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# STRONGLY BOUNDED OPERATORS

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**Recently Dobrakov, Batt and Berg, and Brooks and Lewis have studied the class of strongly bounded operators on continuous function spaces in detail. In many cases, these operators coincide with the weakly compact operators and therefore form a norm-closed two-sided ideal; and it is known in general that the strongly bounded operators form a norm-closed left ideal. In this note an example is presented which shows that the strongly bounded operators do not form a right ideal.**

Let each of $E$ and $F$ denote a $B$-space, let $H$ denote a locally compact Hausdorff space, and let $\boldsymbol{B} \equiv C_0(H, E)$ be the $B$-space (under sup norm) of all continuous $E$-valued functions defined on $H$ which vanish at infinity. It is known that if an operator (=continuous linear transformation) $L$ maps $\boldsymbol{B}$ into $F$ and $\Sigma = \Sigma(H)$ is the Borel $\sigma$-algebra of subsets of $H$, then there is a unique weakly regular finitely additive representing measure $m: \Sigma \rightarrow B(E, F^{**})$ so that $L(f) = \int f dm$, $f \in \boldsymbol{B}$, e.g., see Batt and Berg [1] and Brooks and Lewis [3] for further details.

The representing measure $m: \Sigma \rightarrow B(E, F^{**})$ is said to be *strongly bounded* ($s$-bounded) provided that if $(A_i)$ is a disjoint sequence, then $\tilde{m}(A_i) \rightarrow 0$, where $\tilde{m}$ denotes the semivariation of $m$; an operator will be called $s$-bounded if its representing measure is $s$-bounded. Equivalent formulations are given in the following lemma, which we state for reference purposes. For the details of the proof, one may consult Brooks and Lewis [3].

LEMMA 1. *Suppose that $m$ is a representing measure, $m \leftrightarrow L$. Then the following are equivalent:*

(a) *$m$ is $s$-bounded;*

(b) *$\tilde{m}(A_i) \rightarrow 0$ whenever $A_i \searrow \varnothing$;*

(c) *$\{|m_z|: z \in F_1^*\}$ is conditionally weakly compact in $ca(\Sigma, E^*)$;*

(d) *$\Sigma m(A_i)x_i$ converges (in $F$) for each disjoint sequence $(A_i)$ and $(x_i) \subset E_1$ (=closed unit ball of $E$);*

(e) *If $(A_i) \subset \Sigma$ and $A_i \searrow \varnothing$, then there is a nested sequence $U_n$ of open sets so that $A_n \subset U_n$ and $L(f_n) \rightarrow 0$ uniformly for each sequence $(f_n)$ so that support $(f_n) \subset U_n$, $||f_n|| \leqq 1$.*

The class of $s$-bounded operators which we denote by $S$ has been studied in [1], [2], [3], and [5], with the sharpest results being presented in § 4 of [3]. In particular, if $E$ is reflexive, then $L \in S$ if and only if $L$ is weakly compact. Dobrakov proved in [5] that if $F = \boldsymbol{B}$, then $S$ is a norm-closed left ideal. Part (e) of Lemma 1 also provides a simple proof of this fact. However, the question of whether $S$ is a two-sided ideal is apparently open; the following example answers this question in the negative and thereby settles a problem raised in [3] and [5]. Consequently, while $S$ coincides with the weakly compact operators in many cases, in general it differs not only in terms of mapping properties but also in terms of algebraic structure.

EXAMPLE. For each positive integer $n$, let $E^{2n+1}$ denote Euclidean $2n+1$ dimensional space equipped with the $l^1$-norm, and let $E$ be the substitution or product space $P_{l^\infty}(E^{2n+1})$ [4, p. 31]. Let $e_{i,j_i}$ be that point in $E$ with 1 in the $j_i$-component of the $i$th coordinate and 0 otherwise. For $n$ a positive integer, set $A_n = (1/(n+1), 1/n)$ and partition $A_n$ into $2n+1$ disjoint subintervals of equal length. Denote these sets by $a_{n,i_n}$, $1 \leqq i_n \leqq 2n+1$. Equip $\Gamma = \{a_{i,j_i} : i \geqq 1, 1 \leqq j_i \leqq 2i+1\}$ with the discrete topology, and let $H$ denote the one-point compactification of this set, with the point at infinity denoted by $a_{0,0}$. Let $\nu$ denote Lebesgue measure, where we put $\nu(a_{0,0}) = 0$; and if $A \in \Sigma(H)$, then define $\lambda(A) = \Sigma\nu(a_{i,j})$, $a_{i,j} \subset A$. Notice that if $x \in E$ and we put $x(t) = x$ for each $t \in H$, then $E$ is isometrically isomorphic to a subspace of $C(H, E)$. Now define $L\colon C(H, E) \to E \subsetneq C(H, E)$ by setting $L(f) = \int f d\lambda$; then $L$ is an operator, $L(x) = x$ for each $x \in E$, and $L$ is $s$-bounded since $\lambda$ is countably additive with finite total variation. However, $L$ is not weakly compact since $E$ is not reflexive and thus $E_1 = L(C(H, E)_1)$ is not weakly compact.

If $S_E(\Sigma(\Gamma))$ denotes the $E$-valued simple functions and $U_E(\Sigma(\Gamma))$ denotes their uniform closure, then $C_0(\Gamma, E) \subset \overline{S_E(\Sigma(\Gamma))} = U_E(\Sigma(\Gamma))$. Therefore, to define an operator $U$ on $C_0(\Gamma, E)$, it suffices to define $U$ on $S_E(\Sigma(\Gamma))$. If $\xi_A$ denotes the characteristic function of $A \in \Sigma(\Gamma)$ and $x = (x_{i,j_i}) \in E$, then we define $U(\xi_A x)$ by the following equation:

$$P_{i,j_i}(U(\xi_A x))\begin{cases} x_{i,j_i} & \text{if } a_{i,j_i} \in A, \\ 0 & \text{otherwise}, \end{cases}$$

where $P_{i,j_i}$ is the projection on the $j_i$-component of the $i$th coordinate. Now extend $U$ by linearity to all of $S_E(\Sigma(\Gamma))$; it is not difficult to see that $U$ is well-defined, linear, and $\|U\| \leqq 1$. Hence we may consider $U$ to be defined on all of $C_0(\Gamma, E)$.

Then if $f \in C(H, E)$, let $f_0(\cdot) = f(\cdot) - f(a_{0,0}) \in C_0(\Gamma, E)$, and define $V(f)$ to be $U(f_0)$. It follows that $V$ is linear and continuous. Further, if $\mu$ is the representing measure for $V$, then $L \circ \mu$ is the representing measure for $L \circ V$, and $\sum_{i=1}^{\infty} L[\mu(\{a_{i,1_i}\})e_{i,1_i}] = \sum L[U(\xi_{\{a_{i,1_i}\}} e_{i,1_i})] =$

$\sum L(e_{i,1_i}) = \sum e_{i,1_i}$, and this series clearly does not converge in $E$. Thus $L \circ V$ is not $s$-bounded. In fact, the representing measure for $L \circ V$ is not even countably additive. For if $\varphi \in E_1$ so that $P_{i,1_i}(\varphi) = 1$ for each $i$, then $\sum L[\mu(\{a_{i,1_i}\})\varphi] = \sum L(e_{i,1_i})$, a divergent series.

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