PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

A GENERALIZATION OF $(-1, 1)$ RINGS

ERWIN KLEINFELD

A ring is defined to be a division ring in case the equations $ax = b$, and $ya = b$, have unique solutions for x and y whenever $a \neq 0$. It is shown that division rings of characteristic $\neq 2, 3$ which satisfy the identities (i) $(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x$, (ii) $(x, y, z) + (y, z, x) + (z, x, y) = 0$, and (iii) $((x, y), y, y) = 0$, are associative.

Main section. We consider rings R of characteristic different from 2 and 3 which satisfy the identities

$$(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x \, , \tag{1}$$

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \, , \tag{2}$$

$$((x, y), y, y) = 0 \, , \tag{3}$$

where $(a, b, c) = (ab)c - a(bc)$, and $(a, b) = ab - ba$. The first of these identities holds in right alternative rings and has been investigated in combination with others [5, 6]. In an arbitrary ring it can be verified that $((x, y), z) + ((y, z), x) + ((z, x), y) = (x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (z, y, x) - (y, x, z)$. If the ring is third-power associative, then $(x, y, z) + (y, z, x) + (z, x, y) + (x, z, y) + (z, y, x) + (y, x, z) = 0$, so that $((x, y), z) + ((y, z), x) + ((z, x), y) = 2(x, y, z) + 2(y, z, x) + 2(z, x, y)$. We have shown that in a third-power associative ring of characteristic $\neq 2$, identity (2) is equivalent to Lie-admissibility. Next we will show that a ring which satisfies (1) and (2) is Jordan-admissible if and only if it satisfies (3). Suppose A is a ring of characteristic $\neq 2$ which satisfies (1) and (2) and which is Jordan-admissible. That means that A under a new product $xy + yx$ must satisfy the Jordan identity, so that $(xy + yx)x^2 + x^2(xy + yx) - x(x^2y + yx^2) - (x^2y + yx^2)x = 0$, which implies that $-(x^2, y, x) + (x, y, x^2) - (y, x^2, x) + (y, x, x^2) + (x, x^2, y) - (x^2, x, y) = 0$. Since we have $(yx, x, x) - (y, x^2, x) + (y, x, x^2) = (y, x, x)x$, in every ring, while (1) implies $(yx, x, x) = (y, x, x)x$, we see that $(y, x^2, x) = (y, x, x^2)$. Also (1) implies the two identities $(x^2, y, x) + (x, x, (y, x)) = x(x, y, x) + (x, y, x)x$, and $(x^2, x, y) + (x, x, (x, y)) = x(x, x, y) + (x, x, y)x$. Adding these and use of (2) shows that $(x^2, y, x) + (x^2, x, y) = x(x, y, x) + x(x, x, y) + (x, y, x)x + (x, x, y)x = -x(y, x, x) - (y, x, x)x$. In an arbitrary ring one may verify the identity $(xy, x, x) - (x, yx, x) + (x, y, x^2) = x(y, x, x) + (x, y, x)x$, while (1) implies that $(xy, x, x) = x(y, x, x)$, so that $(x, y, x^2) = (x, yx, x) + (x, y, x)x$. Also in an arbitrary ring $(x^2, x, y) - (x, x^2, y) + (x, x, xy) = x(x, x, y)$, while (1) implies $(x^2, x, y) + (x, x, (x, y)) = x(x, x, y) +$

$(x, x, y)x$. Subtracting the next to the last from the last equation it follows that $(x, x^2, y) = (x, x, yx) + (x, x, y)x$. Putting all of these substitutions in the Jordan-admissible relation, we find that $(y, x, x)x = x(y, x, x)$, as all other terms cancel. But in that case, since we have already verified that $(xy, x, x) = x(y, x, x)$, and $(yx, x, x) = (y, x, x)x$, we must have $((x, y), x, x) = 0$. This is equivalent to identity (3). Similarly one can work through this argument backwards to show that (1), (2), and (3) imply Jordan-admissibility. Of course (3) also holds in every right alternative ring. All three identities are valid in $(-1, 1)$ rings, which have been investigated in detail by Maneri [7] and Hentzel [1, 2].

Nonassociative division rings are of interest in the study of the foundation of projective geometry, as they coordinate non-Desarguesian planes. It is therefore of interest to classify those rings R which are division rings. Our main result is the following.

