqq 0$.

COROLLARY 1. *If* $K \leqq L$, *then* $\Lambda_K(x) \leqq \Lambda_L(x)$ *for* $x > 0$.

Proof of Corollary 1. $K = L + (K - L)$. Note that $K - L \leqq 0$. Let $x > 0$ and let ε be an arbitrary positive number with $\varepsilon < x$. Then $\Lambda_K(x) \leqq \Lambda_L(x - \varepsilon) + \Lambda_{K-L}(\varepsilon)$ by Theorem 2. Since $\Lambda_{K-L}(\varepsilon) \leqq 0$ and the function Λ_L is continuous from the left, $\Lambda_K(x) \leqq \Lambda_L(x)$.

COROLLARY 2. *If* $K \leqq L$ *and* f *is a nondecreasing real-valued function with domain* $(-\infty, \infty)$, *then* $m(f(K)) \leqq m(f(L))$, *provided these quantities are both defined.*

Proof of Corollary 2. $m(f(K)) = \int_{-\infty}^{\infty} f(\lambda) dm(P_K(\lambda))$ and $m(f(L)) =$

$\int_{-\infty}^{\infty} f(\lambda) dm(P_L(\lambda))$. The conclusion is immediate since by Corollary 1, $m(P_K[\lambda, \infty)) \leqq m(P_L[\lambda, \infty))$ and $m(P_L(-\infty, \lambda]) \leqq m(P_K(-\infty, \lambda])$ for all real numbers λ. Note that the hypotheses of Corollary 2 do not imply $f(K) \leqq f(L)$ [1].

THEOREM 3. *$\Lambda_{|K+L|}(x + r) \leqq \Lambda_{|K|}(x) + \Lambda_{|L|}(r)$ for $x > 0$ and $r > 0$, where $|K|$ is the absolute value of the operator K.*

COROLLARY 3. *Let $q = m(\text{support } (L))$ and assume $q < \infty$. Then $\Lambda_{|K|}(x + q) \leqq \Lambda_{|K+L|}(x)$ for $x > 0$; equivalently, $\Lambda_{|K|}(x) \leqq \Lambda_{|K+L|}(x - q)$ for $x > q$.*

Proof of Corollary 3. Let $x > 0$, and let ε be an arbitrary positive number with $\varepsilon < x$. Since $K = (K + L) - L$, by Theorem 3, $\Lambda_{|K|}(x + q) \leqq \Lambda_{|K+L|}(x - \varepsilon) + \Lambda_{|L|}(q + \varepsilon) = \Lambda_{|K+L|}(x - \varepsilon)$ since $\Lambda_{|L|}(q + \varepsilon) = 0$ or $-\infty$. Now apply the left continuity of the function $\Lambda_{|K+L|}$.

THEOREM 4. *Assume L is bounded with norm $\| L \|$. Then $\Lambda_{LKL}(x) \leqq \| L \|^2 \Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$. In particular, if P is a projection in A, then $\Lambda_{PKP}(x) \leqq \Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$.*

COROLLARY 4. *Assume L is invertible and $aI \leqq |L| \leqq bI$ for some positive numbers a and b. Then $a^2\Lambda_K(x) \leqq \Lambda_{LKL}(x) \leqq b^2\Lambda_K(x)$ for $0 < x \leqq m(P_K[0, \infty))$, where I is the identity operator on H.*

Proof of Corollary 4. Clearly $|L| \leqq b$ and $|L^{-1}| \leqq 1/a$. Apply Theorem 4 to LKL and to $L^{-1}(LKL)L^{-1} = K$.

The p-norm of a self-adjoint measurable operator T is defined [3, Definition 3.1] by $|T|_p = (m(|T|^p))^{1/p}$ if $1 \leqq p < \infty$ and $\| T \|_\infty = \sup \{\lambda \mid m(P_{|T|}[\lambda, \infty)) > 0\}$. Note that $\| T \|_\infty$ equals the operator norm of T if the gage space is regular, that is, if the gage of every nonzero projection is positive.

THEOREM 5. *Let $q = m(\text{support } (L))$ and assume $q < \infty$. Let $\mu = \Lambda_{|K|}(q)$. Then $\| K + L \|_p \geqq \| KP_K(-\mu, \mu) \|_p$ for $1 \leqq p \leqq \infty$.*

3. Proof of the theorems.

LEMMA 1. *Let P and Q be projections in A. Let $Y = \{v \in H \mid Pv = v$ and $Qv = 0\}$. Then $Y \in A$ and $Y + \bar{R}(PQ) = P$.*

Proof of Lemma 1. Let $v \in Y$ and $w \in R(PQ), w = PQz$. Then

$\langle v, w\rangle = \langle v, PQz\rangle = \langle Pv, Qz\rangle = \langle v, Qz\rangle = \langle Qv, z\rangle = \langle 0, z\rangle = 0$, so that $Y \perp \bar{R}(PQ)$.