THEOREM. *All division rings of characteristic different from* 2 *and* 3 *which satisfy the identities* (1)-(3) *are associative.*

In order to prove this result we derive a number of other identities, many of which are valid in all R, not necessarily division rings. We begin with the Teichmüller identity

$$(4)\quad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z ,$$

which is valid in every ring and which can be established by use of the definition of the associator (a, b, c). Four applications of the identity (4) lead to the equations

$$\begin{aligned}(wx, y, z) - (w, xy, z) + (w, x, yz) &= w(x, y, z) + (w, x, y)z ,\\ -(xy, z, w) + (x, yz, w) - (x, y, zw) &= -x(y, z, w) - (x, y, z)w ,\\ (yz, w, x) - (y, zw, x) + (y, z, wx) &= y(z, w, x) + (y, z, w)x ,\\ -(zw, x, y) + (z, wx, y) - (z, w, xy) &= -z(w, x, y) - (z, w, x)y ,\end{aligned}$$

so that adding all the left sides we obtain, after repeated applications of (2), the sum of zero. Adding the right hand sides we obtain $(w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y))$. Thus

$$(5)\quad (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0 .$$

Using (1) twice, we discover that $(ab, y, y) + (a, b, (y, y)) = a(b, y, y) + (a, y, y)b$, while $(ba, y, y) + (b, a, (y, y)) = b(a, y, y) + (b, y, y)a$. Subtracting the second equation from the first it follows that

$$(6)\qquad ((a, b), y, y) = (a, (b, y, y)) - (b, (a, y, y)) .$$

If we let $a = x$, and $b = y$, in (6) and use (3), then

$$(7) \qquad (y, (x, y, y)) = 0 .$$

A linearization of (7), obtained by replacing y by $y + z$ and $y - z$, leads to

$$(8) \qquad (y, (x, y, z)) + (y, (x, z, y)) + (z, (x, y, y)) = 0 .$$

Substituting $w = z = y$ in (5) it follows that $(y, (x, y, y)) - (y, (y, x, y)) + (y, (y, y, x)) = 0$, since $(y, y, y) = 0$, follows from (2). But $(x, y, y) + (y, x, y) + (y, y, x) = 0$, as a result of (2). Thus $2(y, (y, x, y)) = 0$, so that

$$(9) \qquad (y, (y, x, y)) = 0 .$$

Linearizing (9), it follows that

$$(10) \qquad (z, (y, x, y)) + (y, (z, x, y)) + (y, (y, x, z)) = 0 .$$

Substituting $x = z = y$, and $w = x$, in (1) it follows that

$$(11) \qquad (xy, y, y) = (x, y, y)y .$$

Expanding $((a, b, c), y, y) = ((ab)c, y, y,) - (a(bc), y, y)$ and applying (1) repeatedly to break up products, we establish that

$$(12) \quad ((a, b, c), y, y) = ((a, y, y), b, c) + (a, (b, y, y), c) + (a, b, (c, y, y)) .$$

Substituting $z = x$, and $w = y$, in (5) it follows that $2(y, (x, y, x)) - 2(x, (y, x, y)) = 0$. Because the characteristic is different from 2 this implies

$$(13) \qquad (y, (x, y, x)) = (x, (y, x, y)) .$$

Substituting $z = y$, and $w = x$, in (5), it follows that $(x, (x, y, y)) - (x, (y, y, x)) + (y, (y, x, x)) - (y, (x, x, y)) = 0$. However, (y, x, x) may be replaced by $-(x, y, x) - (x, x, y)$, because of (2) and $-(y, y, x)$ by $(y, x, y) + (x, y, y)$, so that $(x, (x, y, y)) + (x, (y, x, y)) + (x, (x, y, y)) - (y, (x, y, x)) - (y, (x, x, y)) - (y, (x, x, y)) = 0$. But the second and fourth terms cancel as a consequence of (13), so that the remaining terms become $2(x, (x, y, y)) - 2(y, (x, x, y)) = 0$. Since the characteristic of R is different form 2 this implies $(x, (x, y, y)) = (y, (x, x, y))$. Substituting $z = x$ in (8), it follows that $(x, (x, y, y)) = -(y, (x, y, x)) - (y, (x, x, y)) = (y, (y, x, x))$, since $(y, x, x) + (x, y, x)(x, x, y) = 0$, as a consequence of (2). We can now combine the previous equations to obtain $(y, (x, x, y)) = (y, (y, x, x))$. Since $(y, x, x) + (x, y, x) + (x, x, y) = 0$, this implies

$$(14) \qquad (y, (x, y, x)) = -2(y, (y, x, x)) = -2(y, (x, x, y)) .$$

Substituting $y = a + b$, $x = y$, and $y = a - b$, $x = y$, in (14) it follows that

$$(15)\quad (a, (y, b, y)) + (b, (y, a, y)) = -2(a, (b, y, y)) - 2(b, (a, y, y)) .$$

Substituting $x = xy$ in (3), we derive $((xy, y), y, y) = 0$. Substituting $x = (x, y)$ in (11) it follows that $((x, y)y, y, y) = ((x, y), y, y)y = 0$, using (3). But $(xy, y) - (x, y)y = (xy)y - y(xy) - (xy)y + (yx)y = (y, x, y)$. Consequently $((y, x, y), y, y) = ((xy, y), y, y) - ((x, y)y, y, y) = 0$. We have shown

$$(16)\qquad ((y, x, y), y, y) = 0 .$$

Substituting $x = (z, x, y) + (y, x, z)$ in (3), we see that $(((z, x, y) + (y, x, z), y), y, y) = 0$. But then use of (10) leads to $0 = -((z, (y, x, y)), y, y) = (((y, x, y), z), y, y)$. Substituting $a = (y, x, y)$, $b = z$, in (6), it follows that $(((y, x, y), z), y, y) = ((y, x, y), (z, y, y)) - (z, ((y, x, y), y, y)) = ((y, x, y), (z, y, y))$, because of (16). Thus $((y, x, y), (z, y, y)) = 0$. Permuting x and z in this last inentity we obtain

$$(17)\qquad ((y, z, y), (x, y, y)) = 0 .$$

From (8) it follows that $(z, (x, y, y)) = -(y, (x, y, z)) - (y, (x, z, y)) = ((x, y, z) + (x, z, y), y)$. Substituting $x = (x, y, z) + (x, z, y)$ in (3), it follows that $(((x, y, z) + (x, z, y), y), y, y) = 0$. Thus $((z, (x, y, y), y, y)) = 0$. Substituting $a = z$, $b = (x, y, y)$ in (6), it follows that $((z, (x, y, y)), y, y) = (z, ((x, y, y), y, y)) - ((x, y, y), (z, y, y))$. Hence $0 = (z, ((x, y, y), y, y)) - ((x, y, y), (z, y, y))$, so that

$$(18)\qquad (z, ((x, y, y), y, y)) = ((x, y, y), (z, y, y)) .$$

Substitute $a = c = y$, and $b = x$, in (12). Then $((y, x, y), y, y) = (y, (x, y, y), y)$. As a consequence of (16) it follows that

$$(19)\qquad (y, (x, y, y), y) = 0 .$$

In (15) substitute $a = z$, $b = (x, y, y)$. Then $(z, (y, (x, y, y), y)) + ((x, y, y), (y, z, y)) = -2(z, ((x, y, y), y, y)) - 2((x, y, y), (z, y, y))$. Because of (17) and (19) the left side of the last identity is zero. Thus $-2(z, ((x, y, y), y, y)) - 2((x, y, y), (z, y, y)) = 0$. Using characteristic different from 2, this implies

$$(20)\qquad (z, ((x, y, y), y, y)) + ((x, y, y), (z, y, y)) = 0 .$$

Comparing identities (18) and (20), and using characteristic different from 2 once more, we obtain

$$(21)\qquad (z, ((x, y, y), y, y)) = 0 .$$

Substituting $x = xy$ in (21), we see that $(z, ((xy, y, y), y, y)) = 0$. But (11) implies that $(xy, y, y) = (x, y, y)y$, so that $(z, ((x, y, y)y, y, y)) = 0$. Substituting $x = (x, y, y)$ in (11), it follows that $((x, y, y)y, y, y) =$

$((x, y, y), y, y)y$, so that

$$(z, ((x, y, y), y, y)y) = 0 . \tag{22}$$

By expansion $(a, bc) + (b, ca) + (c, ab) = -(a, b, c) - (b, c, a) - (c, a, b) = 0$, as a consequence of (2). Thus we have established

$$(a, bc) + (b, ca) + (c, ab) = 0 . \tag{23}$$

Substituting $a = ((x, y, y), y, y)$, $b = y$, $c = z$, in (23), it follows that $(((x, y, y), y, y), yz) + (y, z((x, y, y), y, y)) + (z, ((x, y, y), y, y)y) = 0$. Substituting $z = yz$ in (21) implies that the first term of the last identity must vanish. The third term vanishes because of (22). That leaves

$$(y, z((x, y, y), y, y)) = 0 . \tag{24}$$

We are now ready to prove the following.