Now let $Py = y$. Let z be the projection of y on the subspace $\bar{R}(PQ)$. Then $y = (y - z) + z$. Clearly $z \in \bar{R}(PQ)$ and consequently, $P(y - z) = y - z$. In addition, $(y - z) \perp \bar{R}(PQ)$. Let w be any vector. Then $\langle Q(y - z), w\rangle = \langle QP(y - z), w\rangle = \langle y - z, PQw\rangle = 0$, so that $Q(y - z) = 0$.

LEMMA 2. *Let P and Q be projections in A. Let $Y = \{v \in H \mid Pv = v$ and $Qv = 0\}$. Then $m(Y) \geqq m(P) - m(Q)$. In particular, if $Pv = v$ implies $Qv \neq 0$, then $m(Q) \geqq m(P)$.*

Proof of Lemma 2. By additivity of the gage and Lemma 1, $m(Y) + m(\bar{R}(PQ)) = m(P)$. To prove the lemma, it suffices to show that $m(\bar{R}(PQ)) \leqq m(Q)$; this is well-known for factors but we know of no reference for the general case. For later use, this is proved under the assumption that P is self-adjoint but is not necessarily a projection.

The operator PQ has polar decomposition [2, pp. 323–324] $PQ = M(QP^2Q)^{1/2}$, where M is a partial isometry with initial domain support $(QP^2Q)^{1/2} =$ support $(QP^2Q) = \bar{R}(QP^2Q)$ and terminal domain support $(PQP)^{1/2} =$ support $(PQP) = \bar{R}(PQP)$. Consequently, $\bar{R}(PQ) = \bar{R}(PQP)$; also $m(\bar{R}(PQP)) = m(\bar{R}(QPQ))$ since the initial domain and the terminal domain of a partial isomerty have the same gage. Then $m(\bar{R}(PQ)) = m(\bar{R}(PQP)) = m(\bar{R}(QPQ)) \leqq m(Q)$ since $\bar{R}(QPQ)$ is a subspace of Q.

If H is finite dimensional, Lemma 2 states that the dimension of the solution space of a system of $m(Q)$ homogeneous linear equations in $m(P)$ unknowns is at least $m(P) - m(Q)$.

Proof of Theorem 1. Let $x > 0$ and $\varepsilon > 0$ and $\mu = \Lambda_K(x + q)$. Then $m(P_K[\mu - \varepsilon, \infty)) \geqq x + q$. Apply Lemma 2 with $P = P_K[\mu - \varepsilon, \infty)$ and $Q =$ support (L) to obtain $m\{v \in H \mid P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0\} \geqq (x + q) - q = x$. If $P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0$, then $\langle (K + L)v, v\rangle \geqq (\mu - \varepsilon)\|v\|^2$, so that $P_{K+L}[\mu - \varepsilon, \infty)v \neq 0$. By Lemma 2, $m(P_{K+L}[\mu - \varepsilon, \infty)) \geqq m\{v \in H \mid P_K[\mu - \varepsilon, \infty)v = v$ and $Lv = 0\} \geqq x$. Since ε is an arbitrary positive number, $\Lambda_{K+L}(x) \geqq \mu = \Lambda_K(x + q)$.

Proof of Theorem 2. Let $x > 0$ and $r > 0$, and assume $\Lambda_{K+L}(x + r) > \Lambda_K(x) + \Lambda_L(r)$. Let $4\delta = \Lambda_{K+L}(x + r) - \Lambda_K(x) - \Lambda_L(r)$. Let $P = P_{K+L}[\Lambda_{K+L}(x + r) - \delta, \infty)$; then $m(P) \geqq x + r$. Let Q be projection on the subspace of H spanned by $P_K[\Lambda_K(x) + \delta, \infty)$ and $P_L[\Lambda_L(r) + \delta, \infty)$; then $m(Q) < x + r$.

Let $Pv = v$ with $\|v\| = 1$. Then $\langle (K + L)v, v\rangle \geqq \Lambda_{K+L}(x + r) - \delta = \Lambda_K(x) + \Lambda_L(r) + 4\delta - \delta > (\Lambda_K(x) + \delta) + (\Lambda_L(r) + \delta)$, so that either

$\langle Kv, v\rangle > \Lambda_K(x) + \delta$ or $\langle Lv, v\rangle > \Lambda_L(r) + \delta$. Therefore $Qv \neq 0$. By Lemma 2, $m(Q) \geqq m(P)$, which is impossible since $m(P) \geqq x + r$ and $m(Q) < x + r$.

Proof of Theorem 3. Let $\lambda > 0$ and $\psi > 0$. Apply Lemma 2 with $P = P_{K+L}[\lambda + \psi, \infty)$ and $Q =$ projection on the subspace of H spanned by $P_K[\lambda, \infty)$ and $P_L[\psi, \infty)$ to obtain $m(P_{K+L}[\lambda + \psi, \infty)) \leqq m(Q) \leqq m(P_K[\lambda, \infty)) + m(P_L[\psi, \infty))$. Similarly, $m(P_{-(K+L)}[\lambda+\psi, \infty)) \leqq m(P_{-K}[\lambda, \infty)) + m(P_{-L}[\psi, \infty))$. Adding these inequalities yields $m(P_{|K+L|}[\lambda + \psi, \infty)) \leqq m(P_{|K|}[\lambda, \infty)) + m(P_{|L|}[\psi, \infty))$.