LEMMA 1. *If R is a division ring then for every element y in R either $(y, R) = 0$, or $(R, y, y) = 0$.*

Proof. It follows directly from (24) that either $((R, y, y), y, y) = 0$, or $(y, R) = 0$. Suppose that $((R, y, y), y, y) = 0$. Then $((x^2, y, y), y, y) = 0$. Substituting $w = x$, $z = y$, in (1), it follows that $(x^2, y, y) = x(x, y, y) + (x, y, y)x$. Thus $0 = (x(x, y, y) + (x, y, y)x, y, y)$. Substituting $w = (x, y, y)$, and $z = y$ in (1), it follows that $((x, y, y)x, y, y) = (x, y, y)^2 + ((x, y, y), y, y)x = (x, y, y)^2$, as a consequence of $((R, y, y), y, y) = 0$. Also substituting $w = x$, $x = (x, y, y)$, $z = y$, in (1), it follows that $(x(x, y, y), y, y) = x((x, y, y), y, y) + (x, y, y)^2 = (x, y, y)^2$. Thus $0 = (x(x, y, y) + (x, y, y)x, y, y) = 2(x, y, y)^2$. Then characteristic different from 2 implies that $(x, y, y)^2 = 0$, and since R is a division ring, $(x, y, y) = 0$. This completes the proof of the lemma.

LEMMA 2. *If R is a division ring then for all elements x, y, z in R we have the identity $(z, (x, y, y)) = 0$.*

Proof. From (8) it follows that $(z, (x, y, y)) = -(y, (x, y, z)) - (y, (x, z, y))$. Then Lemma 1 suffices to show that either the left hand side or the right hand side vanishes for every fixed y and arbitrary x and z in R. This completes the proof of the lemma.

LEMMA 3. *If R is a division ring then the identity $(x, (x, y, y)z) = 0$, holds for all elements x, y, z in R.*

Proof. Substituting x^2 for x in Lemma 2, it follows that

$(z, (x^2, y, y)) = 0$. But substituting $w = x$, $z = y$, in (1), it follows that $(x^2, y, y) = x(x, y, y) + (x, y, y)x = 2x(x, y, y)$, as a result of substituting $z = x$ in Lemma 2. Thus $0 = (z, 2x(x, y, y))$, so that $(z, x(x, y, y)) = 0$. Substituting $a = z$, $b = x$, $c = (x, y, y)$ in (23), it follows that $(z, x(x, y, y)) + (x, (x, y, y)z) + ((x, y, y), zx) = 0$. We have just shown that the first term of the last equation is zero. Substituting zx for z in Lemma 2 implies that the third term is zero as well. Hence $(x, (x, y, y)z) = 0$. This completes the proof of the lemma.

LEMMA 4. *If R is a division ring then the identity $((x, z), y, y) = 0$, holds for all elements x, y, z in R.*

Proof. Substituting $a = x$, $b = z$ in (6), it follows that $((x, z), y, y) = (x, (z, y, y)) - (z, (x, y, y)) = 0$, as a consequence of Lemma 2. This completes the proof of the lemma.

THEOREM 1. *If R is a division ring then R is either right alternative or the identity $((a, b), c) = 0$, holds for all elements a, b, c in R.*

Proof. Linearizing the identity of Lemma 3, we obtain $(w, (x, y, y)z) = -(x, (w, y, y)z)$. Substituting $w = (a, b)$ in this, it is clear that $((a, b), (x, y, y)z) = -(x, ((a, b), y, y)z) = 0$, using Lemma 4 with $x = a$, $z = b$. If R is not right alternative, then for every c in R we can find a z such that $(x, y, y)z = c$. This completes the proof of the theorem.

We are now ready to prove the main result.

THEOREM 2. *If R is a division ring then R must be associative.*

Proof. If R is right alternative then it is known [3, 8] that R must be alternative and then (2) suffices to make R associative. Then as a result of Theorem 1 we are down to the case where the identity $((a, b), c) = 0$, holds for all elements a, b, c in R. In (4) substitute $w = x$, and $x = y$. Then $(xy, y, z) - (x, y^2, z) + (x, y, yz) = x(y, y, z) + (x, y, y)z$. Substituting $w = x$, and $x = y$, in (1) leads to $(xy, y, z) + (x, y, (y, z)) = x(y, y, z) + (x, y, z)y$. Subtract the left hand sides of the lest two identities to obtain

$$(x, y^2, z) - (x, y, zy) = -(x, y, y)z + (x, y, z)y \; . \tag{25}$$

Substituting $w = x$, $x = z$, $z = y$ in (4), we find that

(26) $(xz, y, y) - (x, zy, y) + (x, z, y^2) = x(z, y, y) + (x, z, y)y$.

Define K to be the set of all elements k in R such that $(k, R) = 0$. Clearly K is closed under subtraction. Also Lemma 2 implies that (x, y, y) belongs to K, for all elements x, y in R. Hence $(x, y, z) + (x, z, y)$ is also an element of K, for all x, y, z in R. Adding the left hand sides of (25) and (26), it is clear that we get an element of K. Thus

(27) $(x, y, z)y + (x, z, y)y + x(z, y, y) - (x, y, y)z$ belongs to K .

Substituting $w = x$, $x = z$, $z = y$ in (1), it follows that $(xz, y, y) = x(z, y, y) + (x, y, y)z$, so that

(28) $x(z, y, y) + (x, y, y)z$ belongs to K .

Subtracting (28) from (27), it is clear that

(29) $(x, y, z)y + (x, z, y)y - 2(x, y, y)z$ belongs to K .

Because of (11) we have $(x, y, y)y = (xy, y, y)$, so that Lemma 2, with xy substituted for x, implies

(30) $(x, y, y)y$ belongs to K .

Linearizing (30), we obtain

(31) $(x, y, z)y + (x, z, y)y + (x, y, y)z$ belongs to K .

Comparing (29) and (31), it is clear that $3(x, y, y)z$ belongs to K, so that $(x, y, y)z$ must also. Thus $(w, (x, y, y)z) = 0$, for all w, x, y, z in R. If R is not right alternative, then clearly R must be commutative. But in that case it is certainly flexible and that was enough to imply that R is alternative and hence associative [6]. This is a contradiction, so R must have been right alternative and hence associative to begin with. This completes the proof of the theorem.

We conclude with a generalization of Theorem 2. Let S be a division ring of characteristic $\neq 2, 3$, which satisfies (1), (3) and a weaker form of identity (2), namely that (2) holds whenever x, y, z lie in a subring which can be generated by two elements. Now we are dealing with a set of identities which are valid in every alternative ring. From the conclusion of Theorem 2 it follows that every subring of S generated by two elements is associative, so that S must in fact be alternative. This yields a worthwhile generalization of the main result, with little additional effort.

References

1. I. R. Hentzel, $(-1, 1)$ *rings*, Proc. Amer. Math. Soc., **22** (1969), 367-374.
2. ———, *Nil semi-simple* $(-1, 1)$ *rings*, J. Algebra, **22** (1972), 442-450.
3. Erwin Kleinfeld, *Right alternative rings*, Proc. Amer. Math. Soc., **4** (1953), 939-944.
4. ———, *On a class of right alternative rings*, Math. Zeitschr., **87** (1965), 12-16.
5. ———, *Generalization of alternative rings, I*, J. Algebra, **18** (1971), 304-325.
6. ———, *Generalization of alternative rings, II*, J. Algebra, **18** (1971), 326-339.
7. Carl Maneri, *Simple* $(-1, 1)$ *rings with an idempotent*, Proc, Amer. Math. Soc., **14** (1963), 110-117.
8. L. A. Skorniakov, *Right alternative fields*, Izvestia Akad. Nauk. SSSR Ser. Mat., **15** (1951), 177-184.

Received June 6, 1973. This research was supported in part by the National Science Foundation under GP-32898 XI.

University of Iowa