Let δ be a small positive number. Then $m(P_{|K+L|}[\Lambda_{|K|}(\lambda) + \delta + \Lambda_{|L|}(\psi) + \delta, \infty)) \leqq m(P_{|K|}[\Lambda_{|K|}(\lambda) + \delta, \infty)) + m(P_{|L|}[\Lambda_{|L|}(\psi) + \delta, \infty)) < \lambda + \psi$, so that $\Lambda_{|K+L|}(\lambda + \psi) < \Lambda_{|K|}(\lambda) + \Lambda_{|L|}(\psi) + 2\delta$.

Proof of Theorem 4. Without loss of generality assume $\|L\| = 1$. We will show that $m(P_{LKL}[\lambda, \infty)) \leqq m(P_K[\lambda, \infty))$ for $\lambda > 0$. Let $\lambda > 0$ and let $P = P_{LKL}[\lambda, \infty)$. Let $v \in H$, $\|v\| = 1$, and $Pv = v$. Then $\langle LKLv, v\rangle \geqq \lambda$, so that $\langle KLv, Lv\rangle \geqq \lambda$. Since $\|Lv\| \leqq 1$, $P_K[\lambda, \infty)Lv \neq 0$ and $LP_K[\lambda, \infty)Lv \neq 0$. Let Q be projection on $\bar{R}(LP_K[\lambda, \infty)L)$; then $Qv \neq 0$. By Lemma 2, $m(P) \leqq m(Q)$. But $m(Q) \leqq m(\bar{R}(LP_K[\lambda, \infty))$ because of set inclusion, and $m(\bar{R}(LP_K[\lambda, \infty)) \leqq m(P_K[\lambda, \infty))$; this is proved in the last paragraph of the proof of Lemma 2.

Proof of Theorem 5. If $p = \infty$, $\|K + L\|_\infty = \lim_{\varepsilon\to 0+} \Lambda_{|K+L|}(\varepsilon) \geqq \lim_{\varepsilon\to 0+} \Lambda_{|K|}(q + \varepsilon)$ by Corollary 3. If $m(P_{|K|}[\mu, \infty)) > q$, then $\lim_{\varepsilon\to 0+} \Lambda_{|K|}(q + \varepsilon) = \mu \geqq \|KP_K(-\mu, \mu)\|_\infty$. If $m(P_{|K|}[\mu, \infty)) = q$, then $\Lambda_{|K|}(q + \varepsilon) = \Lambda_{|KP_K(-\mu,\mu)|}(\varepsilon)$ for all $\varepsilon > 0$ and the result is immediate. If $m(P_{|K|}[\mu, \infty)) < q$, then $m(P_{|K|}(\mu - \varepsilon, \mu)) = \infty$ for every $\varepsilon > 0$, so that $\lim_{\varepsilon\to 0+} \Lambda_{|K|}(q + \varepsilon) = \mu = \|KP_K(-\mu, \mu)\|_\infty$.

Now let $1 \leqq p < \infty$. Since the theorem is trivial for $\mu = 0$, assume $\mu > 0$. If $m(P_{|K+L|}[\lambda, \infty)) = \infty$ for some $\lambda > 0$, then $\|K + L\|_p = \infty$ and the theorem holds trivially. Therefore, assume $m(P_{|K+L|}[\lambda, \infty)) < \infty$ for all $\lambda > 0$. Fix λ, $0 < \lambda < \mu$. Let $\gamma = m(P_{|K+L|}[\lambda, \infty))$ and let $\varepsilon > 0$. Note that if $\lambda = \Lambda_{|K+L|}(\gamma + \varepsilon)$, then $m(P_{|K+L|}(\lambda - \varepsilon_1, \lambda)) = \infty$ for every $\varepsilon_1 > 0$. Therefore, we can assume $\lambda > \Lambda_{|K+L|}(\gamma + \varepsilon)$. By Corollary 3, $\Lambda_{|K+L|}(\gamma + \varepsilon) \geqq \Lambda_{|K|}(\gamma + q + \varepsilon)$. Therefore, $m(P_{|K|}[\lambda, \infty)) \leqq \gamma + q + \varepsilon$. Since ε is arbitrary, $m(P_{|K+L|}[\lambda, \infty)) \geqq m(P_{|K|}[\lambda, \infty)) - q \geqq m(P_{|K|}[\lambda, \mu))$ for $\lambda < \mu$. Now $\|K + L\|_p^p = \int_{-\infty}^{\infty} |\lambda|^p dm(P_{K+L}(\lambda)) = \int_0^{\infty} \lambda^p dm(P_{|K+L|}(\lambda))$ and

$$\|KP_K(-\mu, \mu)\|_p^p = \int_{-\mu<\lambda<\mu} |\lambda|^p dm(P_K(\lambda)) = \int_{0\leqq\lambda<\mu} \lambda^p dm(P_{|K|}(\lambda)) .$$

The conclusion follows immediately.

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